Thomas and Lindner Reply: In the preceding Comment [1], Pikovsky raises three points. (1) A 2D linear system $(\boldsymbol{x},\boldsymbol{y})$ with a stable focus, driven by noise, would not constitute a "limit-cycle-like" oscillator; it would not admit a "good foliation." Therefore, a phase could not meaningfully be assigned to this system. But our method would, nevertheless, assign the phase $\boldsymbol{\phi} = \mathbf{tan}^{-1}(\boldsymbol{y}/\boldsymbol{x})$. (2) For a multirhythmic system, consisting of a noisy limit cycle (the "true" oscillator in Pikovsky's view) and a 2D quasicycle [2], as under (1), our method may pick the "wrong" phase if the 2D perturbation is not as noisy as the limit cycle, with respect to its phase. (3) The method based on the mean first passage time (MFPT) [3] would avoid this problem and always pick out the correct phase. We rebut each of these criticisms in turn.

(1) Our definition of the asymptotic phase [4] applies to a broad class of stochastic oscillations, including, but not limited to, the important example of limit cycles perturbed by noise. In [4], we considered, for instance, a heteroclinic 2D system that does not possess a limit cycle in the deterministic limit. Pikovsky's first example, a spiral focus with additive noise

$$\dot{x} = -\gamma x + \omega y + \xi_x, \qquad \dot{y} = -\omega x - \gamma y + \xi_y, \quad (1)$$

with $\langle \xi_a(t)\xi_b(t')\rangle = 2D\delta_{a,b}\delta(t-t')$, is another system with noise-sustained stochastic oscillations lacking an underlying deterministic limit cycle. In the noisy underdamped case ($\gamma/\omega \ll 1$) such systems are, nevertheless, strongly oscillatory, have power spectra with well-defined peaks, and can be described reasonably well with a phase variable (see the red line in Fig. 1 and, e.g., [5] for a study of the phase dynamics of a related system).

(2) Can a method designed to pick out one phase from a stochastic system succeed if there are several oscillations present? Obviously, neither our method nor the MFPT approach will unambiguously identify a unique phase reduction for arbitrary multirhythmic systems. Nevertheless, it is instructive to look in detail at Pikovsky's example of a rotating Ornstein-Uhlenbeck process, Eq. (1), combined with an additional rotation

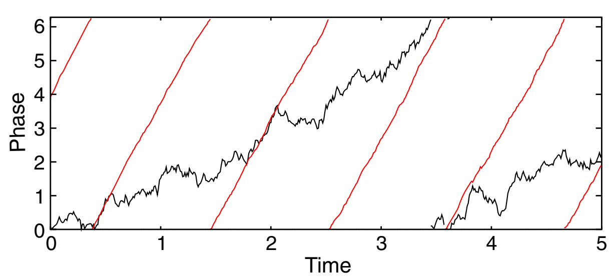

FIG. 1 (color online). Two stochastic "phase" variables for Eqs. (1) and (2). Black: $\theta(t)$. Red: $\phi(t) = \tan^{-1}[y(t)/x(t)]$.

$$\dot{\theta} = \Omega + \xi_\theta. \quad (2)$$

Here, ξ_θ is white Gaussian noise with $\langle \xi_\theta(t)\xi_\theta(t')\rangle = 2D'\delta(t-t')$. Our method applies to "robustly oscillatory:" systems, i.e., those that satisfy three criteria [4] [Pikovsky's Comment [1] notes condition (i), but overlooks conditions (ii) and (iii)]. (i) The nontrivial eigenvalue pair of the adjoint Kolmogorov operator, $\lambda_\pm = \alpha \pm i\beta$, with least negative real part, must have $\beta \neq 0$. (ii) The oscillation is fast relative to the decay, $|\beta| \gg -\alpha$. (iii) For all other nontrivial eigenvalues λ', we require $\Re[\lambda'] \leq 2\alpha$. For the system (1,2) there are two cases satisfying these criteria.

Case 1.—If $\gamma \geq 2D' > 0$ and $\Omega \gg D'$, then $\lambda = -D' \pm i\Omega$. The corresponding eigenfunction (EF) has phase θ.

Case 2.—If $D' \geq 2\gamma > 0$ and $\omega \gg \gamma$, then $\lambda = -\gamma \pm i\omega$. The corresponding EF has phase $\phi = \tan^{-1}(y/x)$.

Figure. 1 plots θ (black trace) and ϕ (red trace) for a trajectory with parameters $\gamma = 0.02$, $\omega = 6$, $D = 0.02$, $\Omega = 1$, and $D' = 0.4$, i.e., falling in case 2. While the system is multirhythmic, ϕ is clearly the more coherent phase variable. What our method does in this case is to pick the least noisy of the two possible phases, which is certainly a reasonable choice if no other constraints are set.

(3) Pikovsky asserts without proof that the MFPT method would identify θ as the phase variable for the system (1), (2). However, for every pair of integers k, k' (excluding $k = k' = 0$), the surfaces $\{k\theta + k'\phi = \text{const}\}$ provide a family of sections satisfying the MFPT property (see our original Reply [6] to Pikovsky's original Comment [7] for details). In particular, the surfaces $\{\phi = \text{const}\}$ define a system satisfying the MFPT property.

Multirhythmicity is common in stochastic physical and biological systems. In many situations, the context dictates what notion of phase is relevant. For instance, in [6] we showed that the persistent sodium-potassium model driven by channel noise [4,8] can exhibit sustained subthreshold oscillations alternating with large amplitude limit cycle oscillations (action potentials). Under these conditions the eigenvalue spectrum of the adjoint equation [4] has two complex eigenvalue pairs with similar real parts. The system violates our criterion (iii), and we do not expect it to have a single well defined phase. Instead, our method points to the coexistence of two phaselike variables, each determined by a different slowly decaying oscillatory eigenmode. Here, the phase associated with the action potentials (APs) may be regarded as the important one because APs are believed to carry information from one neuron to the other one. This type of interpretation, however, lies beyond the scope of our method.

This research was supported by the BMBF (Grant No. FKZ: 01GQ1001A) and the NSF (Grant No. DMS:1413770).

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Received 1 June 2015; published 4 August 2015
DOI: 10.1103/PhysRevLett.115.069402
PACS numbers: 05.45.-a, 05.10.Ln, 87.18.Tt, 87.19.ln

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