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Exponentially distributed noise—its correlation function and its effect on nonlinear dynamics

George N Farah[1,2] **and Benjamin Lindner**[1,3,*]

[1] Bernstein Center for Computational Neuroscience Berlin, Philippstr. 13, Haus 2, 10115 Berlin, Germany
[2] Department of Physics, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany
[3] Physics Department of Humboldt University Berlin, Newtonstr. 15, 12489 Berlin, Germany

E-mail: benjamin.lindner@physik.hu-berlin.de

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Abstract
We propose a simple Langevin equation as a generator for a noise process with Laplace-distributed values (pure exponential decays for both positive and negative values of the noise). We calculate explicit expressions for the correlation function, the noise intensity, and the correlation time of this noise process and formulate a scaled version of the generating Langevin equation such that correlation time and variance or correlation time and noise intensity for the desired noise process can be exactly prescribed. We then test the effect of the noise distribution on a classical escape problem: the Kramers rate of an overdamped particle out of the minimum of a cubic potential. We study the problem both for constant variance and constant intensity scalings and compare to an Ornstein–Uhlenbeck process with the same noise parameters. We demonstrate that specifically at weak fluctuations, the Laplace noise induces more frequent escapes than its Gaussian counterpart while at stronger noise the opposite effect is observed.

Keywords: active particles, escape problem, colored noise, non-Gaussian fluctuations, Langevin equation, Markovian embedding

(Some figures may appear in colour only in the online journal)

*Author to whom any correspondence should be addressed.

1. Introduction

How fluctuations lead to emergent behavior is a recurring theme in statistical physics. The escape of a noise-driven particle over a potential well, the celebrated Kramers problem [17, 20], is a paradigmatic example with many applications in different fields of science. A lot of studies have focussed on the kind of nonlinear dynamics that is driven by the noise: be it bistable, multistable, excitable, or chaotic. For the escape statistics, however, the *nature of the driving fluctuations* has equal importance and so it is crucial to have well-understood noise models that capture the salient features of the dynamical randomness that drives the dynamics of interest.

Basic characteristics of dynamical fluctuations $\eta(t)$ comprise (i) its stationary statistics, i.e. the stationary one-time probability density $p_0(\eta)$, and (ii) its temporal correlations given by the correlation function $c(\tau) = \langle \eta(t)\eta(t+\tau)\rangle - \langle \eta(t)\rangle^2$. Many studies of noise-induced effects make the simple assumption of a white Gaussian noise; differential equations driven by such a noise process then obey a corresponding Fokker–Planck equation for their probability density that is a classical instrument for dealing with escape, transport and first-passage-time problems [13, 34, 46]. If we want to relax the assumption of an uncorrelated noise (which is not met in many systems, see [16]), but the noise still possesses Gaussian statistics, a colored noise process can be incorporated into the Fokker–Planck framework by *Markovian embedding*, i.e. by additional stochastic differential equations that generate temporally correlated noise. By far the most popular example the Ornstein–Uhlenbeck (OU) noise in the scaled version

$$\tau_c \dot{\eta}_{ou} = -\eta_{ou} + \sqrt{2\tau_c \sigma_\eta^2}\,\xi(t), \tag{1}$$

where $\xi(t)$ is Gaussian white noise with zero mean and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. Equation (1) is the stochastic differential equation of the well-known Ornstein–Uhlenbeck process for the velocity of a Brownian particle [45] [equivalently, it can be regarded as a Brownian motion in a harmonic potential $U_{ou}(\eta) = \eta^2/(2\tau_c)$, see figure 1(a)] but serves here as a generator for a Gaussian and temporally exponentially correlated noise with pdf $P_0(\eta) = e^{-\eta^2/(2\sigma_\eta^2)}/\sqrt{2\pi\sigma_\eta^2}$ and correlation function $c(\tau) = \sigma_\eta^2\, e^{-|\tau|/\tau_c}$, where σ_η^2 is the variance and $\tau_c = \int_0^\infty d\tau\, c(\tau)/c(0)$ is the correlation time. More complex correlation functions are possible if we go from one linear SDE to a system of linear SDEs (see e.g. [38, 39, 47]) but we always keep a stationary Gaussian statistics.

On the other end, we can change the Gaussian statistics of the noise to a Lévy noise (e.g. [7]) or Poisson noise (e.g. in [19]) but keep the limit of vanishing correlations, which has strong effects on, for instance, the escape statistics of particles and can be described in the framework of fractional Fokker–Planck equations [27, 42]. For the case of non-Gaussian and temporally correlated noise, there is so far one analytically tractable example: the dichotomous noise or random telegraph process that jumps between two distinct values (having thus a highly non-Gaussian PDF) and possesses exponential correlations $c(\tau) = e^{-|\tau|/\tau_c}$ (identical to those of the OU noise). Although this noise has important applications in neuroscience [9, 10, 30] and other fields [3, 4], not all non-Gaussian processes are well captured by only two values.

Especially for random motility phenomena in biological systems, exponential distributions of noise increments (or at least distributions with pronounced exponential tails) are often seen; examples include the diffusive motion of colloidal beads [49], probe particles in active gels [44], intracellular RNA protein particles [21], and nanoparticles in cells [51] (cf also the review in [50]). These observations call for a colored noise model with exponential distribution. In this paper we introduce such a noise model, calculate its exact correlation function, its correlation

