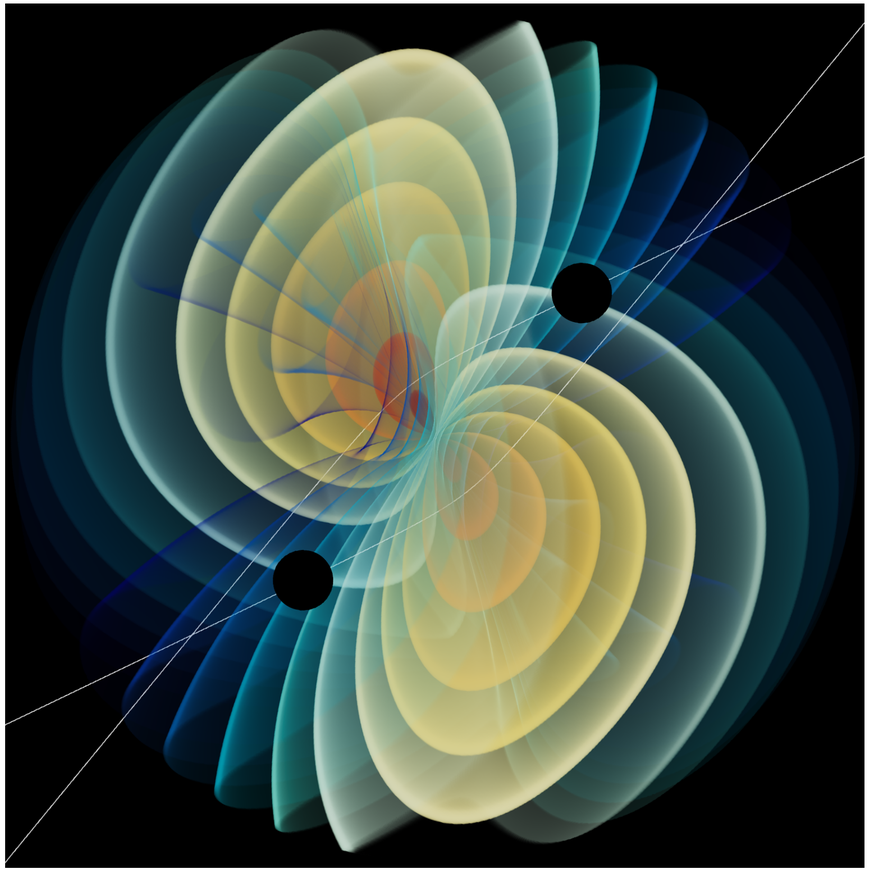
HIGH PRECISION BLACK HOLE SCATTERING FROM WORLDLINE QUANTUM FIELD THEORY



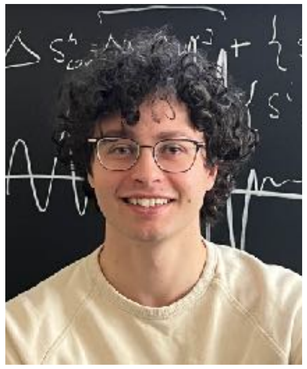
Jan Plefka

Humboldt Universität zu Berlin

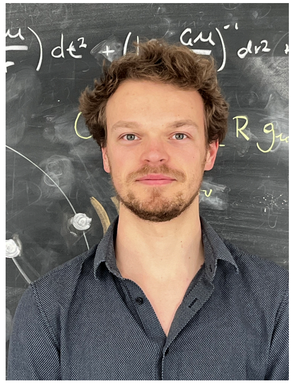
European Research Council
Established by the European Commission

RTG 2575:
Rethinking
Quantum Field Theory

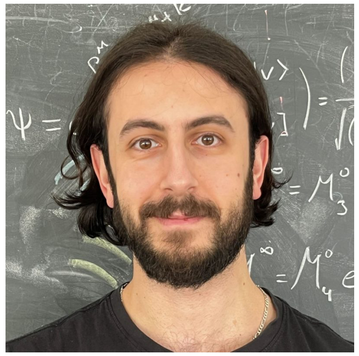
JOINT WORK WITH . . .



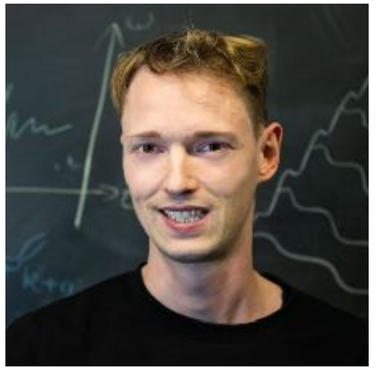
Mathias Driesse



Jitze Hoogeveen



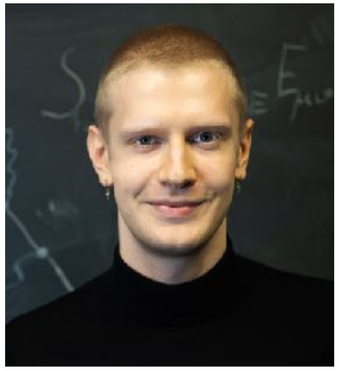
Kays Haddad



Gustav Uhre Jakobsen



Johann Usovitsch



Benjamin Sauer



Kathrin Stoldt



Gustav Mogull

(QMUL)



Christoph Nega

(MPI Potsdam)

