Recent developments in AdS/CFT

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Overview

AdS/CFT correspondence provides a fascinating link between conformal quantum field theories without gravity and string theory with (both classical and quantized) gravity

Major (recent) activities:

1. Integrability in AdS/CFT: Spectral problem solved (?)
2. Scattering amplitudes in maximally susy Yang-Mills, relation to light-like Wilson loops and dual superconformal symmetry
3. Novel well understood AdS_4/CFT_3 duality pair: IIA strings on $AdS_4 \times CP^3$ dual to max susy 3d Chern-Simons theory [Aharony,Bergmann,Jafferis,Maldacena '08]
4. "Applied" AdS/CFT: AdS/QCD and meson spectroscopy, applications to quark-gluon-plasma
5. Use AdS/CFT as tool to study quantum gravity
6. ...

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This talk!

$\mathcal{N}=4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: A_μ, $\mu = 0, 1, 2, 3$, $\Delta = 1$
  - 6 real scalars: Φ_I, $I = 1, \ldots, 6$, $\Delta = 1$
  - 4×4 real fermions: $\Psi_{\alpha\, A}$, $\bar{\Psi}^{\dot\alpha}_{A}$,$\alpha, \dot\alpha = 1, 2$. $A = 1, 2, 3, 4$, $\Delta = 3/2$
  - Covariant derivative: $\mathcal{D}_\mu = \partial_\mu - i[A_\mu, *]$, $\Delta = 1$

- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{{g_{\rm YM}}^2} \int d^4x \, \mathrm{Tr} \Big[\tfrac{1}{4} F^2_{\mu\nu} + \tfrac{1}{2} (D_\mu \Phi_i)^2 - \tfrac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \bar{\Psi}^{A}_{\dot\alpha} \sigma^{\dot\alpha\beta}_{\mu} \mathcal{D}^\mu \Psi_{\beta\, A} - \tfrac{i}{2} \Psi_{\alpha\, A} \sigma^{AB}_{I} \epsilon^{\alpha\beta} [\Phi^I, \Psi_{\beta\, B}] - \tfrac{i}{2} \bar{\Psi}_{\dot\alpha\, A} \sigma^{AB}_{I} \epsilon^{\dot\alpha\dot\beta} [\Phi^I, \bar{\Psi}_{\dot\beta\, B}] \Big]$$

- $\boxed{\beta_{g_{\rm YM}} = 0}$: Quantum Conformal Field Theory, 2 parameters: N & $\lambda = {g_{\rm YM}}^2 N$

- Shall consider 't Hooft planar limit: $N \to \infty$ with λ fixed.

Most symmetric 4d gauge theory!

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

  Poincaré: $p^{\alpha\dot{\alpha}} = p_\mu \, (\sigma^\mu)^{\dot{\alpha}\beta}, \quad m_{\alpha\beta}, \quad \bar{m}_{\dot{\alpha}\dot{\beta}}$

  Conformal: $k_{\alpha\dot{\alpha}}, \quad d \qquad (c$: central charge$)$

  R-symmetry: r_{AB}

  Poncaré Susy: $q^{\alpha\, A}, \bar{q}^{\dot{\alpha}}_A$ Conformal Susy: $s_{\alpha\, A}, \bar{s}^A_{\dot{\alpha}}$

- $4+4$ Supermatrix notation $\bar{A} = (\alpha, \dot{\alpha}|A)$

$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^\alpha{}_\beta - \frac{1}{2}\,\delta^\alpha_\beta\,(d + \frac{1}{2}c) & k^\alpha{}_{\dot{\beta}} & s^\alpha{}_B \\ p^{\dot{\alpha}}{}_\beta & \overline{m}^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{2}\,\delta^{\dot{\alpha}}_{\dot{\beta}}\,(d - \frac{1}{2}c) & \bar{q}^{\dot{\alpha}}{}_B \\ q^A{}_\beta & \bar{s}^A{}_{\dot{\beta}} & -r^A{}_B - \frac{1}{4}\delta^A_B\, c \end{pmatrix}$$

- Algebra:

$$[J_i{}^{\bar{A}}{}_{\bar{B}}\,,\, J_j{}^{\bar{C}}{}_{\bar{D}}\} = \delta_{ij}[\delta^{\bar{C}}_{\bar{B}}\, J_i{}^{\bar{A}}{}_{\bar{D}} - (-1)^{(|\bar{A}|+|\bar{B}|)(\bar{C}|+|\bar{D}|)}\delta^{\bar{A}}_{\bar{D}}\, J_i{}^{\bar{C}}{}_{\bar{B}}]$$

Observables

- **Local operators:** $\mathcal{O}_n(x) = \mathrm{Tr}[\mathcal{W}_1\,\mathcal{W}_2\ldots\mathcal{W}_n]$ with $\mathcal{W}_i \in \{\mathcal{D}^k\Phi, \mathcal{D}^k\Psi, \mathcal{D}^k F\}$

  2 point fct: $\langle\mathcal{O}_a(x_1)\,\mathcal{O}_b(x_2)\rangle = \dfrac{\delta_{ab}}{(x_1-x_2)^{2\,\Delta_a(\lambda)}}$ $\quad\Delta_a(\lambda)$ Scaling Dims

  3 point fct: $\Big\langle\mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_2)\Big\rangle = \dfrac{c_{abc}(\lambda)}{x_{12}^{\Delta_a+\Delta_b-\Delta_c}\,x_{23}^{\Delta_b+\Delta_c-\Delta_a}\,x_{31}^{\Delta_c+\Delta_a-\Delta_b}}$

  n-point functions follow from OPE

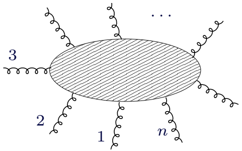
- **Wilson loops:**

$$\mathcal{W}_C = \Big\langle \mathrm{Tr} P \exp i \oint_C ds\,(\dot{x}^\mu A_\mu + i|\dot{x}|\,\theta^I\,\Phi_I)\Big\rangle$$

