Scattering amplitudes and hidden symmetries in supersymmetric gauge theory

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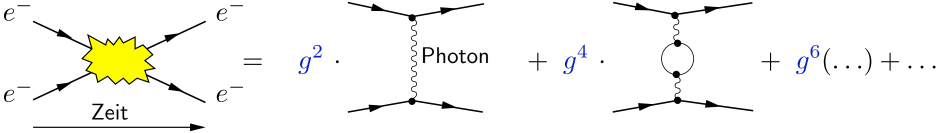
Scattering amplitudes and hidden symmetries in supersymmetric gauge theory

Plan:

1. Quantum field theory
2. Symmetries
3. Supersymmetric gauge field theory
4. String-gauge theory duality
5. Scattering amplitudes in gauge theories and on-shell methods
6. Hidden symmetries of scattering amplitudes
7. Generalized unitarity

Mathematical framework of particle physics: QFT

- Quantum Field Theory: Relativistic many particle quantum theory
- Describes scattering processes of elementary particles



Perturbative description: Series expansion in $g \ll 1$ g: Coupling constant

- Feynman diagrams: Describe particle propagation & interactions
- Symmetries play central role:
  - Determine possible particles & their interactions
  - Can severly constrain results for scattering probabilities
- Exact analytic methods beyond perturbation theory are sparse
- Desirable to innovate our ability to compute & advance our fundamental understanding of quantum field theory

The Standard Model of Particle Physics

Three fundamental forces described by **Gauge Field Theories** [1955,1971]

Forces: $\mathbf{SU(3)} \times \mathbf{SU(2)} \times \mathbf{U(1)}$ $\hat{=}$ Gauge Field Theories

- $\mathbf{U(1)}$ → Electromagnetism (photons)
- $\mathbf{SU(2)}$ → Weak Force (W & Z bosons)
- $\mathbf{SU(3)}$ → Strong Force (gluons) $\hat{=}$ Quantum Chromodynamics (QCD)

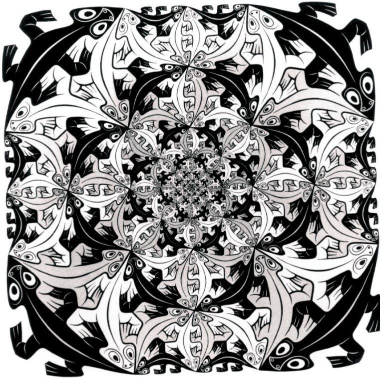
$\mathbf{SU(N)}$ Gauge Field Theory: Fields are $N \times N$ matrices: $\mathbf{A}_\mu^{\mathrm{SU(2)}}(x) = \begin{pmatrix} Z & W^+ \\ W^- & -Z \end{pmatrix}$

Spectrum:

Leptons	Quarks	Vector bosons	Higgs
$e,\ \nu_e$	$u,\ d$	A_μ	
$\mu,\ \nu_\mu$	$s,\ c$	$W^\pm,\ Z$	Φ
$\tau,\ \nu_\tau$	$t,\ b$	A_μ^a	

Gravity is not contained!

Symmetries



Symmetries

Symmetries lie at the heart of our understanding of physics. They constrain or even determine physical theories and their observables.

- Mathematically symmetry transformations form a group

$$G_1 \circ G_2 = G_3 \qquad \{\mathbb{1}, G_i, G_i^{-1}\} \in \text{group}$$

- Continous transf.: Lie group $G(\phi) = e^{i\,\phi^a\,\hat{J}_a}$ $\quad \hat{J}_a$: Generator $\quad \phi^a \in \mathbb{R}$

- Group property entails commutation relations

$$[\hat{J}_a, \hat{J}_b] = i f_{ab}{}^c\,\hat{J}_c \qquad \text{Lie algebra} \qquad a, b, c = 1, \ldots, \dim(g)$$

- Symmetries may be obvious or hidden
- Example for obvious symmetries: Rotations and translations

$$R(\vec{\phi}) = e^{i\vec{\phi}\cdot\hat{\vec{L}}} \qquad T(\vec{a}) = e^{i\vec{\alpha}\cdot\hat{\vec{P}}}$$

$\vec{L}$: Angular momentum $\qquad \vec{P}$: Momentum

$$[L_i, L_j] = i\hbar\,\epsilon_{ijk}\,L_k \qquad [P_i, P_j] = 0 \qquad [L_i, P_k] = i\hbar\,\epsilon_{ijk}\,P_k$$

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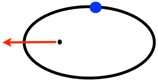
Example of a hidden symmetry: The Hydrogen atom

- Hamiltonian $\boxed{H = \frac{\vec{p}^2}{2m} - \frac{k}{r}}$
- Obvious rotational symmetry: $[H, L_i] = 0 \quad \Rightarrow \quad H\,|n,l,m\rangle = E_{n,l}|n,l,m\rangle$
- Hidden symmetry in H-atom: Pauli-Lenz vector

$$\vec{A} = \tfrac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - m\,k\,\frac{\vec{r}}{r}$$