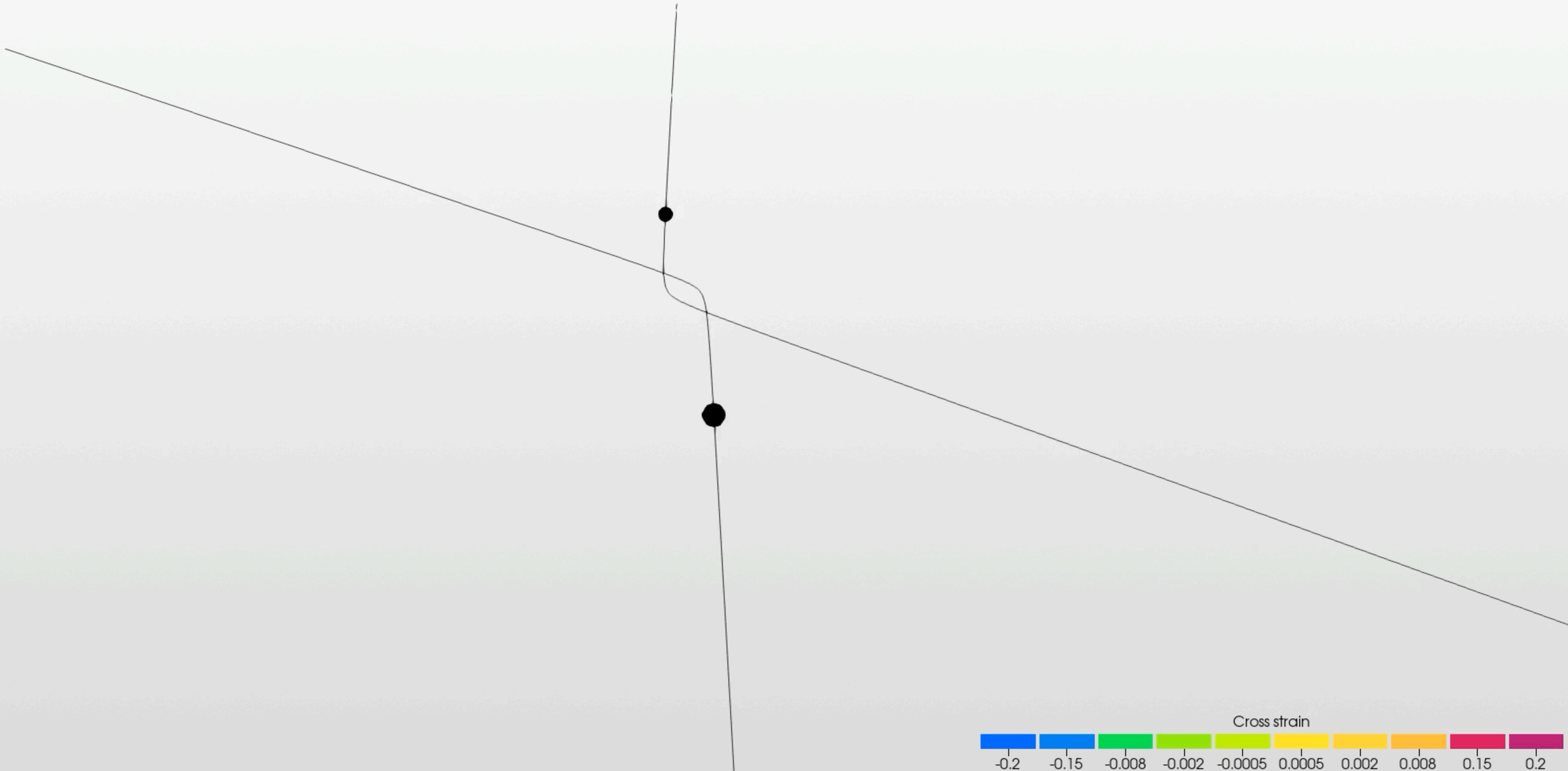


Albrecht Klemm

(U Bonn/Shefield)

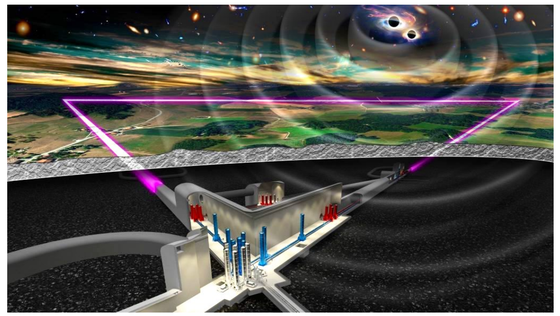
$\gamma = 1.1,\ \frac{m_1}{m_2} = 1,\ \frac{b}{m_1} = 50$

Time: -10.000000

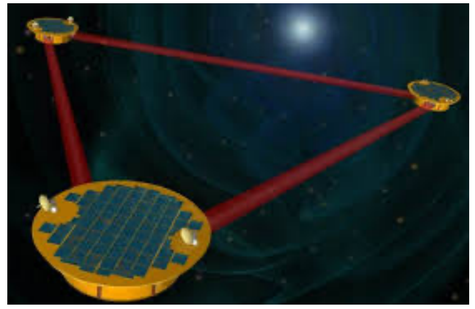


ERA OF GRAVITATIONAL WAVE PHYSICS: NEED FOR HIGH-PRECISION PREDICTIONS

- Upcoming 3rd generation of gravitational wave observatories with 10^2 sensitivity increase
- Need for accurate waveform predictions well beyond state-of-the art