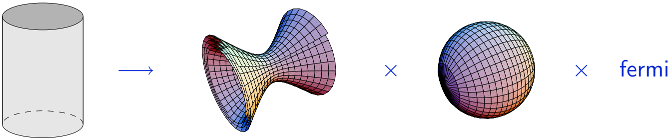
- **Scattering amplitudes:**

$$\mathcal{A}_n(\{p_i, h_i, a_i\};\lambda) = \begin{Bmatrix}\text{UV-finite}\\ \text{IR-divergent}\end{Bmatrix}$$

  helicities: $h_i \in \{0, \pm\frac{1}{2}, \pm 1\}$



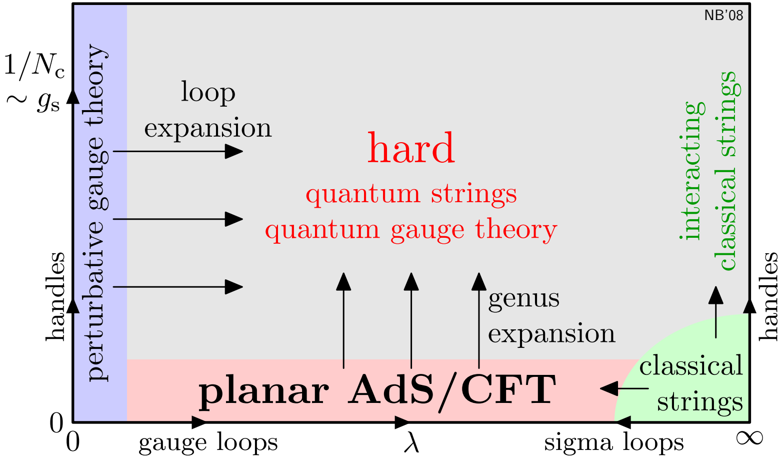
Superstring in $AdS_5 \times S^5$



$$I = \sqrt{\lambda} \int d\tau\, d\sigma \left[G^{(\mathrm{AdS}_5)}_{mn}\, \partial_a X^m \partial^a X^n + G^{(\mathrm{S}_5)}_{mn}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$

- $ds^2_{AdS} = R^2 \dfrac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Quantization unsolved!**
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- Isometries: $\mathfrak{so}(2,4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$
- Include fermions: Formulate as $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset model [Metsaev,Tseytlin]

The AdS/CFT landscape



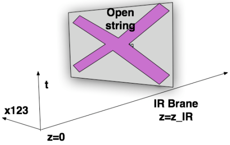
(Picture by N. Beisert)

Gauge Theory - String Theory Dictionary of Observables

$\Delta_a(\lambda)$ spectrum of scaling dimensions $\Leftrightarrow$ $E(\lambda)$ string excitation spectrum — solved (?)

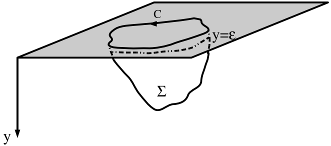
$c_{abc}(\lambda)$ structure constants $(\Leftrightarrow)$ Only SUGRA: $\mathcal{Z}_{AdS}[\phi|_{\partial \text{AdS}} = J] = \mathcal{Z}_{\text{CFT}}[J]$

$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ $(\Leftrightarrow)$



open string amplitude

Wilson loop $\mathcal{W}_C$ $\Leftrightarrow$



minimal surface

The spectral problem and integrability

The spectral problem of AdS/CFT

string energy	$\leftrightarrow$	scaling dimension
$E(\lambda)$	$=$	$\Delta(\lambda)$

- String states resp. gauge theory local operators classified by conserved Cartan charges (E, S_1, S_2) of $\mathfrak{so}(2,4)$ (energy and "spins") and (J_1, J_2, J_3) of $\mathfrak{so}(6)$ ("angular momenta")
- Geometrical picture:

$$AdS_5: \quad -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2$$

$$S^5: \quad Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2$$

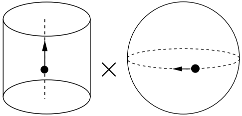
$Z_0 + i\,Z_5 = \rho_3\, e^{i\,t}$, $Z_1 + i\,Z_2 = \rho_1\, e^{i\,\alpha_1}$, $Z_3 + i\,Z_4 = \rho_2\, e^{i\,\alpha_2}$:
3 angles t, α_1, α_2 $\longrightarrow$ 3 conserved quantities E, S_1, S_2. E is the energy.

$Y_1 + i\,Y_2 = r_1\, e^{i\,\phi_1}$, $Y_3 + i\,Y_4 = r_2\, e^{i\,\phi_2}$, $Y_5 + i\,Y_6 = r_3\, e^{i\,\phi}$:
3 angles ϕ_1, ϕ_2, ϕ $\longrightarrow$ 3 conserved angular momenta J_1, J_2, J_3.

Spinning string solutions vs. Local Operators

- **Example 1:** Rotating point particle on S^5

$$t = \kappa\,\tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2} \quad \phi_1 = \kappa\,\tau \quad \phi_2 = \phi_3 = \psi = 0$$



Solves eqs. of motion & Virasoro constraint (here $S_1, S_2, J_2, J_3 = 0$)

$$E = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\dot{X}_0 = \sqrt{\lambda}\,\kappa \qquad \boxed{E = J} \quad \text{classical}$$

$$J_1 = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,(Y_1\,\dot{Y}_2 - Y_2\,\dot{Y}_1) = \sqrt{\lambda}\,\kappa =: J$$

- Dual gauge theory operator: $Z = \Phi_1 + i\Phi_2$ [Berenstein,Madacena,Nastase]

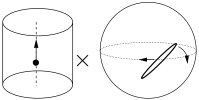
$$\mathcal{O}_J = \mathrm{Tr}[Z^J] \qquad \text{with} \quad \Delta(\lambda) = \Delta(\lambda = 0) = J$$