- Conserved quantity: $[H, A_i] = 0$
- Algebra:

$$[A_i, A_j] = -i\frac{2\hbar}{m}\epsilon_{ijk}\,H\,L_k\,, \quad [L_i, A_j] = i\hbar\,\epsilon_{ijk}\,A_k\,, \quad [L_i, L_j] = i\hbar\,\epsilon_{ijk}\,L_k$$

- Closes on eigenspace $\mathcal{H}_E$ of fixed energy eigenvalue E.
- Operator algebra determines spectrum ($\hat{=}$ representation theory of $SU(2)$)

$$E_n = -\frac{mk^2}{2\hbar^2}\,\frac{1}{n^2} \qquad \text{(explains degeneracy in } l\text{)}$$

Fundamental symmetry of QFT: Poincaré group

Poincare symmetry: Lorentz transformations + translations in space & time

$$M_{\mu\nu}\,,\; P_\mu \qquad \vec{L} \qquad \vec{K} \qquad P_\mu$$

rotations (Lorentz transformations → $\vec{L}$) boosts (Lorentz transformations → $\vec{K}$)

$$L_i = \tfrac{1}{2}\epsilon_{ijk}\,M_{jk} \qquad K_i = M_{0i} \qquad \mu,\nu,\ldots = \{0,i\}$$

- Representations of the 4d Poincaré group $\hat{=}$ possible particles in nature

spin	field	example
0	scalar $\phi(x)$	Higgs
$1/2$	left handed spinor $\chi_\alpha(x)$	leptons, quarks
$1/2$	right handed spinor $\bar{\psi}_{\dot{\alpha}}(x)$	leptons, quarks
1	vector $A_\mu(x)$	photon, gauge bosons
$3/2$	$\psi_\mu^\alpha(x)$	gravitino
2	$h_{\mu\nu}(x)$	graviton

- Massless fields/particles classified by helicity $h = \dfrac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$ with $\boxed{h = \pm s}$

Extension I: Conformal symmetry

- Relativistic QFTs without intrinsic mass scale ($\hat{=}$ massless or at very high energies) have an enlarged space-time symmetry: **Conformal symmetry**
- Additional transformations: Dilatations and inversions

Dilatation transf.:	$D: \quad x^\mu \to \kappa\, x^\mu \quad \kappa \in \mathbb{R}$	
Special conformal transf.:	$K^\mu = I \circ P^\mu \circ I$	with I : Inversion $\quad x^\mu \to \frac{x^\mu}{x^2}$

- Conformal group in 4d is $SO(2,4)$ with algebra:

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu})\,, \quad [D, P_\mu] = iP_\mu\,, \quad [D, K_\mu] = -iK_\mu\,,$$

$$[K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \qquad \text{\& Poincaré algebra}$$

- Prominent examples:

Maxwell's theory	$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$\lambda\phi^4$ theory	$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \lambda\, \phi^4$
Standard model	$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi + \psi_i Y_{ij} \psi_j\, \phi$
up to Higgs mass term	$+ \lvert D_\mu \phi \rvert^2 - \lambda \lvert\phi\rvert^4 + m^2\, \lvert\phi\rvert^2$

Extension II: Supersymmetry

Supersymmetry is a unique extension of space-time symmetries [1971,1974]

"Square root" of the momentum: $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\,(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu$

- Graded Lie algebra: Generators Q_α & $\bar{Q}_{\dot{\alpha}}$ are fermionic generators.
  $\Rightarrow$ Super-Poincaré algebra with generators $\{M_\mu, P_\mu; Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$
- Relates bosons and fermions:

$$\bar{Q}_{\dot{\alpha}}\,|\text{spin} = s\rangle = |\text{spin} = s + 1/2\rangle$$

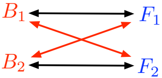
- SUSY:

Boson	$\longleftrightarrow$	Fermion
Gluon	$\longleftrightarrow$	Gluino

  Superpartners are degenerate in all quantum numbers (mass, charge, ...)