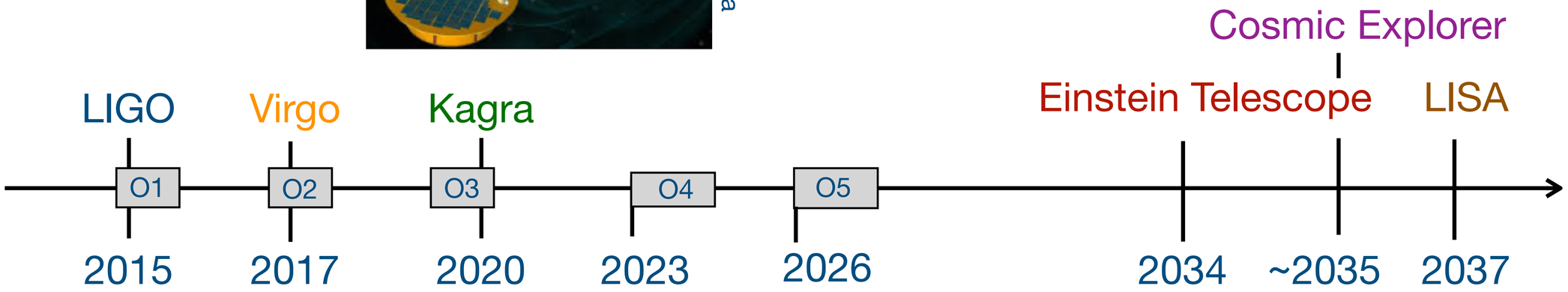
ET©Nikhef



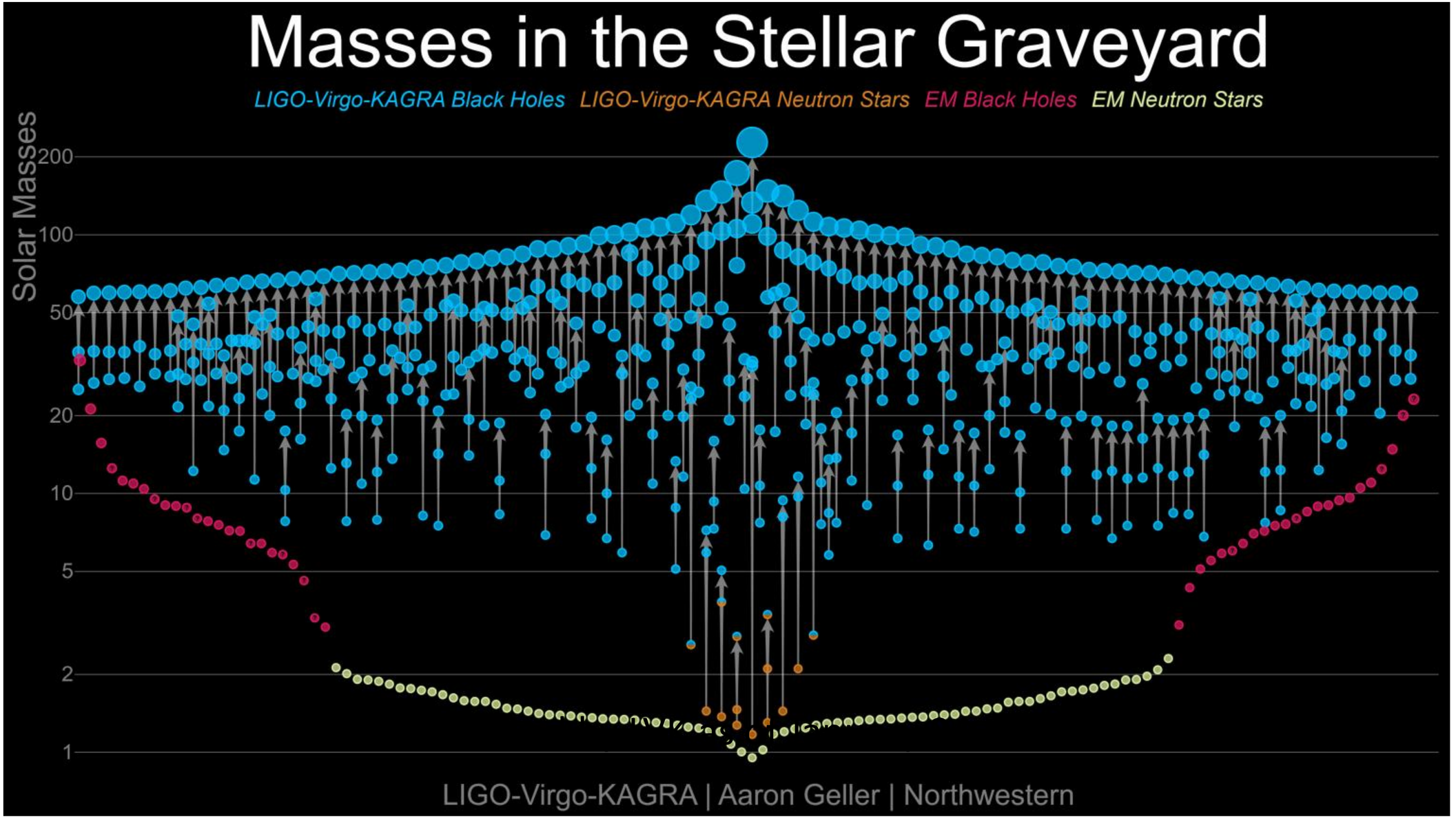
LISA©Nasa

High-precision predictions necessary basis to study fundamental questions in physics:

- Is Einstein's theory correct?
- Black hole formation & population?
- Neutron star properties?
- Physics beyond the standard model?