- Actually classical picture only good for $J \to \infty$

- **Example 2:** Folded spinning string: $\boxed{J_1 \ \& \ J_2 \neq 0}$ [Frolov,Tseytlin]

Ansatz:

$$t = \kappa\,\tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2}$$



$$\phi_1 = \omega_1\,\tau \quad \phi_2 = \omega_2\,\tau \quad \phi_3 = 0 \quad \psi = \psi(\sigma)$$

- Solution yields Charges and Energy

$$J_1 = \sqrt{\lambda}\,\omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi}\,\cos^2\psi(\sigma) \qquad J_2 = \sqrt{\lambda}\,\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi}\,\sin^2\psi(\sigma)\,.$$

$$E = J\left(1 + \frac{\lambda}{J^2}\,E_1 + \frac{\lambda^2}{J^4}\,E_2 + \ldots\right) \qquad J = J_1 + J_2$$

where $E_1 = \frac{2}{\pi^2}\,K(q_0)\Big(E(q_0) - (1-q_0)\,K(q_0)\Big)$ with $\frac{J_2}{J} = 1 - \frac{E(q_0)}{K(q_0)}$

Similarly E_l: l-loop gauge theory prediction.

- Dual gauge theory operator: $Z = \Phi_1 + i\Phi_2 \quad W = \Phi_3 + i\Phi_4$

$$\boxed{\mathcal{O}_J = \mathrm{Tr}[Z^{J_1} W^{J_2}] + \ldots} \qquad \text{with} \quad \Delta(\lambda) = J_1 + J_2 + \lambda\,\Delta_1(J_1, J_2) + \ldots$$

Indeed $\lim_{J\to\infty} \Delta_1(J_1, J_2) = \frac{\lambda}{J^2}\,E_1$!

Operator mixing and the dilatation operator

- Composite operators are renormalized and operators with degenerate $(\Delta^0, S_1, S_2; J_1, J_2, J_3)$ charges mix:

$$\mathcal{O}^A_{\text{ren}} = \mathcal{Z}^A{}_B\, \mathcal{O}^B_{\text{bare}}$$

Mixing matrix (**dilatation operator** $\hat{=}\ d \in \mathfrak{psu}(2,2|4)$)

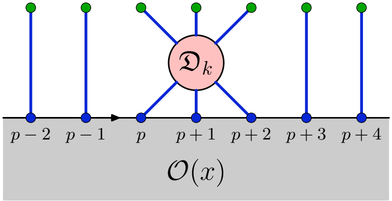
$$(\mathfrak{D})^A{}_B = (\mathcal{Z}^{-1})^A{}_C\, \frac{d}{\log \Lambda}\, Z^C{}_B$$

- Acts on composite operators: $\mathcal{O}(x) = \mathrm{Tr}[\,\Phi_{i_1}\,\Phi_{i_2}\ldots\Phi_{i_n}\,]$

Eigenvalues yield scaling dims. $\boxed{\mathfrak{D}\circ\mathcal{O}(x) = \Delta_{\mathcal{O}}\,\mathcal{O}(x)}$ [Beisert,Kristjansen,Plefka,Staudacher]

- $\mathfrak{D}$ is perturbatively defined:

$$\mathfrak{D} = \Delta^0 + \sum_{l=1}^{\infty} \lambda^l\, \mathfrak{D}_{l+1} \qquad \mathfrak{D}_k = \sum_{p=1}^{L}$$

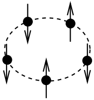


The dilatation operator and spin chains

- For simplicity: Consider $\mathfrak{su}(2)$ subsector

$$Z = \Phi_1 + i\,\Phi_2 \qquad \text{and} \qquad W = \Phi_3 + i\,\Phi_4$$

  & consider operators $\mathcal{O} = \mathrm{Tr}(\text{word in } Z \,\&\, W)$

- Spin chain picture: Operator $\mathrm{Tr}(ZZWZW) \hat{=}$ State $|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle \quad \hat{=}$ 

- One-loop structure: $\mathfrak{D}_2$ is Hamiltonian of the Heisenberg spin chain, an integrable system! [Minahan,Zarembo]

$$\boxed{\mathfrak{D}_2 = 2\sum_{l=1}^{L}(1 - P_{l,l+1})} \qquad P_{i,j} : \text{permutation operator}$$

- Ground state: $|\downarrow\downarrow\ldots\downarrow\rangle \hat{=} \mathrm{Tr}(Z^J)$ with $\Delta = 0$
- Excitations: "Magnons": $|m\rangle = |\underbrace{\uparrow\downarrow\ldots\downarrow\uparrow}_{m}\downarrow\rangle\rangle \hat{=} \mathrm{Tr}(WZ^mWZ^{J-m})$

Integrability

- Heisenberg spin chain is integrable: Existence of L commuting charges Q_n: $\boxed{[Q_m, Q_n] = 0}$ $\forall (m,n)$!
- Spectrum determined by **Bethe equations**:

$$e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i) \quad k = 1, \ldots, M$$

With S-Matrix:

$$S(p_i, p_k) = \frac{x^+(p_i) - x^-(p_k)}{x^-(p_i) - x^+(p_k)} \quad \text{with} \quad x^{\pm}(p) = \frac{1}{2}(\cot(\frac{p}{2}) \pm i)$$

Energy (one loop scaling dimensions) additive:

$$\Delta = L + \lambda\, E_2 \qquad \text{with} \qquad E_2(p_1, \ldots, p_M) = \sum_{k=1}^{M} 4 \sin^2 \frac{p_k}{2}$$

+ Cyclicity of trace condition: $\sum_{k=1}^{M} p_k = 0$

The asymptotic Bethe Ansatz

What happens at higher loops?