- **Extended supersymmetry:** Can have more than one set of supercharges

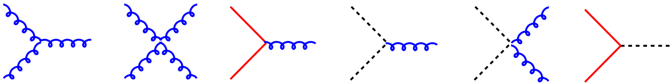
  $\rightarrow Q^A_\alpha$ & $\bar{Q}_{\dot{\alpha}\,A}$ with $A = 1, \ldots, \mathcal{N}$:



  Gluon $\longleftrightarrow$ $\mathcal{N}$ Gluinos

- Maximal SUSY: (massless, spin ≤ 1) $\mathcal{N} = 4$ $\Rightarrow$ Gluon, 4 gluinos, 6 scalars

Gauge field theory



Gauge Field Theory (or Yang-Mills-Theory)

- Builds upon internal (non-space-time) symmetry
- $\mathbf{SU(N)}$ Gauge theory: [1954]

  Generalization of Maxwell's theory of electromagnetism: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

  Vector potential now $N \times N$ hermitian matrix: $(\mathbf{A}_\mu)_{ab}(x) \quad {}_{a,b=1,\dots,N}$
- **Local** gauge symmetry: $\mathbf{A}_\mu(x) \to \mathbf{U}\,\mathbf{A}_\mu\,\mathbf{U}^\dagger + \frac{i}{\mathbf{g}}\,\mathbf{U}\partial_\mu\mathbf{U}^\dagger \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$

  with $\mathbf{U} \in SU(N)$, i.e. unitary $N \times N$ matrix, $\mathbf{U}\,\mathbf{U}^\dagger = 1$
- Invariant action

$$S_{\rm YM} = \frac{1}{4}\int d^4x\,\mathrm{Tr}(\,\mathbf{F}_{\mu\nu}\,\mathbf{F}^{\mu\nu}\,) \qquad \mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + i\,\mathbf{g}\,[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

  $\mathbf{g}$: Coupling constant.
- $N = 1$: Maxwell theory!

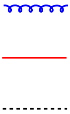
$\mathcal{N} = 4$ super Yang-Mills theory (SYM)

Can we have everything?

- Poincaré symmetry $\to$ relativistic QFT
- Conformal symmetry $\to$ scale-invariant QFT
- Maximal supersymmetry ($\mathcal{N} = 4$)
- $SU(N)$ local gauge symmetry (with $N \to \infty$)

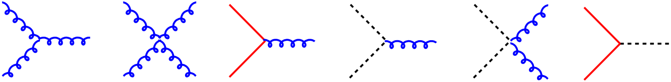
$\Rightarrow$ $\mathcal{N} = 4$ SYM

A_μ	1 Gluon	spin=1	(= same as in QCD)
ψ_α^A	4 Gluinos	spin=1/2	(= cousin of the quarks)
ϕ_I	6 Scalars	spin=0	



$$\mathcal{L}_{\text{SYM}} = \frac{N}{\lambda}\,\mathrm{Tr}\left[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu\Phi_I)^2 - \tfrac{1}{4}[\Phi_I,\Phi_J][\Phi_I,\Phi_J] + \bar{\psi}\not{D}\psi + \psi^A\psi^B\Phi_{AB} + h.c.\right]$$

Interactions:



All fields $N \times N$ matrices. In $N \to \infty$ (planar) limit: One parameter $\lambda = g^2 N$

The simplest quantum field theory

$\mathcal{N}=4$ SYM has remarkably rich properties:

- Uniquely determined by g & N, exactly scale invariant at any coupling, no UV divergences $\Rightarrow$ $g=$ const [1980's]
- Dual to string theory $\to$ AdS/CFT correspondence. [1997]

  Strong coupling limit ($\lambda = g^2 \to \infty$): Classical string on $AdS_5 \times S^5$.
- Appears to be integrable in $N \to \infty$ limit: [since 2003]
  - Exact results for two-point correlation functions $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = (x-y)^{-2\Delta-\gamma(\lambda)}$
  - Hidden symmetries beyond super-conformal group: Yangian algebra
  - Deep mathematical understanding of scattering amplitudes

$\Rightarrow$ An ideal theoretical laboratory to study gauge theories (and string theory)!

$\Rightarrow$ Could be the first exactly solvable interacting 4d QFT.

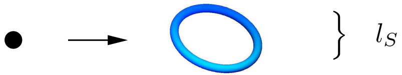
$\Rightarrow$ Non-physical! But possible starting point for novel perturbative approach.

$\Rightarrow$ Already now application to massless QCD exist.

The world as a hologram

String theory in a nut-shell

- **Idea:** Replace particle by extended 1d object: **string**



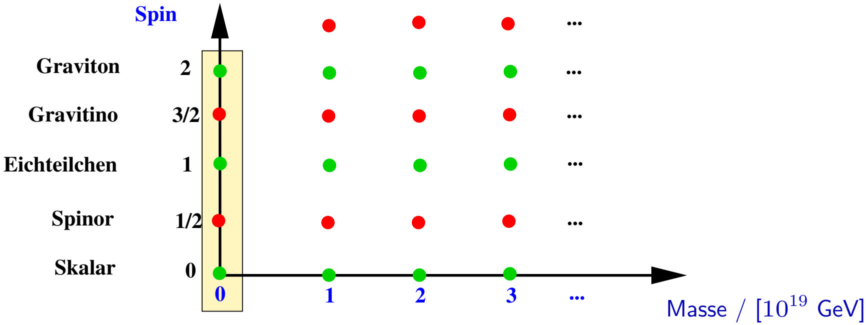
- Quantum mechanics of a relativistic string in flat space-time $\mathbb{R}^{1,d-1}$:

Graviton   Gauge boson   Matter particle

- Oscillation spectrum $\hat{=}$ spectrum of “elementary particles”
- Strings must propagate in d=9+1. Theory depends on background geometry
- Consistent theory of quantum gravity
- Unification of matter and force particles as excitation of one entity: the fundamental string

Spectrum of string excitations in flat (Minkowski) space-time

String theory $\hat{=}$ QFT with infinite number of particle species

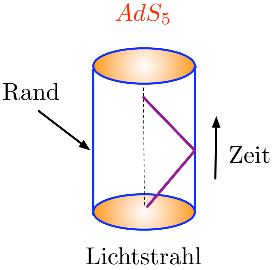


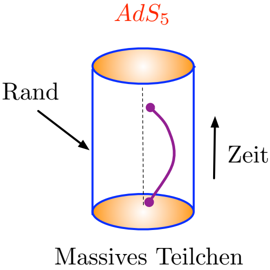
- Low energy limit: Recovers known interacting particle field theories
- This is the well understood situation in flat $\mathbb{R}^{1,9}$
- Now: Take curved space-time geometry with boundary!

Quantum gravity in a box

- Space-time with negative curvature: anti-de-Sitter space (AdS_d)

  [Willem de Sitter, 1872-1934]
- AdS_5 is (4+1)-dimensional space-time with boundary of geometry $\mathbb{R}^{1,3}$

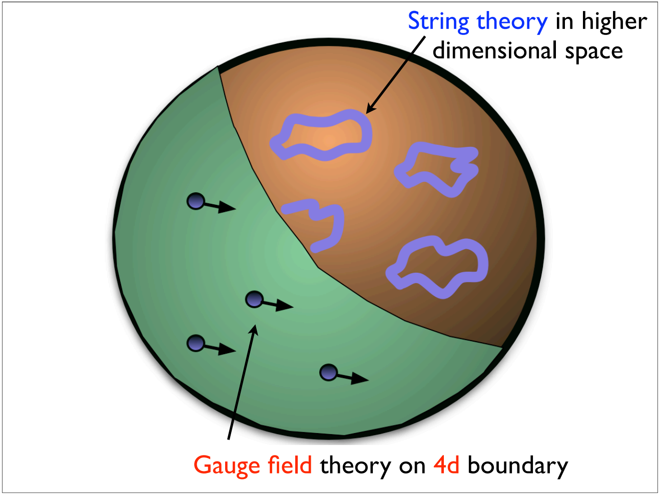




- $ds^2_{AdS} = R^2 \frac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- Quantum gravity in a box!
- String theory well defined on $AdS_5 \times M_5$, e.g. $M_5 = S^5$ (5d-sphere).
- Quantization of strings on $AdS_5 \times S_5$ unsolved!
- Isometry group of $AdS_5 \hat{=}$ conformal group in 4d

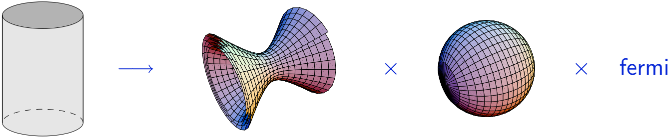
The String-Gauge Theory (or AdS/CFT) duality [Maldacena, 1997]

- **Holographic duality:** Strings in $AdS_5 \times S_5$ are dual to $\mathcal{N} = 4$ SYM



- Claim: Two alternative mathematical descriptions of one physical object!

The dual model: Superstring in $AdS_5 \times S^5$



$$S_{\text{string}} = \sqrt{\lambda} \int d\tau\, d\sigma \left[G_{mn}^{(\mathrm{AdS}_5)}\, \partial_a X^m \partial^a X^n + G_{mn}^{(\mathrm{S}_5)}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$

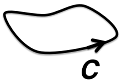
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, semiclassical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $\lambda \ll 1$: perturbative SYM vs. $\lambda \gg 1$: semiclassical ST
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- Isometries: $\mathfrak{so}(2,4) \times \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$
- **Quantization unsolved!**

- **Local operators:** $\mathcal{O}_n(x) = \mathrm{Tr}[\mathcal{W}_1\, \mathcal{W}_2 \ldots \mathcal{W}_n]$ with $\mathcal{W}_i \in \{\mathcal{D}^k \Phi, \mathcal{D}^k \Psi, \mathcal{D}^k F\}$

  2 point fct: $$\langle \mathcal{O}_a(x_1)\, \mathcal{O}_b(x_2)\rangle = \frac{\delta_{ab}}{(x_1 - x_2)^{2\,\Delta_a(\lambda)}}$$ $\Delta_a(\lambda)$ Scaling Dims