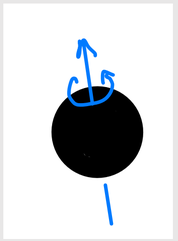
GRAVITATIONAL WAVES: A NEW OBSERVATIONAL ERA



To date O(200) binary mergers detected by LIGO-Virgo-Karga Collaboration

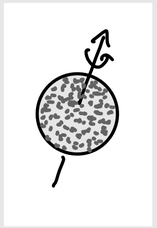
GRAVITATIONAL TWO-BODY PROBLEM

Black Hole



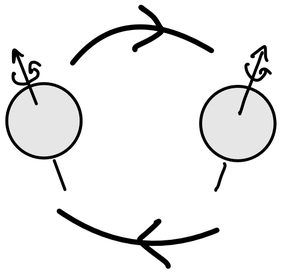
mass, spin

Neutron Star

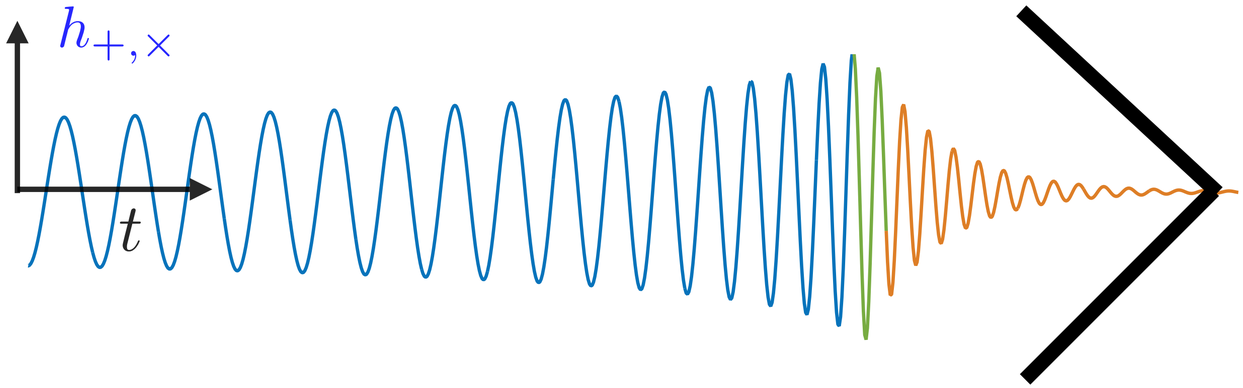


mass, spin, radius,
tidal deformability

Black Hole/Neutron Star Binaries:



Bound state



inspiral merger

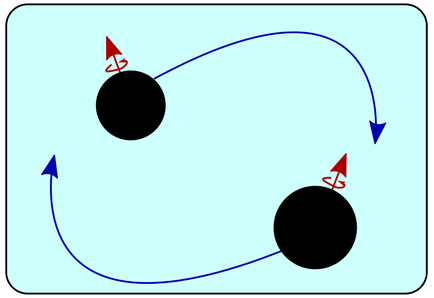
Einstein's eqs. cannot be solved exactly ⇒ approximation schemes

- **Numerical relativity:** good for **merger** (strong gravity, short duration)
- **Perturbation theory**: good for **inspiral** (weak fields, long duration)

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}$$ ⇔ import **Quantum Field Theory** tools!

PERTURBATIVE SCHEMES

Post-Newtonian (PN)