λ deformed variables: $$x^{\pm}(p) = \frac{e^{\pm i\,p/2}}{4\,\sin\frac{p}{2}}\,\Big(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{p}{2}}\,\Big) \quad \Leftrightarrow \quad e^{ip} = \frac{x^+(p)}{x^-(p)}$$

Asymptotic all loop conjecture: $x_k^{\pm} := x^{\pm}(p_k)$ [Beisert,Staudacher]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j=1, j\neq k}^{M} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\,\frac{1 - \frac{\lambda}{16\pi^2\,x_k^+\,x_j^-}}{1 - \frac{\lambda}{16\pi^2\,x_k^-\,x_j^+}} \cdot S_0(\{p_k\},\lambda)^2$$

S_0 : dressing factor

- Valid for $L >$loop order, completely fixed by $\mathfrak{psu}(2,2|4)$ symmetry up to S_0.
- Conjectured all loop form of S_0 exists [Beisert,Hernandez,Lopez;Beisert,Eden,Staudacher]
- Perturbatively: $S_0 \sim \mathcal{O}(\lambda^4)$ [Bern,Czakon,Dixon,Kosower,Smirnov]

Scaling dimensions then $$\Delta = \Delta_0 + \sum_{k=1}^{M} \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{p_k}{2}} - 1 .$$

Integrability in the AdS/CFT system

- $AdS_5 \times S^5$ string σ-model is classically integrable [Bena,Polchinski,Roiban]
  Can be solved completely in terms of algebraic curve

  [Kazakov,Marshakov,Minahan,Zarembo; Beisert,Kasazkov,Sakai,Zarembo]
- Full one-loop dilatation operator has been constructed in terms of an integrable super-spin chain and diagonalized by Bethe ansatz. [Minahan,Zarembo;Beisert,Staudacher]
  Super-magnon excitations scatter according to matrix Bethe equations:

$$e^{ip_k L}\,|\Psi\rangle = \left(\prod_{j=1,j\neq i}^{M} S(p_k,p_j)\right)\cdot|\Psi\rangle\,, \qquad E = \sum_{k=1}^{M} q_2(p_k)\,.$$

  (Asymptotic) S-matrix is assumed to be factorized. So far only proven at one-loop for all and up to four-loop for some operators.
- Wrapping problem: For finite size chains and long-range interactions not allowed to assume exactness of S-matrix!

Full set of conjectured nested $\mathfrak{psu}(2,2|4)$ Bethe equations

[Beisert,Staudacher]

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,k}}{x^-_{4,k}} \qquad \text{Spectral parameter: } x^{\pm}_{4,k} = \frac{1}{4}(\cot p_k/2 \pm i)\,\left(1 + \sqrt{1+16g^2\sin^2 p_k/2}\right)$$

↑ magnon momentum

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i\eta_1}{u_{2,k}-u_{2,j}+i\eta_1} \prod_{j=1}^{K_3+K_1} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{3,j}-\frac{i}{2}\eta_1} \qquad g := \sqrt{\lambda}/4\pi$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{3,k}-u_{2,j}-\frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x^{+\eta_1}_{4,j}-x_{3,k}}{x^{-\eta_1}_{4,j}-x_{3,k}}$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{L-\eta_1K_1-\eta_2K_7} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x^{+\eta_1}_{4,k}-x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k}-x^{+\eta_2}_{4,j}}\,\frac{1-g^2/(x^+_{4,k}x^-_{4,j})}{1-g^2/(x^-_{4,k}x^+_{4,j})}\,\mathbf{S_0^2}\right)$$

↑ Dressing factor

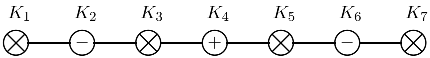
$$\times \prod_{j=1}^{K_3+K_1} \frac{x^{-\eta_1}_{4,k}-x_{3,j}}{x^{+\eta_1}_{4,k}-x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x^{-\eta_2}_{4,k}-x_{5,j}}{x^{+\eta_2}_{4,k}-x_{5,j}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{5,k}-u_{6,j}-\frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x^{+\eta_2}_{4,j}-x_{5,k}}{x^{-\eta_2}_{4,j}-x_{5,k}} \qquad u_{i,k} := x_{i,k} + g^2/x_{i,k}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i\eta_2}{u_{6,k}-u_{6,j}+i\eta_2} \prod_{j=1}^{K_5+K_7} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{5,j}-\frac{i}{2}\eta_2}\,.$$

with η_1, η_2 related to four different choices of $\mathfrak{psu}(2,2|4)$ Dynkin labels, e.g.

$\{\eta_1,\eta_2\} = \{+1,+1\}$:



The AdS/CFT (internal) S-matrix

- Describes scattering of two super-magnons, should be unitary and satisfy Yang-Baxter equation:

  [Arutyunov,Frolov,Staudacher '04; Beisert, Staudacher '05 + '06; Beisert, Hernandez,Lopez '06, Beisert,Eden,Staudacher '06]

$$S_{12}\,S_{21} = 1\,, \qquad S_{12}\,S_{13}\,S_{23} = S_{23}\,S_{13}\,S_{12}$$

- Was (ad hoc) conjectured to possess crossing symmetry: [Janik, '06]

$$S_{12}\,S_{\bar{1}2} = f_{12}^2$$

  $\Rightarrow$ can be used to fix dressing factor S_0.

- AdS/CFT S-matrix has the structure [Beisert '05]

$$S_{12} = \left(S_{12}^{\mathfrak{psu}(2|2)_L} \otimes S_{12}^{\mathfrak{psu}(2|2)_R}\right)\, S_0^2$$

- First motivated from gauge theory spin chain, subsequently found in light-cone quantized string theory [Arutyunov,Frolov,Plefka,Zamaklar '06]

Large Spin Limit of Twist Operators

- Consider twist operators: : Spin J_3: "twist"

$$\mathcal{O}_{S_1,J_3} = \mathrm{Tr}(\mathcal{D}^{S_1}\, Z^{J_3}) + \ldots$$

with $\mathcal{D} = \mathcal{D}_+$ covariant derivative in light-cone direction.