- **Wilson loops:**

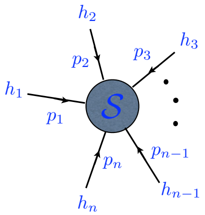
$$\mathcal{W}_C = \langle \mathrm{Tr}\, P \exp i \oint_C ds\, (\dot{x}^\mu A_\mu + i|\dot{x}|\, \theta^I\, \Phi_I)\rangle$$



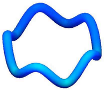
- **Scattering amplitudes:**

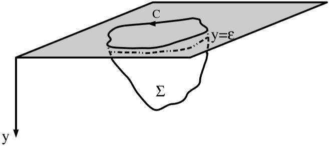
$$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda) = \begin{Bmatrix} \text{UV-finite} \\ \text{IR-divergent} \end{Bmatrix}$$

helicities: $h_i \in \{0, \pm\frac{1}{2}, \pm 1\}$

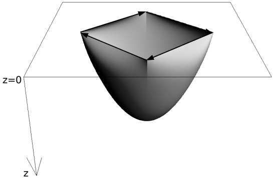


Gauge Theory - String Theory Dictionary: String side

$\Delta_a(\lambda)$ spectrum of scaling dimensions ⇔  $E(\lambda)$ string excitation spectrum (solved?)

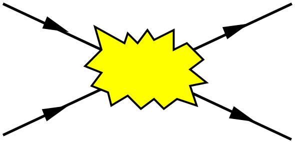
Wilson loop $\mathcal{W}_C$ ⇔ 

minimal surface

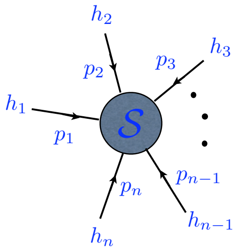
$\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda)$ T-dual (⇔) 

light-like boundary

Scattering amplitudes



Scattering amplitudes

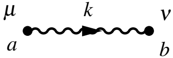


$\mathcal{A}_n(\{p_i, h_i\}) =$	probability amplitude for scattering process

Central quantum field theory prediction for collider experiments

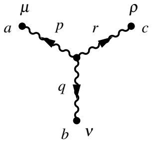
Computed via Feynman diagrams:

Propagator

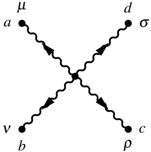


$$= \frac{\delta^{ab}\eta_{\mu\nu}}{k^2 + i\epsilon} \qquad \text{(gluons)}$$

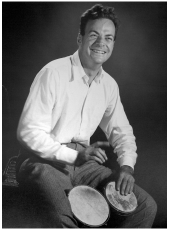
Vertices



$$= g\, f^{abc}\Big[(q-r)_\mu\, \eta_{\nu\rho} + (r-p)_\nu\, \eta_{\rho\mu} + (p-q)_\rho\, \eta_{\mu\nu}\Big]$$



$$= -ig^2\Big[\, f^{abe} f^{cde}\,(\eta_{\mu\rho}\,\eta_{\nu\sigma} - \eta_{\mu\sigma}\,\eta_{\nu\rho}) + f^{ace} f^{dbe}\,(\eta_{\mu\sigma}\,\eta_{\rho\nu} - \eta_{\mu\nu}\,\eta_{\rho\sigma}) + f^{ade} f^{bce}\,(\eta_{\mu\nu}\,\eta_{\sigma\rho} - \eta_{\mu\rho}\,\eta_{\sigma\nu})\Big]$$



Feynman diagramatic expansion

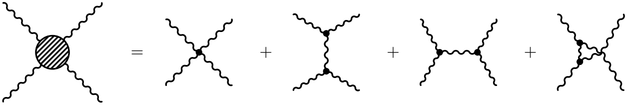
Task:

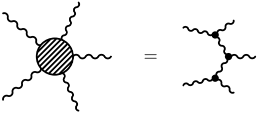
a) Draw all Feynman diagrams contributing to a given process

b) Integrate over all internal (off-shell) momenta $\int d^{4-2\epsilon} l$ imposing momentum conservation $\delta^{(4)}(\sum_i p_i)$ at each vertex

c) $\mathcal{A}_n = \sum$ all diagrams

Can rapidly get out of hand: (even at tree-level)

4 gluons: 