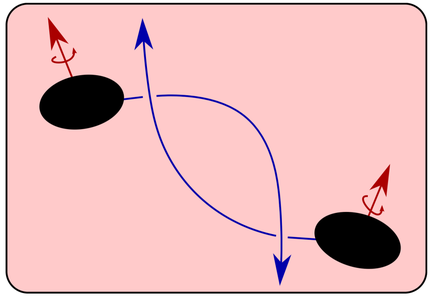


- Large-separation & slow-velocity expansion

$$\epsilon_{\rm PN} = \frac{GM}{r} \sim v^2$$

- Good for quasi-circular bound orbits

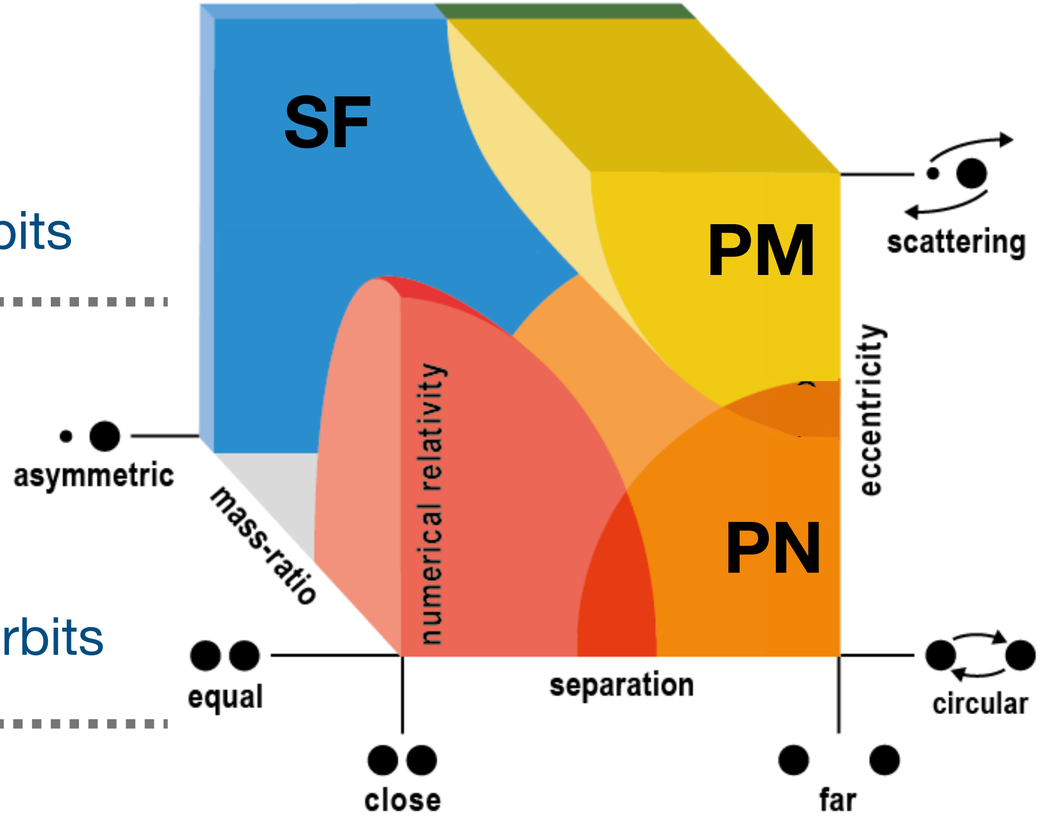
Post-Minkowskian (PM)



- Large-separation expansion

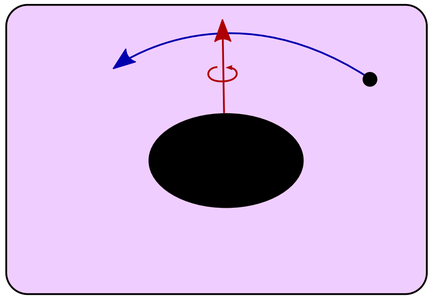
$$\epsilon_{\rm PM} = \frac{GM}{r} \quad \text{exact in } v!$$

- Good for scattering & eccentric orbits



(credit: Ana Carvalho)

Gravitational Self-Force (SF)



- Large-mass ratio expansion

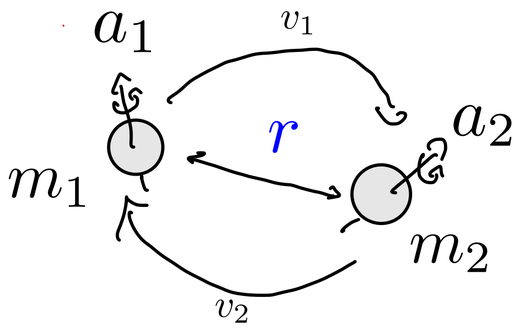
$$\epsilon_{\rm GSF} = \frac{m_1}{m_2} \quad \text{semi-analytic, exact in } G \;\&\; v!$$

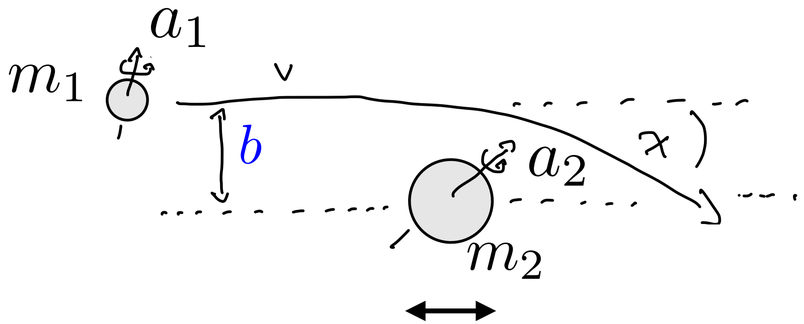
- Good for extreme-mass ratios, scattering and bound orbits

WORLDLINE EFFECTIVE FIELD THEORY OF COMPACT OBJECTS

[Goldberger,Rothstein][Porto][Levi,Steinhoff]

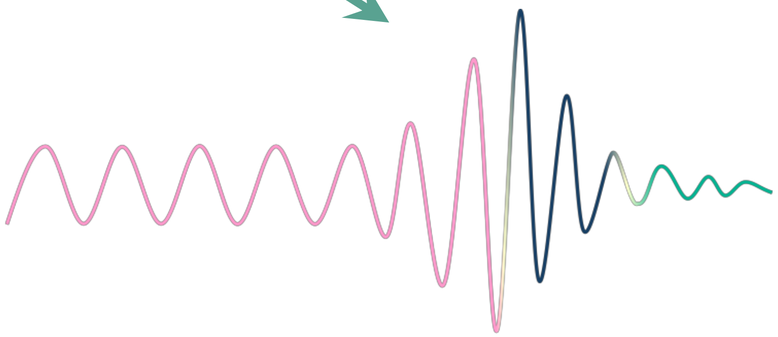
During **inspiral** or **scattering**: Separation of scales: $R_S \sim Gm \ll b$ or $r \ll R$