- General spin chain state of length J_3 is $\mathrm{Tr}\big((\mathcal{D}^{s_1} Z\, (\mathcal{D}^{s_2} Z) \ldots (\mathcal{D}^{s_{J_3}} Z\big)$ where $S_1 = s_1 + s_2 + \ldots s_{J_3} =: M =$ Magnon number.
- Scaling dims in $S_1 \to \infty$ limit:

$$\Delta_{\mathcal{O}_{S_1,J_3}} - S_1 - J_3 = \gamma(\lambda)\, \log S_1 + \mathcal{O}({S_1}^0)$$

$\gamma(\lambda)$: Universal scaling function, aka cusp anomalous dimension.

- $\gamma(\lambda)$ also appears in 4 gluon MHV amplitudes $\mathcal{A}_{4,MHV}$ and in light-cone segmented Wilson loops $\mathcal{W}$! [Bern,Dixon,Smirnov]

$$\mathcal{A}_{4,MHV}^{\text{all- loop}} \sim \exp\left[\gamma(\lambda)\, \mathcal{A}_{4,MHV}^{\text{one-loop}}\right] , \qquad \mathcal{A}_{4,MHV}^{\text{all- loop}} \sim \langle \mathcal{W} \rangle$$

The Beisert-Eden-Staudacher Integral Equation

- Asymptotic Bethe equations reduce in $S_1 \to \infty$, $L = J_3 \to \infty$ with $L \ll \log S_1$ to integral equation for density $\hat{\sigma}$ of Bethe roots: $(g = \sqrt{\lambda/4\pi})$

$$\hat{\sigma}(t) = \frac{t}{e^{\,t} - 1} \left[\hat{K}(2\,g\,t, 0) - 4\,g^2 \int_0^\infty dt' \, \hat{K}(2\,g\,t, 2\,g\,t') \, \hat{\sigma}(t') \right].$$

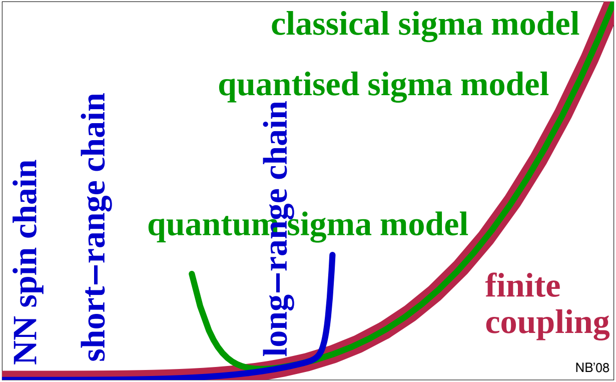
Cusp anomalous dimensions: $\gamma(g) = 16\, g^2\, \hat{\sigma}(0)$ **All loop prediction!**

- Solution yields weak and strong coupling predictions: [BES, Basso, Korchemsky, Kotanski '07]

$$\gamma(g) = \begin{cases} 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16\left(\frac{73}{630}\pi^6 + 4\,\zeta(3)^2\right) g^8 + \ldots & g \ll 1 \\ 4g - \frac{3\log 2}{\pi} - \frac{K}{4\pi^2}\,\frac{1}{g - 3\log 2/4\pi} - \frac{27\,\zeta(3)}{2^9\,\pi^3}\,\frac{1}{g^2} - \ldots & g \gg 1 \end{cases}$$

- Agrees with:
  1) Four loop gauge theory calculation [Bern,Czakon,Dixon,Kosower,Smirnov '06]
  2) 2 loop superstring calculation [Roiban,Tseytlin '07]

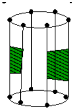
Cusp anomalous dimension of $\mathcal{N} = 4$ SYM:



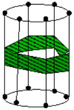
(Plot by N. Beisert)

Asymptotic Bethe equations yield 'half' of the perturbative spectrum of $\mathcal{N} = 4$ SYM:

length \ loops	1	2	3	4	5	6	7...
⋮							
6	√	√	√	√	√	√	√
5	√	√	√	√	√	√	?
4	√	√	√	√	√	?	?
3	√	√	√	√	?	?	?
2	√	√	√	Janik's TBA	?	?	?



incorporated Feynman graphs



missing wrapping interactions

- Wrapping graphs contribute generically at order g^{2L}.
- Asymptotic Bethe eqs. describes $L \to \infty$ spin chain or string with worldsheet geometry $\mathbb{R}^2$ $\Rightarrow$ Exsistence of S-Matrix and asymptotic states

Thermodynamic Bethe Ansatz

- Magnitude of finite size corrections: $\boxed{\sim e^{-E_{\mathsf{TBA}}(p_{\mathsf{TBA}})\,L}}$ with $E_{\mathsf{TBA}} = -ip$ and $p_{\mathsf{TBA}} = -iE$ in 'mirror' theory, i.e. original theory with space and time interchanged
- Approach was successfully implemented by generalization of Lüscher's formulas for 2d Lorentz invariant FT: Computation of four loop scaling dimension of Konishi operator $\mathrm{Tr}([Z,W][Z,W])$ from asymptotic S-matrix [Bajnok,Janik '08]
- Agrees with perturbative four loop supergraph calculation!

  [Fiamberti,Santambrogio,Sieg,Zanon '08]

$$\Delta = \Delta_{\mathsf{aBE}} + \Delta_{\mathsf{wrapping}} \qquad \Delta_{\mathsf{wrapping}} = (324 + 864\zeta(3)1440\zeta(5))g^8$$

- Highly nontrivial test of AdS/CFT!!