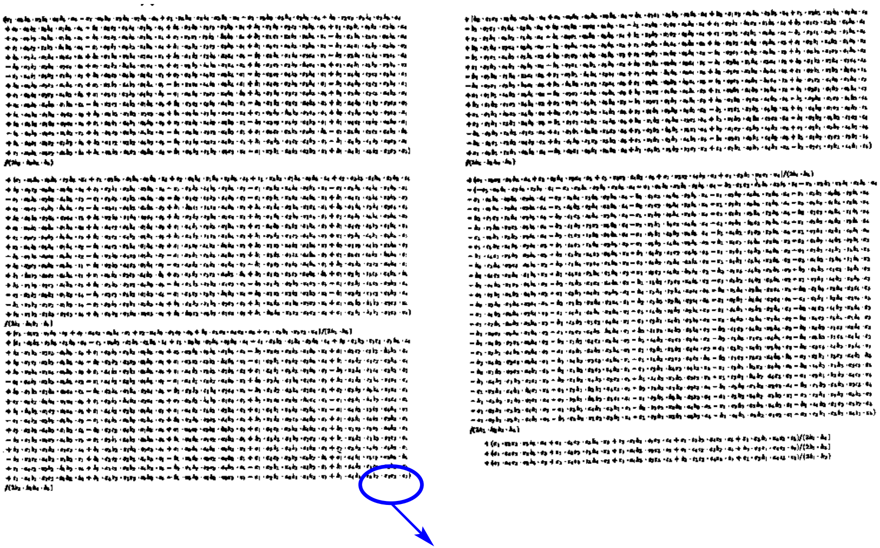
5 gluons:  $+$ 24 more $\rightarrow$ result

number of external gluons	4	5	6	7	8	9	10
number of diagrams	4	25	220	2485	34300	559405	10525900

[Mangano,Parke]

- From Z. Bern's KITP Colloquium:

Result of a brute force calculation (actually only a small part of it):



$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

Simplicity of the result

When expressed in right variables the result is remarkably simple: [Parke, Taylor]

$$\mathcal{A}_5(1^\pm,2^+,3^+,4^+,5^+) = 0$$

$$\mathcal{A}_5(1^-,2^-,3^+,4^+,5^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

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(all others from cyclicity and parity)

Spinor helicity: $\boxed{p^\mu \to p^{\alpha\dot\alpha} = \bar\sigma_\mu^{\alpha\dot\alpha}\, p^\mu = \lambda^\alpha\tilde\lambda^{\dot\alpha}}$ (makes $p^\mu p_\mu = 0$ manifest)

$$\lambda^\alpha = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1+ip^2\end{pmatrix}, \quad \tilde\lambda^{\dot\alpha} = (\lambda^\alpha)^\dagger, \quad \langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$$

What is the reason for this simplicity?

- Hidden symmetries ($\rightarrow$ hidden super-conformal invariance & more)
- Analytic structure of the amplitude ($\rightarrow$ factorization, soft & collinear limits)

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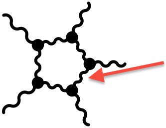
Basic problem of Feynman diagramatic approach

In Feynman graph techniques one sums and integrates over non-physical terms:

$$\int \frac{d^3p\,dE}{(2\pi)^4}$$



$$\boxed{E^2 - \vec{p}^2 \neq m^2}$$

Internal states are off-shell, violate mass-shell condition

Similarly individual diagrams are gauge variant, but final result is gauge invariant!

On-shell approaches:

Since 2005 tremendous progress in our understanding of scattering amplitudes based on on-shell formulations:

- On-shell recursion relations ✓
- Hidden symmetries ✓
- Generalized unitarity ✓
- Twistors & the Grassmannian & Integrability ×

The $\mathcal{N} = 4$ SYM theory has been instrumental in this progress!

Britto-Cachazo-Feng-Witten (BCFW) recursion

- Idea: Complexify momenta but stay on-shell $\quad z \in \mathbb{C}$

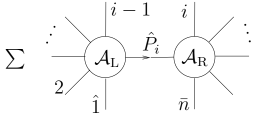
$$p_1 \to \hat{p}_1(z) = \lambda_1 \, (\tilde{\lambda}_1 - z \, \tilde{\lambda}_n) \qquad p_n \to \hat{p}_n(z) = (\lambda_n + z \, \lambda_1) \, \tilde{\lambda}_n$$

Obeys $\hat{p}_i(z)^2 = 0$ and $\hat{p}_1(z) + p_2 + \ldots p_{n-1} + \hat{p}_n(z) = 0$.

$$\text{Deformation} \quad \mathcal{A}_n \to \mathcal{A}_n(z) \qquad \mathcal{A}_n(z=0) = \oint dz \, \mathcal{A}_n(z)/z$$

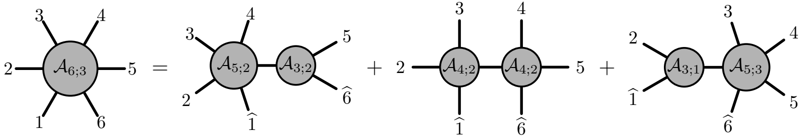
- Cauchy's theorem yields recursive relation for on-shell amplitudes

$$\mathcal{A}_n = \sum_{i=3}^{n-1} \mathcal{A}_i \, \frac{1}{P_i^2} \, \mathcal{A}_{n-i+2}$$