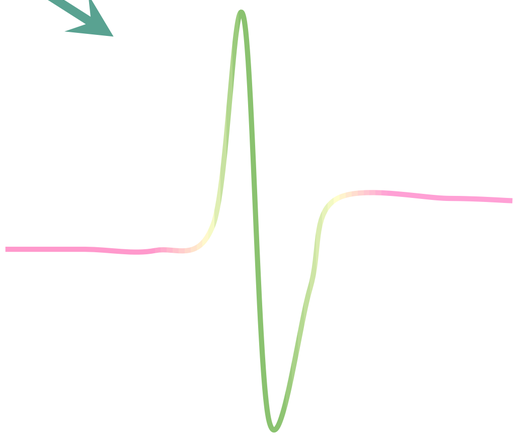
$R_S \sim Gm$

R



Far field waveform

R

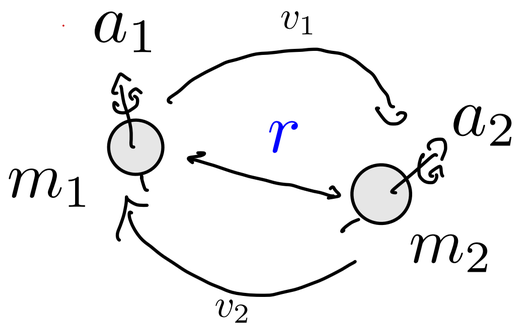


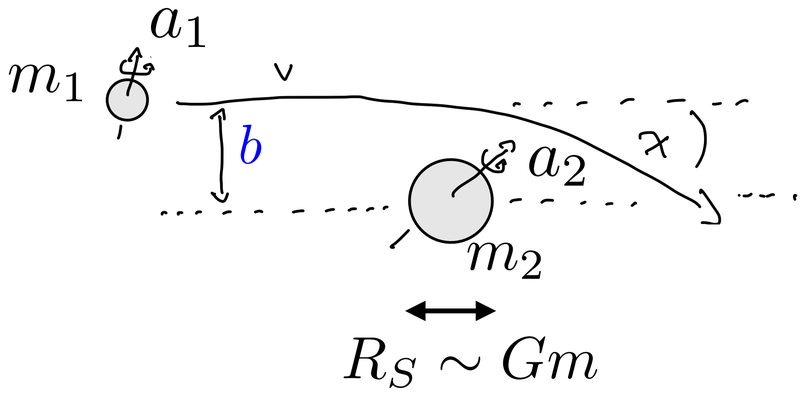
Far field waveform

WORLDLINE EFFECTIVE FIELD THEORY OF COMPACT OBJECTS

[Goldberger,Rothstein][Porto][Levi,Steinhoff]

During **inspiral** or **scattering**: Separation of scales: $R_S \sim Gm \ll b$ or $r \ll R$





Effective Field Theory description: BH?NS as **spinning point particle** moving on its **worldline**

$$S_{\rm EFT} = -m \int d\tau \left(\sqrt{g_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau)} + \sum_{n=1}^{\infty} \mathcal{L}^{(n)}_{\rm EFT} \right)$$

Black-hole spin a^μ:

$$|a^\mu| \leq m G_N$$

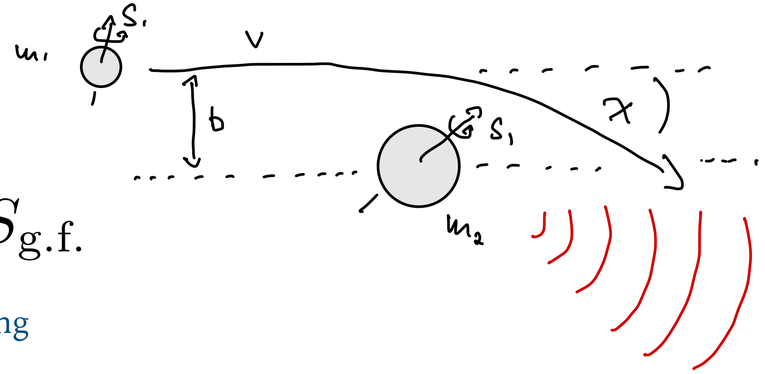
$$\mathcal{L}^{(1)}_{\rm EFT} \sim \frac{m G_N}{b} \sim a\, \partial g$$

$$\mathcal{L}^{(2)}_{\rm EFT} \sim \left(\frac{m G_N}{b} \right)^2 \sim a^2 \partial^2 g \quad \dots$$

$$\mathcal{L}^{(4)}_{\rm EFT} \sim \left(\frac{m G_N}{b} \right)^4 \sim a^4 \partial^4 g + (m G_N)^4 (\partial^2 g)^2 + \dots$$

RELATIVISTIC TWO BODY PROBLEM IN PM: TRADITIONAL APPROACH

Point-particle approximation for BHs (or NSs) $\quad |b| \gg Gm_*$



$$S = -\sum_{i=1}^{2} m_i \int d\tau_i \sqrt{g_{\mu\nu}\dot{x}_i^\mu(\tau_i)\dot{x}_i^\nu(\tau_i)} + \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{\text{g.f.}}$$

Point particle approximation — Bulk gravity & gauge fixing

1) Equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{\kappa^2}{8}T_{\mu\nu}$$

Einstein's eqs.

$$\ddot{x}_i^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}_i^\nu\dot{x}_i^\rho = 0$$

Geodesic eqs.