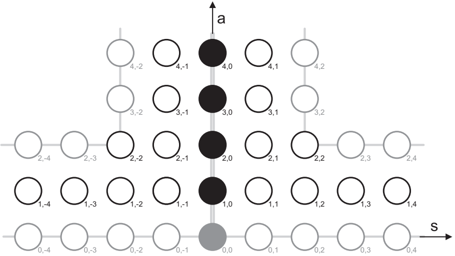
Recent conjecture: Implementation of TBA through a "Y-system" to describe planar AdS/CFT at finite size. Passes all known tests! [Gromov, Kazakov, Vieira '09]

Result:

- Y-system

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

$Y_{a,s}(u)$



- Asymptotics

$$Y_{a,s\neq 0}(u\to\infty) \;\to\; \mathrm{const}_{a,s}$$

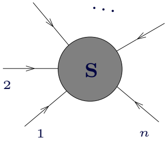
$$Y_{a,0}(u\to\infty) \;\to\; \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^{L} \times \mathrm{const}_{a}$$

(from talk of V. Kazakov at KITP 02/09)

Scattering Amplitudes

Scattering amplitudes in $\mathcal{N} = 4$ SYM I

- N-particle scattering amplitude



$$A_n(\{p_i, h_i, a_i\}) = (2\pi)^4\, \delta^{(4)}(\sum_{i=1}^{n} p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2}\, \mathrm{tr}[t^{a_1} \ldots t^{a_n}]$$
$$\times\, \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\}; \lambda = g^2\, N)$$

$\mathcal{A}_n$: Color ordered, planar amplitude
Helicities: $h = 0$ scalar, $h = \pm 1$ gluon, $h = \pm\frac{1}{2}$ gluino

- Commuting spinor helicity formalism:

$$p^{\alpha\dot\alpha} = (\sigma^\mu)^{\alpha\dot\alpha}\, p_\mu = \lambda^\alpha \tilde\lambda^{\dot\alpha} \quad \Leftrightarrow \quad p_\mu\, p^\mu = \det p^{\alpha\dot\alpha} = 0$$

2 spinors + choice of helicity determines polarization vector ε^μ of gluon

$$h = +1 \qquad \varepsilon^{\alpha\dot\alpha} = \frac{\lambda^\alpha \tilde\mu^{\dot\alpha}}{[\tilde\lambda\, \tilde\mu]} \qquad [\tilde\lambda\, \tilde\mu] := \epsilon_{\dot\alpha\dot\beta}\, \tilde\lambda^{\dot\alpha} \tilde\mu^{\dot\beta}$$

$$h = -1 \qquad \tilde\varepsilon^{\alpha\dot\alpha} = \frac{\mu^{\alpha\tilde{\dot\alpha}}}{\langle\lambda\, \mu\rangle} \qquad \langle\lambda\, \mu\rangle := \epsilon_{\alpha\beta}\, \lambda^\alpha \mu^\beta \quad \mu, \bar\mu \text{ arbitrary}$$

Scattering amplitudes in $\mathcal{N}=4$ SYM II

- Gluon amplitudes: $\mathcal{A}_n(1^+,2^+,\dots,n^+) = 0 = \mathcal{A}_n(1^-,2^+,\dots,n^+)$
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^-2^+,\dots j^- \dots n^+) = \mathcal{A}^{\text{MHV}}_{n;0} + \lambda \cdot \mathcal{A}^{\text{MHV}}_{n;1} + \dots = \mathcal{A}^{\text{MHV}}_{n;0} \cdot \mathcal{M}^{\text{MHV}}_n(\{p_i \cdot p_j\};\lambda)$$

Park-Taylor formula: $$\boxed{\mathcal{A}^{\text{MHV}}_{n;0} = i\frac{\langle 1,j\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle\dots\langle n,1\rangle}}$$ [Park, Taylor]

- BDS conjecture [Bern, Dixon, Smirnov]

$$\log \mathcal{M}^{\text{MHV}}_n = \gamma(\lambda)\cdot \mathcal{M}^{\text{MHV}}_{n,\text{1-loop}} + \text{“}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\text{”}$$

(true for $n=4,5$ known to fail for $n \geq 6$)

- N^{k}MHV amplitudes have rather complicated structure!
  $\Rightarrow$ Could there be a better formulation?

On-shell superspace

- Introduce Grassmann variables $\eta_i^A \quad A = 1,2,3,4 \quad i = 1,\ldots,n$ [Nair]
- Superwavefunction:

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

- Express amplitudes compactly in on-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta_i^A)$

$$\mathbb{A}_{n;0}^{\text{MHV}}(\lambda_1,\tilde{\lambda}_1,\eta_1;\ldots;\lambda_n,\tilde{\lambda}_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}})\,\delta^{(8)}(\sum_i \lambda_i^\alpha \eta_i^A)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle}$$

- MHV-superamplitude: General gluon$^{\pm}$-gluino$^{\pm 1/2}$-scalar amplitude
  Factor $\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$
  η-expansion associates $(\eta_i)^n := \prod_{k=1}^n \eta_i^{A_k}$ with ith particle of helicity $1 - h/2$

$$\Rightarrow \quad \mathbb{A}_n^{\text{MHV}} = i(2\pi)^4\,\delta^{(4)}(\sum_i p_i)\sum_{j\neq k}(\eta_j)^4\,(\eta_k)^4\,\mathcal{A}_n^{\text{MHV}}(1^+\ldots j^-\ldots k^-\ldots n^+)$$

Superamplitudes

- General form of superamplitudes:

$$\mathbb{A}_n = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

- $\mathbb{A}_n$ is invariant under full superconformal group $\mathfrak{psu}(2,2|4)$: $p, m, \bar{m}, k, d \oplus r \oplus q, \bar{q}, s, \bar{s} \oplus (c)$
- Realization of $\mathfrak{psu}(2,2|4)$ generators in on-shell superspace, e.g. [Witten]

$$p^{\alpha\dot{\alpha}} = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \qquad q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A \qquad \text{obvious symmetries}$$

$$k_{\alpha\dot{\alpha}} = \sum \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \qquad s_{\alpha A} = \sum_i \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \eta_i^A} \qquad \text{less obvious sym}$$