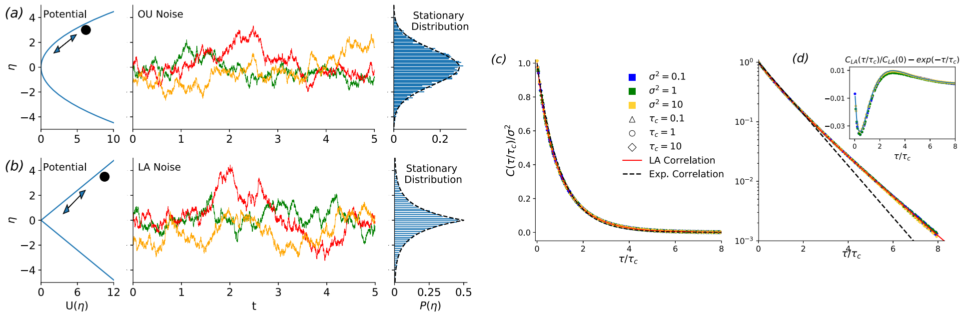

Figure 1. Ornstein–Uhlenbeck (OU) noise compared to Laplace (LA) noise potentials (left), example traces (middle) and probability densities (right) for the OU noise (a) and LA noise (b) for $\tau_c = 1$, $\sigma^2 = 2$. Correlation function, equation (3) in linear (c) and logarithmic (d) scale compared to simulation results at different variances and correlation times as indicated (in the shown scaling, all data collapse to a single curve); the purely exponential correlation function of the OU noise is also displayed (c) and (d); the deviation of LA correlation function from a pure exponential is shown in the inset in (d) (simulation data for $\tau_c = 1$ and various values of the variance).

time and noise intensity and demonstrate that it is very close but not identical to an exponential function. We furthermore inspect how such an exponentially distributed noise drives a particle over a potential barrier. We compare our results with that for an OU noise with the same correlation time, variance and noise intensity and find that the exponential noise distribution can greatly facilitate the escape rate over a barrier for an intermediate value of the fluctuations' correlation time.

Methodologically, our approach is similar in spirit to previous suggestions to incorporate non-Gaussian noise by a Markovian embedding: Fuentes *et al* [12] and Sen and Bag [41] have considered fluctuations with strong deviations from Gaussian statistics (yet these fluctuations are not exponentially distributed). An obvious advantage of the noise model considered here is that it is simple enough to allow for an exact calculation of the noise correlation function and for a convenient control of the noise parameters correlation time and noise intensity in an appropriately scaled version of the generating stochastic differential equation.

2. Model for an exponentially distributed noise

The basic model for the exponentially distributed noise is the overdamped Brownian motion in a piecewise linear potential $U_{\ell a}(\eta) = \mu|\eta|$ (see figure 1(b))

$$\dot{\eta}_{\ell a} = -\mu \cdot \mathrm{sgn}(\eta_{\ell a}) + \sqrt{2Q}\,\xi(t), \tag{2}$$

where $\mathrm{sgn}(x)$ is the signum function and $\xi(t)$ is again a white Gaussian noise with the aforementioned properties. It is an elementary task to find the stationary probability density $P_0(\eta) = \exp(-\sqrt{2}|\eta|/\sigma)/\sqrt{2\sigma^2}$ and variance $\sigma^2 = 2Q^2/\mu^2$. More difficult is the derivation of the autocorrelation function, for which we have found an explicit expression (the derivation is outlined in the appendix)

$$C_{\ell a}(\tau) = \sigma^2 \left[\left(1 - \frac{2}{3}\alpha|\tau| - \frac{1}{3}(\alpha\tau)^2\right) \sqrt{\frac{2}{\pi}\alpha|\tau|}\, \mathrm{e}^{-\alpha|\tau|/2} + \left(1 - \alpha|\tau| + (\alpha\tau)^2 + \frac{(\alpha\tau)^3}{3}\right) \mathrm{erfc}(\sqrt{\alpha\tau/2}) \right]. \quad (3)$$

Here the abbreviation $\alpha = 5/(4\tau_c)$ is given in terms of the correlation time, which reads (see appendix)

$$\tau_c = \frac{5Q}{2\mu^2}. \quad (4)$$

Evidently, if we plot equation (3) scaled by the variance and in terms of a rescaled lag time τ/τ_c, correlation functions for different noise parameters should fall onto one universal curve, which is confirmed by comparison with numerical simulations (cf figures 1(c) and (d)). Remarkably, although the expression for the correlation function is not elementary (involving a Gaussian and a complementary error function), the resulting curve is very close to an exponential function (the deviation from a pure exponential is shown in the inset of figure 1(d)). Thus, if properly scaled, the main difference of the Laplace noise to the Ornstein–Uhlenbeck noise will be the stationary distribution but not the second-order temporal correlation structure.

The simple expressions for variance and correlation time suggest to use the process in the following parameterization

$$\tau_c \dot{\eta}_{\mathrm{LA}} = -\frac{5\sigma}{2\sqrt{2}} \,\mathrm{sgn}(\eta_{\mathrm{LA}}) + \sqrt{\frac{5}{2}\sigma^2\tau_c}\, \xi(t). \quad (5)$$

If we want to generate a noise with fixed intensity D instead of fixed variance σ^2, we simply substitute $\sigma^2 = D/\tau_c$ in these expressions.

3. Escape out of a cubic potential well

We now explore a nonlinear dynamical system driven by either Ornstein–Uhlenbeck or Laplace noise. Specifically, we would like to study the escape rate from a cubic potential driven by either OU of LA colored noise. The dynamics reads

$$\dot{x} = \gamma x^2 - \beta + \eta, \quad (6)$$

in which η is either given by the OU noise in equation (1) or by the LA noise generated by equation (5). The dynamics can be regarded as that of an overdamped Brownian particle[4] that moves in a cubic potential $U(x) = -\gamma x^3/3 + \beta x$. For $\beta > 0$, as we will assume throughout the following, the potential possesses a stable minimum at $x_{\min} = -\sqrt{\beta/\gamma}$ and a potential barrier (maximum) at $x_{\max} = \sqrt{\beta/\gamma}$ (see figure 2(a) for a sketch of the potential shape); the barrier

[4] Taking into account the Stokes friction coefficient Γ in the overdamped dynamics $\Gamma\dot{x} = -U'(x) + \eta(t)$ could be easily done by rescaling the parameters γ, β, and σ, i.e. by using $\gamma/\Gamma, \beta/\Gamma, \sigma/\Gamma$ instead. For the ease of notation, we omit the additional parameter Γ here.