2) Solve iteratively in G

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\sum_{n=0}^{\infty} G^n h_{\mu\nu}^{(n)}(x)$$

emitted radiation

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu\,\tau + \sum_{n=1}^{\infty} G^n z_i^{(n)\,\mu}(\tau)$$

straight line: „in" state — deflections

3) Construct observables

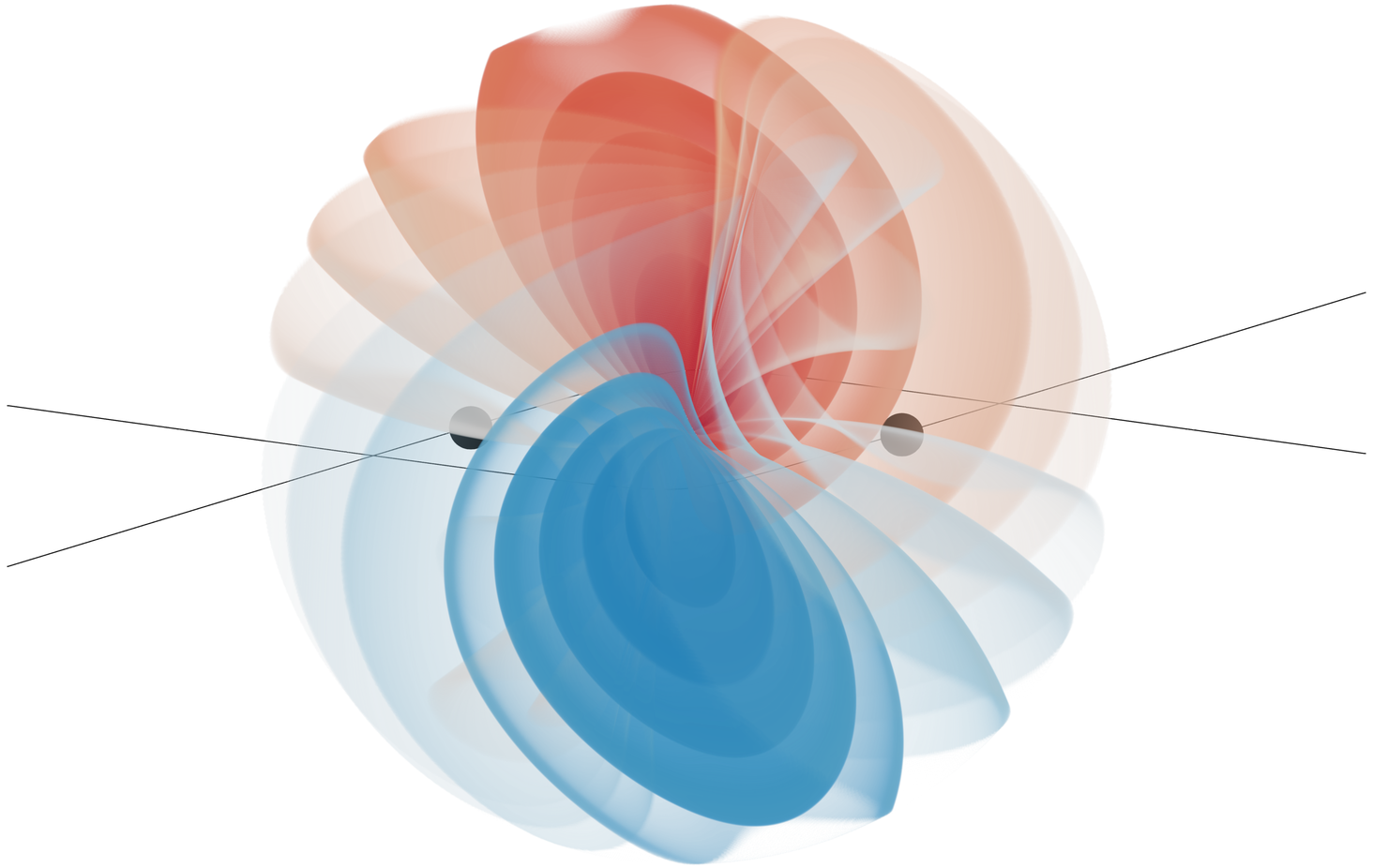
Far field waveform:

$$\lim_{r\to\infty} h_{\mu\nu} = \frac{f_{\mu\nu}(t-r,\theta,\varphi)}{r} + \mathcal{O}(\frac{1}{r^2})$$

„Impulse" (change in momentum):

$$\Delta p_i^\mu = m_i\dot{x}_i^\mu\Big|_{\tau=-\infty}^{\tau=+\infty} = m_i\int d\tau\,\ddot{x}_i^\mu(\tau)$$

WORLDLINE QUANTUM FIELD THEORY



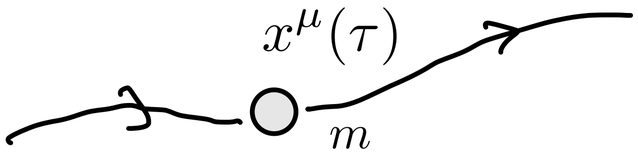
[Jakobsen,Mogull,JP,Sauer]