- We have $\boxed{p^{\alpha\dot{\alpha}}\, \mathbb{A}_n = q^{\alpha A}\, \mathbb{A}_n = k_{\alpha\dot{\alpha}}\, \mathbb{A}_n = s_{\alpha A} \mathbb{A}_n = 0}$
- Also: Local relation $h_i\, \mathbb{A}_n = 1 \cdot \mathbb{A}_n$

  Helicity operator: $h_i = -\frac{1}{2} \lambda_i^\alpha \partial_{i\,\alpha} + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \partial_{i\,\dot{\alpha}} + \frac{1}{2} \eta_i^A \partial_{i\,A} = 1 - c_i$

  $\Rightarrow$ (Tree) amplitudes are $\mathfrak{su}(2,2|4)$ invariant

On shell recursion techniques

- Efficient way of computing tree level gluon amplitudes: BCFW On shell recursion techniques [Britto,Cachazo,Feng+Witten '04,05]

  Closed formula for 'split helicity' gluon amplitudes $(+\ldots+-\ldots-)$

  [Roiban,Spradlin,Volovich,Britto,Feng]

- Reformulation of recursion relations in on-shell superspace through shift in $(\lambda_i,\tilde{\lambda})$ and η_i [Elvang et al 08, Arkani-Hamed et al 08, Brandhuber et al 08]

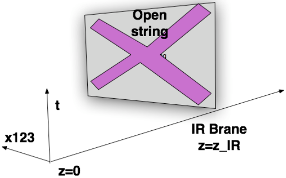
- Recursion much simpler and can be solved! [Drummond,Henn]

$$\mathbb{A}_n = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i\,\tilde{\lambda}_i)\,\delta^{(8)}(\sum_i \lambda_i\,\eta_i)}{\langle 1,2\rangle\,\langle 2,3\rangle\ldots\langle n,1\rangle}\,\mathcal{P}_n^{\text{tree}}(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$$

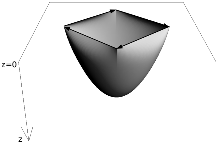
$\Rightarrow$ $\mathcal{P}_n^{\text{tree}}$ now known analytically (implies in particular pure Yang-Mills result).

MHV Scattering amplitudes in AdS/CFT

- Dual string description of scattering amplitudes [Alday,Maldacena '07]



T-dual $\Leftrightarrow$



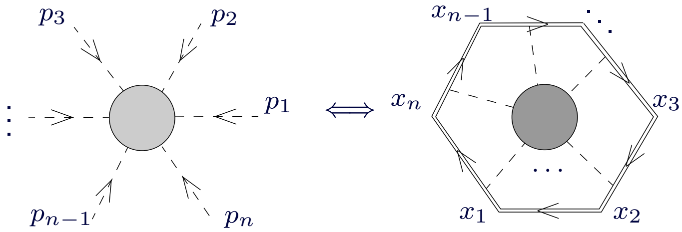
Open string amplitude on IR-brane $\overset{\text{T-dual}}{\Leftrightarrow}$ Wilson loop with light-like segments

- Cusp points determined by gluon momenta via key relation

$$\boxed{p_i^\mu = x_{i+1}^\mu - x_i^\mu}$$

- Yields strong coupling prediction for four-gluon MHV amplitude via classical string theory!
- Indeed BDS conjecture for $n = 4$ gluons tested:

$$\lim_{g\to\infty} \log \mathcal{M}_4^{\text{MHV}} = \underbrace{4g}_{\gamma(\lambda\to\infty)} \cdot \mathcal{M}_{n,\text{1-loop}}^{\text{MHV}} + \text{“}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\text{”}$$



$$x^\mu_{i+1} - x^\mu_i = p^\mu_i$$

[Drummond,Henn,Korchemsky, Sokatchev]

Planar relation:

$$\ln \mathcal{M}_n^{\mathsf{MHV}} = \ln \mathcal{W}_n + \mathsf{div} + \mathcal{O}(\epsilon)$$

$$\mathcal{W}_n = \frac{1}{N} \left\langle \mathrm{Tr} P \exp[ig \oint_{C_n} dx^\mu A_\mu] \right\rangle$$

Checked up to two loops and $n \leq 6$ points

[Drummond,Henn,Korchemsky,Sokatchev;Brandhuber,Heslop,Travaglini; Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

String interpretation: Combination of bosonic and 'fermionic' T-duality transformation for $AdS_5 \times S^5$ superstring.

[Beisert,Ricci,Tseytlin,Wolf;Berkovits,Maldacena]

⇒ Conformal invariance in dual space

⇒ Dual conformal covariance of scattering amplitudes!

Dual Superconformal symmetry

- Dual superspace

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha \, \tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha \, \eta_i^A$$

Then $x_i^{\alpha\dot\alpha}$ and $\theta_i^{\alpha\, A}$ have standard transformation law under (dual) conformal transformations

- Dual superconformal algebra, with generators $P, M, \bar M, K, D \oplus R \oplus Q, \bar Q, S, \bar S$, e.g.

$$K^{\alpha\dot\alpha} = \sum_{i=1}^{n} x_i^{\alpha\dot\beta} \, x_i^{\dot\alpha\beta} \, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta} \, \theta_i^{\alpha\, B} \, \frac{\partial}{\partial \theta_i^{\beta\, B}}$$

Structure: [Drummond,Henn,Korchemsky,Sokatchev '08]

$$\begin{array}{ccccc} & p & & K & \\ q & & \bar q = \bar S & & S \\ s & & \bar s = \bar Q & & Q \\ & k & & P & \end{array}$$

Also observed in dual string theory [Beisert,Ricci,Tseytlin,Wolf;Berkovits,Maldacena '08]

Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Indeed $\boxed{K^{\alpha\dot{\alpha}}\,\mathbb{A}_n = -\sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}}\,\mathbb{A}_n}$ $\Rightarrow$ $K' = K + \sum_i x_i$ annihilates the amplitude.
- Beyond tree-level: Dual superconformal symmetry brroken by IR divergences. However, breaking is under control and proportional to $\gamma(g)$ for MHV amplitudes. Conjecture: Dual superconformal 'anomaly' is the same for MHV and non-MHV amplitudes [Drummond,Henn,Korchemsky,Sokatchev '08]
- Question: What algebraic structure emerges when one commutes conformal with dual conformal generators? [Drummond,Henn,Plefka]

  Task: Tranform dual superconformal generators expressed in (x_i, θ_i) into original on-shell superspace $(\lambda_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$.