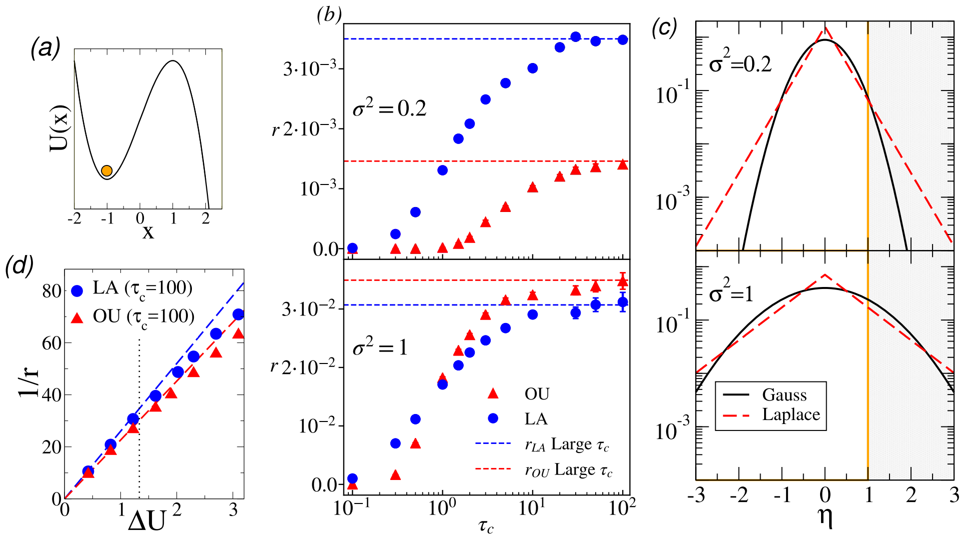

Figure 2. (a): the cubic potential for $\beta = 1$ and $\gamma = 1$ possesses a minimum at $x = -1$ and a maximum (the potential barrier) at $x = 1$. (b): the escape rate with OU and LA noise for a fixed variance as a function of the correlation time compared with the asymptotic limit as in equation equation (11). (c): comparison of Laplace and Gaussian distributions for the two values of σ as indicated. The shaded area to the right of $\eta = 1$ shows the relevant range of integration in equation (11) ($\eta > \beta$): for $\sigma^2 = 0.2$ (upper panel), the fatter tail of the Laplace distribution leads to higher escape rates whereas for $\sigma^2 = 1$ (lower panel) the Gaussian has more probability, at least if η is still close to the critical value, which results in higher escape rates for the OU noise. Parameters in (b) and (c): $\beta = 1, \gamma = 1$ (corresponding to a barrier height of $\Delta U = 4/3$). (d): inverse rate as a function of the potential barrier ΔU (varied solely by changing γ) for $\beta = 1, \tau_c = 100$ from simulations (symbols) compared to the theoretical formulas for the large-τ_c limit (dashed lines), equation (12) (LA noise) and equation (13) (OU noise); the dotted line marks the potential barrier used in (b) and (c). In all numerical simulations of this figure we used a simple Euler–Maruyama integration scheme with a time step of $\Delta t = 10^{-3}$, initial and final points at $x_\pm = \pm 10$.

height is given by

$$\Delta U = \frac{4}{3}\sqrt{\frac{\beta^3}{\gamma}}. \tag{7}$$

The system is also known as the normal form of a saddle-node bifurcation (occurring at $\beta = 0$ and usually defined with the parameter β having the opposite sign) and the escape-time problem has been studied in computational neuroscience under the label of the firing rate of the quadratic integrate-and-fire neuron [5, 24].

We start the dynamics at $x(t) = x_- = -10$ and consider an escape to have taken place at time $t + T$ if $x(t + T)$ reaches $x_+ = 10$; we then reset x to x_- (the noise η is not reset), register T as one realization of the first-passage time from x_- to x_+, and continue simulating the dynamics in order to measure the next escape time. The escape rate is then given by the inverse of the mean first-passage time:

$$r = 1/\langle T\rangle\,; \tag{8}$$

the average $\langle . . \rangle$ is taken over a long sequence of subsequent escape times.

With a correlated noise, different choices for the initial value of the noise at the beginning of the escape process are possible: one can start the noise for each escape realization at a fixed value (e.g. in [48]), draw it for each escape initially from the stationary probability density of the noise process (e.g. in [18]), or let it evolve unaffected by the escape process itself (e.g. in [29]). As mentioned above, we follow the latter procedure which is adequate when the colored noise is an external one and when there is an intrinsic mechanism that brings the variable right after escape quickly back to its starting point so that we can measure the escape rate from a temporal sequence of escape events.

Thanks to our rescaled dynamics and the knowledge of noise variance, intensity and correlation time for both processes, we can compare the effect of the statistical distribution on the escape rate—does an exponential distribution of noise values generally lead to higher or lower values of the escape rate than a Gaussian noise with the same variance, intensity and correlation time? Or is the role of this statistical feature negligible? We study the rates as functions of the correlation time τ_c separately in two scenarios: for a fixed noise variance σ^2 (or, equivalently, standard deviation) and for a fixed noise intensity $D = \sigma^2\tau_c = \text{const}$ because these different scalings admit different limit cases for vanishing and infinite correlation time.

3.1. Escape rate with fixed variance

The escape rates with OU and LA noise for two different values of the variance are shown in figure 2(b) and as functions of the noise correlation time.

In this setting with fixed noise variance, both rates become monotonically increasing functions of the correlation time. We can easily understand that in the limit $\tau_c \to 0$ the escape rate has to go to zero: in this limit, the noise process does not affect the driven dynamics because its effect averages out; the combination of a limited standard deviation and a vanishing temporal correlation is equivalent to zero noise intensity.

In the other limit of $\tau_c \to \infty$, the escape rate saturates for both noise processes at finite values. Interestingly, for a small noise variance (upper panel in figure 2(b)), the rate for the LA-driven system is larger than that of the OU-driven system—for the larger noise variance it is the other way around. We can understand this and also gain an analytical approximation of the rate in the limit of large correlation time (shown in figure 2(b) by dashed horizontal lines) as follows.