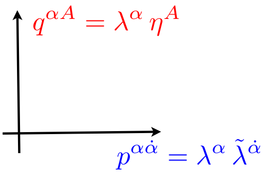


$$\boxed{\hat{P}_i^2 = 0}$$

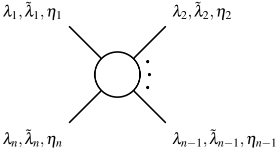
- "Atoms" are the 3-point amplitudes: $\quad A_3(i^-, j^-) = \dfrac{\langle ij \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$
- Example: $N = 6$



$\mathcal{N}=4$ SYM: Superamplitudes and Super-BCFW recursion



- Consider super momentum-space using 4 anti-commuting coordinates η^A:
- Define superamplitudes $\mathbb{A}_n$ in this formal space: [Nair]



$$\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i p_i)\, \delta^{(8)}(\sum_i q_i)}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde\lambda_i, \eta_i\})$$

  Superamplitudes package all gluon-gluino-scalar amplitudes.

- Super-BCFW recursion exists: [Brandhuber,Heslop,Travaglini][Arkani-Hamed,Cachazo,Cheung,Kaplan]

$$\mathbb{A}_n(1,\ldots,n) = \sum_{i=3}^{n-1} \int d^4\eta_{\hat P_i}\, \mathbb{A}^L_i\Big(\hat 1, \ldots, -\hat P_i\Big) \frac{1}{P_i^2} \mathbb{A}^R_{n-i+2}\Big(\hat P_i, \ldots, \hat n\Big)$$

- May be solved analytically!
  $\Rightarrow$ **All** tree-amplitudes in $\mathcal{N}=4$ SYM known in analytic form. [Drummond,Henn]

Symmetries of scattering amplitudes

- Superconformal symmetry of $\mathcal{N}=4$ SYM constrains superamplitudes

$$\boxed{\mathbb{A}_n^{\text{tree}} = \frac{\delta^{(4)}(\sum_i p_i)\,\delta^{(8)}(\sum_i q_i)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\,\mathcal{P}_n(\lambda_i,\tilde{\lambda}_i,\eta_i)}$$

- Obvious symmetries:

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^{\alpha}\,\tilde{\lambda}_i^{\dot{\alpha}} \qquad q^{\alpha\,A} = \sum_{i=1}^{n} \lambda_i^{\alpha}\,\eta_i^{A} \qquad \Rightarrow\ p^{\alpha\dot{\alpha}}\,\mathbb{A}^{\text{tree}} = 0 = q^{\alpha A}\,\mathbb{A}^{\text{tree}}$$

explains vanishing of $A_n(1^{\pm},2^{+},\ldots,n^{+})$

- Less obvious symmetries: [Witten]

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i^{\alpha}}\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}} \qquad s_{\alpha\,A} = \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i^{\alpha}}\,\frac{\partial}{\partial\eta_i^{A}} \qquad \Rightarrow\ k_{\alpha\dot{\alpha}}\,\mathbb{A}^{\text{tree}} = 0 = s_{\alpha A}\,\mathbb{A}^{\text{tree}}$$

explains form of $A_n(1^{-},2^{-},3^{+}\ldots,n^{+})$

- Super-conformal invariance of tree-amplitudes (32+32 generators):

$$\boxed{J^a\,\mathbb{A}_n^{\text{tree}} = 0 \quad \text{with} \quad J^a \in \{\,p,k,\bar{m},m,d,r,q,\bar{q},s,\bar{s},c_i\,\}}$$

Infinite dimensional hidden symmetry

Tree superamplitudes are invariant under additional hidden symmetry (as in H-atom)

[Drummond,Henn,Korchemsky,Sokatchev][Drummond,Henn,JP]

- Mathematical structure: Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$ [Drinfeld]

$$J^a = \sum_{i=1}^{n} J_i^a \quad \text{(level 0)} \qquad J_{(1)}^a = f^a{}_{bc} \sum_{i<j}^{n} J_i^b \, J_j^c \quad \text{(level 1)}$$

An ∞-dim non-local symmetry algebra $\quad J_{(n)}^a \;\; n = 0, 1, 2, \dots$

$$[J^a, J^b] = i f^{ab}{}_c \, J^c$$
$$[J^a, J_{(1)}^b] = i f^{ab}{}_c \, J_{(1)}^c$$
$$[J_{(1)}^a, J_{(1)}^b] = i f^{ab}{}_c \, J_{(2)}^c + g_{ab}(J^a, J_{(1)}^a)$$

$$\boxed{J_{(n)}^a \, \mathbb{A}_n^{\text{tree}} = 0} \quad \forall n$$

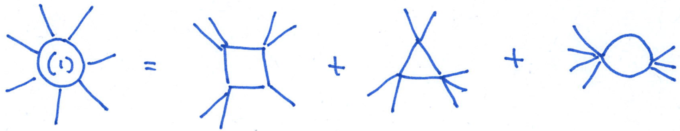
[Drummond,Henn,JP]