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Open chain by droping $x_{n+1}=x_1$ and $\theta_{n+1}=\theta_1$ conditions, implemented via δ-fcts: $\delta^{(4)}(p)\,\delta^{(8)}(q)=\delta^{(4)}(x_1-x_{n+1})\,\delta^{(8)}(\theta_1-\theta_{n+1})$
- "Non-local' relations:

$$x_i^{\alpha\dot\alpha}=x_1^{\alpha\dot\alpha}+\sum_{j<i}\lambda_j^{\alpha}\,\tilde\lambda_j^{\dot\alpha}\qquad \theta_i^{\alpha A}=\theta_1^{\alpha A}+\sum_{j<i}\lambda_j^{\alpha}\,\eta_j^{A}$$

  Set $x_1=\theta_1=0$ by dual translation and susy.
- Can show that dual superconformal generator may be lifted to level 1 generators of a Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$:

$$[J_a^{(0)},J_b^{(0)}\}=f_{ab}{}^{c}\,J_c^{(0)}\qquad \text{conventional superconformal symmetry}$$
$$[J_a^{(1)},J_b^{(0)}\}=f_{ab}{}^{c}\,J_c^{(1)}\qquad \text{from dual conformal symmetry}$$

  with nonlocal generators $\boxed{J_a^{(1)}=f^{cb}{}_a\sum_{1<j<i<n}J_{i,b}^{(0)}\,J_{j,c}^{(0)}}$ and super Serre relations (representation dependent).

Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- E.g. $p^{(1)}_{\alpha\dot\alpha}\,\mathbb{A}_n = 0$ with

$$p^{(1)}_{\alpha\dot\alpha} = K'_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha} = \frac{1}{2}\sum_{i<j}(m_{i,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar m_{i,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\,\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\,p_{j,\gamma\dot\gamma} + \bar q_{i,\dot\alpha C}\,q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

- In supermatrix notation: $\bar A = (\alpha, \dot\alpha | A)$

$$J^{\bar A}{}_{\bar B} = \begin{pmatrix} m^{\alpha}{}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(d + \frac{1}{2}c) & k^{\alpha}{}_{\dot\beta} & s^{\alpha}{}_{B} \\ p^{\dot\alpha}{}_{\beta} & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar q^{\dot\alpha}{}_{B} \\ q^{A}{}_{\beta} & \bar s^{A}{}_{\dot\beta} & -r^{A}{}_{B} - \frac{1}{4}\delta^{A}_{B}\,c \end{pmatrix}$$

and

$$\boxed{J^{(1)\,\bar A}{}_{\bar B} := -\sum_{i>j}(-1)^{|\bar C|}(J_i^{\bar A}{}_{\bar C}\,J_j^{\bar C}{}_{\bar B} - J_j^{\bar A}{}_{\bar C}\,J_i^{\bar C}{}_{\bar B})}$$

- Integrable spin chain picture also for colour ordered scattering amplitudes!
- Implies an infinite-dimensional symmetry algebra for $\mathcal{N}=4$ SYM scattering amplitudes!

Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Same Yangian symmetry appears in the spectral problem of AdS/CFT!

  [Dolan,Nappi,Witten;Beisert, Zwiebel, Torrieli, de Leeuw,. . .]

- Strong hint for integrability in scattering amplitudes!

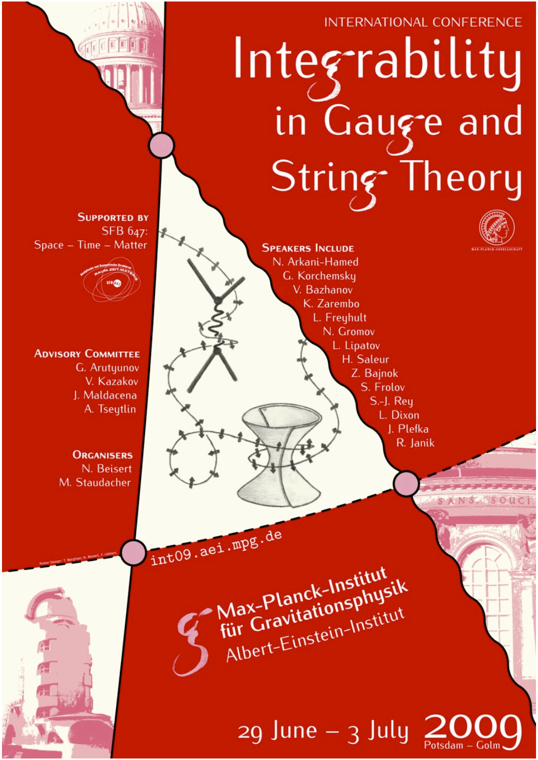
- Some Questions:

  Does this generalize to higher loops? Most certainly yes, from string picture
  Can it constrain the form of the higher loop amplitudes? In particular the 'remainder' function for MHV amplitudes . . .

Conclusions

Great progress in our understanding of the maximally supersymmetric AdS_4/CFT_3 system

- Spectral problem (close) to exact solution!
- Integrability in scattering amplitudes at higher loops?
- What can be said about gauge theory three-point functions?



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