If the correlation time is very long in comparison with the typical escape time, we can assume that the external noise remains approximately constant during the escape (similar approaches have been pursuit in computational neuroscience for integrate-and-fire models driven by colored noise, see e.g. [28, 29]). If η is constant, we can integrate the dynamics over x and obtain by standard methods

$$T(\eta) = \int_{x_-}^{x_+} \mathrm{d}x \, \frac{1}{\gamma x^2 - \beta + \eta} = \frac{\arctan\left(\frac{x_+}{\sqrt{\gamma(\eta-\beta)}}\right) - \arctan\left(\frac{x_-}{\sqrt{\gamma(\eta-\beta)}}\right)}{\sqrt{\gamma(\eta-\beta)}}, \qquad (9)$$

which obviously has a finite solution only if $\eta - \beta > 0$. With a frozen value of η, we obtain a particularly simple result if we let initial and threshold values go to infinity ($x_\pm \to \pm\infty$); in this case the rate (the inverse of the first-passage time) is given by

$$r(\eta) = \frac{1}{T} = \frac{\sqrt{\gamma(\eta-\beta)}}{\pi}. \qquad (10)$$

This expression describes the escape rate in the asymptotic limit only for values of $\eta \geqslant \eta_{\text{critical}} = \beta$. In order to find the escape rate for all values we have to average this expression with its corresponding stationary distribution (see [29] for a thorough discussion of why one has to average the inverse time to obtain the rate):

$$r = \int_{\beta}^{\infty} \mathrm{d}\eta\, P_0(\eta) r(\eta). \tag{11}$$

This is the point at which the distribution of noise can make a difference.

For the Laplace noise, we can calculate the corresponding integral explicitly and obtain

$$r_{\text{LA}} = \frac{1}{4}\sqrt{\frac{\gamma\sigma}{\sqrt{2}\pi}}\,\mathrm{e}^{-\sqrt{2}\beta/\sigma}. \tag{12}$$

For the OU noise, equation (11) leads to an expression in terms of modified Bessel functions of the second kind $K_\nu(x)$ [1]

$$r_{\text{OU}} = \frac{\sqrt{\gamma}}{4\sigma}\left(\frac{\beta}{\pi}\right)^{3/2} \mathrm{e}^{-\beta^2/4\sigma^2}\left[K_{3/4}\left(\frac{\beta^2}{4\sigma^2}\right) - K_{1/4}\left(\frac{\beta^2}{4\sigma^2}\right)\right]. \tag{13}$$

The two expressions equations (12) and (13) match well the numerical simulations results for correlation times $\tau_{\text{c}} \gg 10^1$.

Considering the noise distributions (cf figure 2(c)), we can also qualitatively understand, why at low noise variance the OU noise leads to a larger escape rate and why this effect is reversed at larger noise variance. At weak noise ($\sigma^2 = 0.2$) there is for most of the relevant range over which we integrate in equation (11) a higher probability for the exponential noise because it decays slower than the Gaussian noise in the relevant tail. For the larger noise, the picture is more complicated: initially (and these are the values that contribute most probability to the integral), the Gaussian distribution is larger—only at very high values the exponential tail exceeds the Gaussian function. In consequence, the rate is higher for the Gaussian noise and will be even higher if we further increase the noise variance.

We can also find the values of the variance σ_*^2 at which the escape rates for both noise sources are equal. Setting the rates equal, we obtain a transcendental equation in β/σ, which has a unique solution $\beta/\sigma \approx 1.3206$ and thus we get the following relation between variance and the potential parameter β (independent of the potential parameter γ):

$$\sigma_*^2 \approx 0.573\,39\beta^2. \tag{14}$$

This result is plausible: the further minimum and maximum are apart and the higher the potential barrier, the larger will be the noise amplitude at which the Gaussian-noise-induced escape rate exceeds the rate of escapes due to exponential noise. Of course, the principle scaling relation (noise standard deviation is proportional to absolute value of β) can be obtained by renormalizing the parameters of the system (similar to what is done in [22]).

We note that in both equations (12) and (13), there is no Arrhenius- or Kramers-like dependence on the potential barrier ΔU as in $r \sim \exp(-\Delta U/D)$. This is to be expected because first of all in the strict static limit $\tau_{\text{c}} \to \infty$, the noise intensity $D = \sigma^2\tau_{\text{c}}$ diverges if we keep the variance constant—the Arrhenius dependence holds only true, however, for $\Delta U \gg D$ and thus cannot be a valid description of the rate anymore. Another way to see that an Arrhenius-like dependence does not work in this limit is to change the barrier height by varying the parameter γ. According to equation (7), we have $\Delta U \sim 1/\sqrt{\gamma}$, by means of which we obtain for both LA and OU noise $r \sim 1/\Delta U$, a much weaker dependence on ΔU than the exponential relation in the classical Kramers rate. In figure 2(d) we compare simulation results for the inverse of the rate in the case of a long-correlated noise ($\tau_c = 100$) as a function of the potential barrier to the approximations equations (12) and (13); plotted in this way, the data follow indeed roughly the theoretically predicted linear relationship on ΔU. Small sublinear deviations for higher barriers can be explained by the fact that the theory requires $1/r \ll \tau_c$, which is not obeyed for larger barriers in figure 2(d) (for any large but finite value of the correlation time, there is an upper bound for the barrier height, for which $r \sim 1/\Delta U$). In any case, the relation is very different than the exponential function $1/r \sim \exp(\Delta U/D)$ (Arrhenius- or Kramers-like dependence).

3.2. Escape rate with fixed noise intensity

We now turn to a different scaling, in which we use in all of the above equations and formulas $D = \sigma^2\tau_c = \text{const}$ when varying τ_c. We will see that this permits a non-trivial white-noise limit $\tau_c \to 0$ and an entirely different physical effect.

The rate for constant noise intensity is shown as a function of the correlation time in figure 3. Both curves start at the same rate for very small τ_c, the firing rate for the exponential noise then starts increasing strongly and finally drops down to zero at large correlation time. The firing rate for the system with OU noise decreases monotonically. Put differently, the exponential noise statistics leads to a *strong amplification of the Kramers rate* at intermediate correlation times. This effect is more pronounced for a smaller value of the noise intensity. We start with the analysis of the limit cases of vanishing and infinite correlation times and then explain the cause for the maximum.