- Signature of integrable field theory. Explains simplicity of $\mathbb{A}_n^{\text{tree}}$
  $\Leftrightarrow$ Determines form of $\mathbb{A}_n^{\text{tree}}$ [Bargheer,Beisert,McLoughlin,Loebbert,Galleas]
- AdS/CFT: T-duality of dual string theory. [Alday,Maldacena][Beisert,Ricci,Tseytlin][Berkovits,Maldacena]

Generalized unitarity

- On-shell methods also constructive at 1-loop (NLO-order) [Bern,Dixon,Dunbar,Kosower]
- General 1-loop amplitude may be decomposed in basis integrals

  [Passarino,Veltman][Ossola,Papadopoulos,Pittau][Giele,Kunszt,Melnikov]

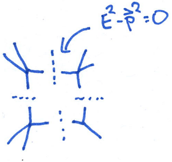


- In $\mathcal{N}=4$ SYM: Only box integrals occur due to hidden symmetry.

$$A_n^{\text{1-loop}} = \sum_i c_i \text{Box}_i$$

- Find c_i by putting internal propagators on-shell [Bern,Dixon,Kosower,Smirnov]

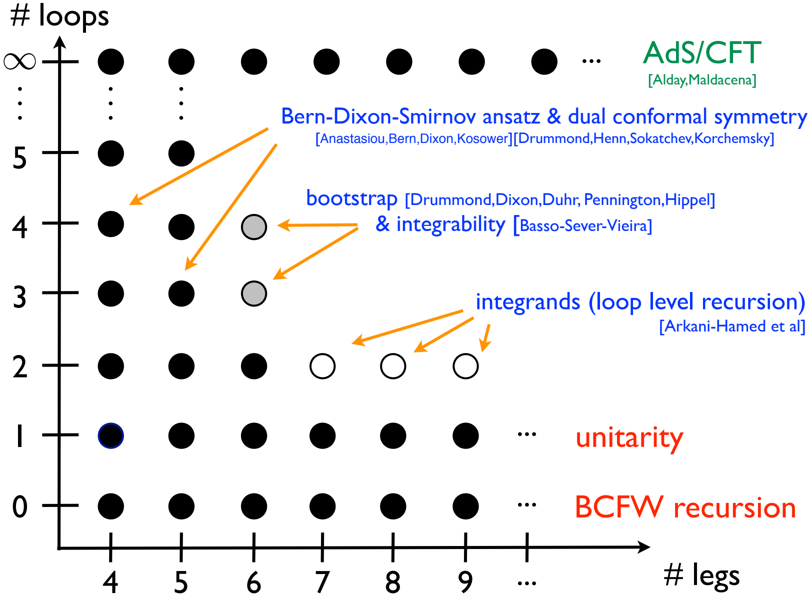
$$c_i =$$



$$= \frac{1}{2}\sum_{l_\pm} A_1^{\text{tree}}(l_\pm)\, A_2^{\text{tree}}(l_\pm)\, A_3^{\text{tree}}(l_\pm)\, A_4^{\text{tree}}(l_\pm)$$

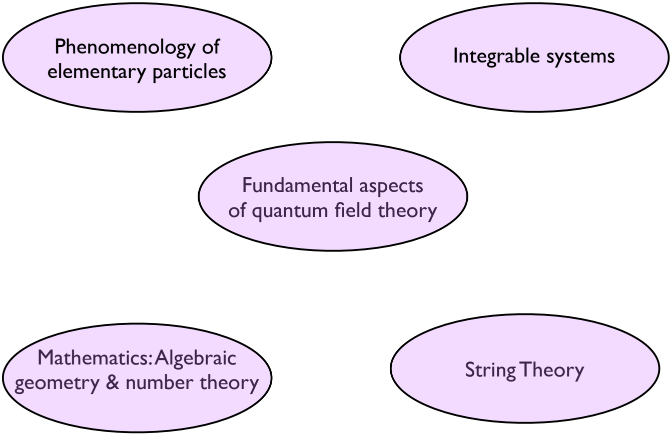
State of the art

Known MHV amplitudes: $A_n(1^-, 2^-, 3^+, \ldots, n^+)$ in $\mathcal{N} = 4$ SYM



Summary

Field combines a multitude of areas in theoretical and mathematical physics:



$\Rightarrow$ Intellectually rich and fascinating research area with “real physics” applications!

Thank you for your attention

Literature:

Bern, Dixon, Kosower, Scientific American 2012
Beisert et. al. „Review of AdS/CFT integrability“, Lett.Math.Phys.99
Ellis, Kunszt, Melnikov, Zanderighi, Phys. Rep. 518 (2012)
Henn & Plefka, „Scattering Amplitudes in Gauge Theories“
LNP 883, Springer