First of all, it is not difficult to understand why the rate has to drop to zero in the limit of infinite noise correlation time. For a fixed noise intensity, the variance is given by $\sigma^2 = D/\tau_c$ and has to go to zero as $\tau_c \to \infty$. A zero-variance noise will not cause any escape independent of its correlation time.

In the opposite limit $\tau_c \to 0$ we deal with Gaussian white noise in *both* cases of Ornstein–Uhlenbeck and Laplace noise. While it is well-explored and well-known that the Ornstein–Uhlenbeck process converges to white Gaussian noise for $\tau_c \to 0$, the same convergence might be surprising for the Laplace noise—after all, for *any* value of the correlation time, a snapshot of noise values over different realizations of the Laplace noise would be of course still exponential. So why would this noise act effectively like a Gaussian white noise in the limit $\tau_c \to 0$? This problem has already been thoroughly addressed for other non-Gaussian noise processes [12]; here we would like to discuss it briefly for our model in a qualitative manner.

The answer to the above question is best understood if we think in terms of a finite integration scheme for equation (6):

$$x(t + \Delta t) = x(t) + (\gamma x(t)^2 - \beta)\Delta t + \int_t^{t+\Delta t} \mathrm{d}t'\, \eta(t'), \tag{15}$$

which describes the trajectory sufficiently well if we choose a sufficiently small but fixed value of Δt. If we let τ_c become smaller and smaller, the last term turns into an integral over a rapidly changing function i.e. over a sum over many independent function values, and thus will attain Gaussian statistics according to the central limit theorem. It is not hard to calculate that the variance of the integral term will approach $2\Delta t\sigma^2\tau_c$. Thus, independent of the instantaneous distribution statistics of the noise values $\eta(t)$, the noise increments will in the limit $\tau_c \to 0$

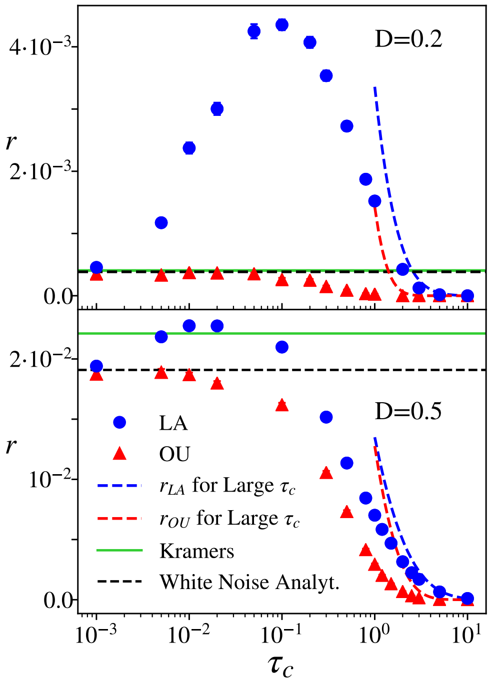

Figure 3. The escape rate with OU and LA noise for a fixed noise intensity as a function of the correlation time. The time step used is between $\mathrm{d}t = 10^{-3}$ and $\mathrm{d}t = 10^{-5}$ for small τ_c. The white-noise limit for small τ_c is compared with the quadrature result, equation (17), and the white-noise Kramers rate, equation (18); the asymptotic limit is compared with the theoretical results in equations (12) and (13). Remaining parameters: $\beta = 1, \gamma = 1$.

correspond to that in the white-Gaussian-noise driven dynamics

$$\dot{x} = \gamma x^2 - \beta + \sqrt{2D}\xi(t). \tag{16}$$

Indeed, in both cases (LA-noise- and OU-noise-driven escape problem), the rates approach the same value for $\tau_\mathrm{c} \to 0$ that can be exactly calculated by a numerical evaluation of a double integral or approximated by classical Kramers formula as follows. First of all, one can employ the first-passage-time approach for white-noise driven systems [13]. The mean first-passage time can be expressed by the following double integral [24]:

$$\langle T(x_- \to x_+) \rangle = \frac{1}{D} \int_{x-}^{x+} \mathrm{d}x\, \mathrm{e}^{-\frac{\gamma x^3/3 - \beta x}{D}} \int_{-\infty}^{x} \mathrm{d}y\, \mathrm{e}^{\frac{\gamma y^3/3 - \beta y}{D}}. \tag{17}$$

In this paper we are interested in the noise with small intensity, so we can also consider the Kramers escape rate which gives a good analytical approximation of the escape rate with white noise for small values of D, and it is given by [8, 24]:

$$r \approx \frac{\sqrt{\gamma\beta}}{\pi} \exp\left[-\frac{4}{3D}\sqrt{\frac{\beta^3}{\gamma}}\right] = \frac{\sqrt{\gamma\beta}}{\pi} \exp\left[-\frac{\Delta U}{D}\right]. \tag{18}$$

While this approximation is a reasonable one for $D = 0.2$ as in figure 3 (top), there are clearly deviations for larger noise intensity, e.g. $D = 0.5$ in figure 3 (bottom).

The most interesting effect is the emergence of a strong maximum for intermediate values of the correlation time. How can we understand this effect, at least qualitatively? It is reasonable that for a nonvanishing correlation time the exponential tails in the Laplace noise will lead to noise increments that are larger than the typical Gaussian increments. Half of the times these increments are also positive and may contribute to an escape event. The effect of these non-Gaussian outliers in the noise statistics becomes especially important in the weak-noise limit—which is the classical limit for the Kramers rate. Hence, we can also understand why the effect is more pronounced at weaker noise.

4. Summary and conclusions

We have developed a simple model for an exponentially distributed noise process. We have found exact expressions for its correlation function, noise intensity, and correlation time. A closer inspection revealed that the correlation function of the exponential Laplace noise is very close to the purely exponential correlation function of the Ornstein–Uhlenbeck process, the most often used model for a Gaussian colored noise. This means, that the main difference to the latter is the exponential distribution of the noise values. It is thus very simple to explore the effect of the stationary one-time noise statistics in nonlinear dynamical systems by comparing the effect of the Laplace and Ornstein–Uhlenbeck noise sources, especially if the properly scaled dynamical equations (1) and (5) are used.

As a first example we considered the escape problem over a potential barrier. We found that indeed in several respects the distribution of the noise made a difference for the value of the escape rate. In the constant-variance scaling, we could derive a simple relation telling us for which noise variances the Laplace noise leads to a higher rate than the OU noise does. In this scaling the differences between the rates are most pronounced in the slow-noise limit $\tau_c \to \infty$, where we can calculate the rate for both noise models and understand why at small noise variance the exponential tail of the Laplace noise leads to higher escape rates.

An even more interesting related (but not identical) effect was observed in the constant-noise-intensity scaling. Here the Laplace noise evokes a much stronger escape rate than the Gaussian noise but it is still highly dynamic at the point of maximal amplification. Unfortunately, we have not found yet a way to estimate the rate in this limit. The most promising approach to this problem seems to be the two-dimensional Fokker–Planck equation as used in [5] for the OU-noise and to solve the problem for small values of the correlation time by a perturbation calculation. This is certainly an interesting exploration for future studies but goes beyond the scope of our paper.

Most promising are applications of the Laplace noise to models of active motion, e.g. active Brownian particles [11, 35] that, already in one dimension, show interesting diffusion properties [25], uncommon behavior in the escape over a barrier [6, 14] and in the rectification of fluctuations in ratchet potentials [2, 33, 40]. In order to take into account increment statistics with exponential tails seen experimentally (see e.g. [26, 31]), exponentially distributed noise (or a combination of such noise with Gaussian fluctuations that model thermal noise) should be used. So far, only the effects of dichotomous and Gaussian colored fluctuations on the diffusion coefficient of active particles have been explored [23]. The noise process put forward here allows for a more systematic investigation of the effects of colored non-Gaussian noise in active motility models. This could also be of interest for systems of interacting active particles in higher spatial dimensions, where structure formation [15, 32, 36], alternations between ballistic and diffusive motion [43], and nontrivial behavior close to surfaces [37] are observed.

5. Dedication

I would like to dedicate this paper to my teacher, collaborator, and friend Lutz Schimansky-Geier (1950–2020), whose ideas, insights, and passion for stochastic physics had a deep impact on my scientific thinking (BL).

Appendix. Derivation of the correlation function

The correlation function $C(\tau)$ of a stochastic process $\eta(t)$ can be expressed by its stationary distribution $P_0(\eta)$ and the transition probability $P(\eta, t|\eta', 0)$ as follows [13]:

$$C(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\eta' \int_{-\infty}^{\infty} \mathrm{d}\eta\, \eta' \eta\, P_0(\eta')[P(\eta, \tau|\eta') - P_0(\eta)], \quad \tau > 0. \tag{A.1}$$

The transition probability for the exponentially distributed noise given in terms of the Langevin equation (2) obeys the following Fokker–Planck equation (FPE)

$$\frac{\partial P(\eta, t|\eta')}{\partial t} = \left(\frac{\partial}{\partial \eta} \mu \ \mathrm{sgn}(\eta) + Q \frac{\partial^2}{\partial \eta^2} \right) P(\eta, t|\eta'), \tag{A.2}$$

which has to be solved with natural boundary conditions ($\lim_{\eta \to \pm\infty} P(\eta, t|\eta') = 0$) and the initial condition $P(\eta, t = 0|\eta') = \delta(\eta - \eta')$. First of all, the asymptotic solution of the FPE is given by the Laplace distribution; it is obtained by setting the time derivative to zero and solving for the zero-flux solution (which is implied by the natural boundary condition):

$$P_0(\eta) = \frac{\mu}{2Q} \exp\left(-\frac{\mu}{Q} |\eta| \right) \tag{A.3}$$

(the prefactor follows from the normalization of the density).

As shown in [34], the solution can be expressed by an eigenfunction expansion with continuous spectrum that can be found by mapping the FP equation to a Schrödinger equation with an attractive delta function potential of the form: $V_s(x) = \frac{\mu^2}{4Q} - \mu\delta(x)$. Besides the eigenvalue zero (corresponding as usual to the stationary solution), the eigenvalues read $\lambda_k = \frac{\mu^2}{4Q} + Qk^2$ for $0 \leqslant k \leqslant \infty$ with even and odd eigenfunctions given by

$$\psi_k^{\mathrm{e}}(\eta) = \frac{2k\ \cos(k\eta) - (\mu/Q) \sin(k|\eta|))}{\sqrt{\pi[4k^2 + (\mu/Q)^2]}}$$

$$\psi_k^{\mathrm{o}}(\eta) = \pi^{-\frac{1}{2}}\ \sin(k\eta).$$

In terms of these functions and the steady state distribution, the full transition probability density reads

$$P(\eta, t|\eta') = P_0(\eta) + \mathrm{e}^{-\frac{\mu}{2Q}(|\eta| - |\eta'|)} \int_0^{\infty} \mathrm{d}k\, \mathrm{e}^{-\lambda_k t} \psi_k^{\mathrm{o}}(\eta) \psi_k^{\mathrm{o}}(\eta')$$
$$+ \mathrm{e}^{-\frac{\mu}{2Q}(|\eta| - |\eta'|)} \int_0^{\infty} \mathrm{d}k\, \mathrm{e}^{-\lambda_k t} \psi_k^{\mathrm{e}}(\eta) \psi_k^{\mathrm{e}}(\eta'). \tag{A.4}$$

Now we can use this formula to find the correlation function using equation (A.1). Evidently, the term $P_0(\eta)$ drops out immediately and we are left with two separate integrals corresponding to the even and to the odd eigenfunctions:

$$C(\tau) = C^{\mathrm{e}}(\tau) + C^{\mathrm{o}}(\tau). \tag{A.5}$$

The first term involves the even eigenfunctions:

$$\begin{aligned} C^{\mathrm{e}}(\tau) = &\int_{-\infty}^{\infty} \mathrm{d}\eta'\, \eta' \int_{-\infty}^{\infty} \mathrm{d}\eta\, \eta \frac{\mu}{2Q}\, \mathrm{e}^{-\frac{\mu}{2Q}(|\eta|-|\eta'|)-\frac{\mu}{Q}|\eta'|} \\ &\times \int_0^{\infty} \mathrm{d}k \frac{\left(2k\, \cos\, k\eta - \frac{\mu}{Q}\, \sin\, k|\eta|\right)\left(2k\, \cos\, k\eta' - \frac{\mu}{Q}\, \sin\, k|\eta'|\right) \mathrm{e}^{-\lambda_k t}}{\left(4k^2 + \left(\frac{\mu}{Q}\right)^2\right)\pi} \\ = &\ 0. \end{aligned} \tag{A.6}$$

Because the integrand is odd in η, the corresponding integral yields zero.

So only the odd eigenfunctions contribute to the correlation function; inserting the probability densities, changing the order of integration in a similar fashion as above, using the symmetry of the integrands in the integrals over η and η', and performing the elementary integral over the product of linear function and complex-values exponential functions, we arrive at

$$\begin{aligned} C^{\mathrm{o}}(\tau) &= \int_{-\infty}^{\infty} \mathrm{d}\eta \int_{-\infty}^{\infty} \mathrm{d}\eta'\, \eta\eta' \frac{\mu}{2Q}\, \mathrm{e}^{-\frac{\mu}{2Q}(|\eta|+|\eta'|)} \int_0^{\infty} \mathrm{d}k \frac{\sin\, k\eta\, \sin\, k\eta'}{\pi}\, \mathrm{e}^{-\lambda_k t} \\ &= \int_0^{\infty} \mathrm{d}k \frac{\mu\, \mathrm{e}^{-\lambda_k \tau}}{2Q\pi} \left(2 \int_0^{\infty} \mathrm{d}\eta\, \eta\, \sin\, k\eta\, \mathrm{e}^{-\frac{\mu}{2Q}|\eta|}\right)^2 \\ &= \int_0^{\infty} \mathrm{d}k \frac{\mu\, \mathrm{e}^{-\lambda_k \tau}}{2Q\pi} \left(\frac{32\mu Q^3 k}{(\mu^2 + 4Q^2k^2)^2}\right)^2 \\ &= \frac{2\mu^3}{\pi Q^3}\, \mathrm{e}^{-\frac{\mu^2}{4Q}\tau} \int_0^{\infty} \mathrm{d}k \frac{k^2\, \exp\left[-Q\tau k^2\right]}{\left(\frac{\mu^2}{4Q^2} + k^2\right)^4}. \end{aligned} \tag{A.7}$$

The last integral has been carried out using the computing software *Mathematica* and yields the final expression for the correlation function:

$$\begin{aligned} C^{\mathrm{o}}(\tau) = &\left(2\frac{Q^2}{\mu^2} - Q\tau + \frac{\mu^2\tau^2}{2} + \frac{\tau^3\mu^4}{12Q}\right) \mathrm{erfc}\left(\frac{\mu\sqrt{\tau}}{2\sqrt{Q}}\right) \\ &+ \left(\frac{2Q}{\mu} - \frac{2\mu\tau}{3} - \frac{\mu^3\tau^2}{6Q}\right) \sqrt{\frac{Q\tau}{\pi}}\, \mathrm{e}^{\frac{-\mu^2\tau}{4Q}} \end{aligned} \tag{A.8}$$

which, after expressing parameters with $\sigma^2 = 2Q/\mu^2$, $\tau_{\mathrm{c}} = 5Q/(2\mu^2)$, and $\alpha = 5/(4\tau_{\mathrm{c}})$ used in the main text is identical with equation (3).

We can use this expression together with that for the variance σ^2 to determine the correlation time, defined by

$$\tau_{\mathrm{c}} = \int_0^{\infty} \mathrm{d}\tau\, \frac{C(\tau)}{\sigma^2}. \tag{A.9}$$

Inserting the above expression, the integral can be evaluated explicitly. As an alternative way to calculate the correlation time (and as a test of our analytical result), we can use Risken's integral formula for the correlation time of an overdamped Brownian motion in a potential $U(x)$ (see the supplement of the second edition of Risken's text book [34]). The result is the same in both cases and given in equation (4).

ORCID iDs

Benjamin Lindner https://orcid.org/0000-0001-5617-127X

References

[1] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
[2] Ai B-Q and Wu J-C 2014 Transport of active ellipsoidal particles in ratchet potentials *J. Chem. Phys.* **140** 094103
[3] Bena I 2006 Dichotomous markov noise: exact results for out-of-equilibrium systems *Int. J. Mod. Phys.* B **20** 2825
[4] Bena I, Van den Broeck C, Kawai R and Lindenberg K 2003 Drift by dichotomous markov noise *Phys. Rev.* E **68** 041111
[5] Brunel N and Latham P E 2003 Firing rate of the noisy quadratic integrate-and-fire neuron *Neural Comput.* **15** 2281
[6] Burada P S and Lindner B 2012 Escape rate of an active Brownian particle over a potential barrier *Phys. Rev.* E **85** 032102
[7] Capała K, Dybiec B and Gudowska-Nowak E 2020 Peculiarities of escape kinetics in the presence of athermal noises *Chaos* **30** 013127
[8] Colet P, San Miguel M, Casademunt J and Sancho J M 1989 Relaxation from a marginal state in optical bistability *Phys. Rev.* A **39** 149
[9] Droste F and Lindner B 2017 Exact analytical results for integrate-and-fire neurons driven by excitatory shot noise *J. Comput. Neurosci.* **43** 81
[10] Droste F and Lindner B 2017 Exact results for power spectrum and susceptibility of a leaky integrate-and-fire neuron with two-state noise *Phys. Rev.* E **95** 012411
[11] Erdmann U, Ebeling W, Schimansky-Geier L and Schweitzer F 2000 Brownian particles far from equilibrium *Eur. Phys. J.* B **15** 105
[12] Fuentes M A, Wio H S and Toral R 2002 Effective Markovian approximation for non-Gaussian noises: a path integral approach *Phys.* A **303** 91
[13] Gardiner C W 1985 *Handbook of Stochastic Methods* (Berlin: Springer)
[14] Geiseler A, Hänggi P and Schmid G J 2016 Kramers escape of a self-propelled particle *Eur. J. Phys.* B **89** 175
[15] Grossmann R, Schimansky-Geier L and Romanczuk P 2013 Self-propelled particles with selective attraction-repulsion interaction: from microscopic dynamics to coarse-grained theories *New J. Phys.* **15** 085014
[16] Hänggi P and Jung P 1995 Colored noise in dynamical systems *Adv. Chem. Phys.* **89** 239
[17] Hänggi P, Talkner P and Borkovec M 1990 Reaction-rate theory: fifty years after Kramers *Rev. Mod. Phys.* **62** 251
[18] Jung P and Hänggi P 1988 Bistability and colored noise in nonequilibrium systems: theory versus precise numerics *Phys. Rev. Lett.* **61** 11
[19] Khovanov I A and Khovanova N A 2014 Numerical simulations versus theoretical predictions for a non-Gaussian noise induced escape problem in application to full counting statistics *Phys. Rev.* B **89** 085419
[20] Kramers H A 1940 Brownian motion in a field of force and the diffusion model of chemical reactions *Physica* **7** 284
[21] Lampo T J, Stylianidou S, Backlund M P, Wiggins P A and Spakowitz A J 2017 Cytoplasmic RNA-Protein particles exhibit non-Gaussian subdiffusive behavior *Biophys. J.* **112** 532
[22] Lindner B 2008 Diffusion coefficient of a Brownian particle with a friction function given by a power law *J. Stat. Phys.* **130** 523
[23] Lindner B 2010 Diffusion of particles subject to nonlinear friction and a colored noise *New J. Phys.* **12** 063026

[24] Lindner B, Longtin A and Bulsara A 2003 Analytic expressions for rate and CV of a type I neuron driven by white Gaussian noise *Neural Comput.* **15** 1761

[25] Lindner B and Nicola E M 2008 Diffusion in different models of active Brownian motion *Eur. Phys. J.: Spec. Top.* **157** 43

[26] Maggi C, Paoluzzi M, Angelani L and Di Leonardo R 2017 Memory-less response and violation of the fluctuation-dissipation theorem in colloids suspended in an active bath *Sci. Rep.* **7** 17588

[27] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 1

[28] Middleton J W, Chacron M J, Lindner B and Longtin A 2003 Firing statistics of a neuron model driven by long-range correlated noise *Phys. Rev.* E **68** 021920

[29] Moreno-Bote R and Parga N 2010 Response of integrate-and-fire neurons to noisy inputs filtered by synapses with arbitrary timescales: firing rate and correlations *Neural Comput.* **22** 1528

[30] Müller-Hansen F, Droste F and Lindner B 2015 Statistics of a neuron model driven by asymmetric colored noise *Phys. Rev.* E **91** 022718

[31] Otten M, Nandi A, Arcizet D, Gorelashvili M, Lindner B and Heinrich D 2012 Local motion analysis reveals impact of the dynamic cytoskeleton on intracellular subdiffusion *Biophys. J.* **102** 758

[32] Pohl O and Stark H 2014 Dynamic clustering and chemotactic collapse of self-phoretic active particles *Phys. Rev. Lett.* **112** 238303

[33] Pototsky A, Hahn A M and Stark H 2013 Rectification of self-propelled particles by symmetric barriers *Phys. Rev.* E **87** 042124

[34] Risken H 1989 *The Fokker–Planck Equation* 2nd edn (Berlin: Springer)

[35] Romanczuk P, Bär M, Ebeling W, Lindner B and Schimansky-Geier L 2012 Active Brownian particles *Eur. Phys. J. Spec. Top.* **202** 1

[36] Romanczuk P, Couzin I D and Schimansky-Geier L 2009 Collective motion due to individual escape and pursuit response *Phys. Rev. Lett.* **102** 010602

[37] Schaar K, Zöttl A and Stark H 2015 Detention times of microswimmers close to surfaces *Phys. Rev. Lett.* **115** 038101

[38] Schimansky-Geier L and Zülicke C 1990 Harmonic noise: effect on bistable systems *Z. Phys. B: Condens. Matter* **79** 451

[39] Schwalger T, Droste F and Lindner B 2015 Statistical structure of neural spiking under non-Poissonian or other non-white stimulation *J. Comput. Neurosci.* **39** 29

[40] Schweitzer F, Ebeling W and Tilch B 1998 Complex motion of Brownian particles with energy depots *Phys. Rev. Lett.* **80** 5044

[41] Sen M K and Bag B C 2009 Generalization of barrier crossing rate for coloured non Gaussian noise driven open systems *Eur. Phys. J.* B **68** 253

[42] Sliusarenko O Y, Gonchar V Y, Chechkin A V, Sokolov I M and Metzler R 2010 Kramers-like escape driven by fractional Gaussian noise *Phys. Rev.* E **81** 041119

[43] Strefler J, Erdmann U and Schimansky-Geier L 2008 Swarming in three dimensions *Phys. Rev.* E **78** 031927

[44] Toyota T, Head D A, Schmidt C F and Mizuno D 2011 Non-Gaussian athermal fluctuations in active gels *Soft Matter* **7** 3234

[45] Uhlenbeck G E and Ornstein L S 1930 On the theory of the Brownian motion *Phys. Rev.* **36** 823

[46] van Kampen N G 1992 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)

[47] Vellmer S and Lindner B 2019 Theory of spike-train power spectra for multidimensional integrate-and-fire models *Phys. Rev. Res.* **1** 023024

[48] Verechtchaguina T, Sokolov I M and Schimansky-Geier L 2006 First passage time densities in non-Markovian models with subthreshold oscillations *Europhys. Lett.* **73** 691

[49] Wang B, Anthony S M, Bae S C and Granick S 2009 Anomalous yet Brownian *Proc. Natl Acad. Sci. USA* **106** 15160

[50] Wang B, Kuo J, Bae S C and Granick S 2012 When Brownian diffusion is not Gaussian *Nat. Mater.* **11** 481

[51] Witzel P, Götz M, Lanoiselée Y, Franosch T, Grebenkov D S and Heinrich D 2019 Heterogeneities shape passive intracellular transport *Biophys. J.* **117** 203