WORLDLINE QUANTUM FIELD THEORY

Mogull, **JP**, Steinhoff *JHEP* 02 (2021) 048

- Model Black Holes/Neutron Stars as point particles

$$S_{\rm BH/NS} = -\frac{m}{2}\int d\tau g_{\mu\nu}\,\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau) \quad + \text{[spin \& tidal effects]}$$



They interact through Einstein's gravity: $$S = S_{\rm BH/NS} + \frac{1}{16\pi\, G}\int d^4x\sqrt{-g}\,R(g)$$

- Scattering scenario: $x_i^\mu(\tau) = b_i^\mu + v_i^\mu\tau + z^\mu(\tau)$ $\qquad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu}$

- Path integral quantisation perturbative in Newton's constant G but exact in velocity

$$\langle\mathcal{O}\rangle_{\rm WQFT} = \int D[h,z]\mathcal{O}\,e^{-\frac{i}{\hbar}S[z,h]} \xrightarrow{\hbar\to 0}$$

Tree-level one-point functions $\langle h_{\mu\nu}\rangle$ and $\langle z^\mu\rangle$ solve classical equations of motion

$\Rightarrow$ Advanced quantum field theory technology for classical gravitational wave physics

WORLDLINE QUANTUM FIELD THEORY: PROPAGATORS

$$S_{\rm WQFT} = -\frac{m}{2}\int d\tau g_{\mu\nu}\,\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau) + \frac{1}{16\pi\,G}\int d^4x\sqrt{-g}\,R(g)$$

- Scattering scenario: $x_i^\mu(\tau) = b_i^\mu + v_i^\mu\tau + z^\mu(\tau)$ $\qquad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu}$

- Worldline propagators:



$$\langle z^\mu(\omega)z^\nu(-\omega)\rangle = -\frac{i}{m}\frac{\eta_{\mu\nu}}{(\omega+i0)^2}$$

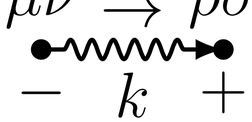
- Perturbative (quantum) gravity (in the bulk):

$$\sqrt{-g}\,R(g) \;=\; -\frac{1}{2}h_{\mu\nu}(P^{-1})^{\mu\nu;\rho\sigma}\Box h_{\rho\sigma} + \sqrt{G}[\partial^2 h^3] + \sqrt{G}^2[\partial^2 h^4] + \sqrt{G}^3[\partial^2 h^5] + \ldots$$

$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu}$

$$P_{\mu\nu;\rho\sigma} = \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}$$

⇒ Graviton propagator:

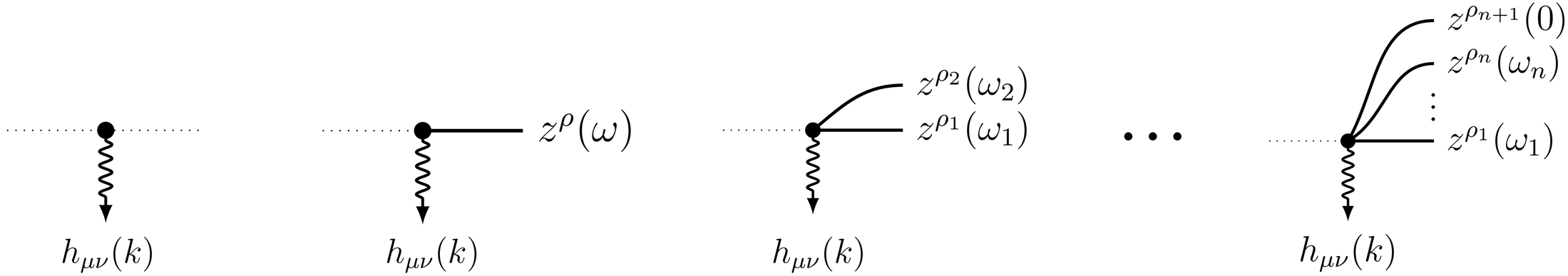


$$= i\frac{P_{\mu\nu;\rho\sigma}}{(k^0+i0)^2 - \mathbf{k}^2}$$

Dissipation: need to take retarded propagator **(in-in formalism)**

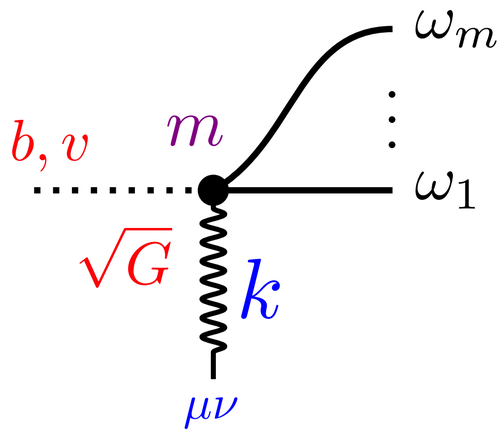
WORLD LINE VERTICES

$$-\frac{m\sqrt{G}}{2}\int d\tau\, h_{\mu\nu}[x(\tau)]\,\dot{x}^{\mu}(\tau)\,\dot{x}^{\nu}(\tau)$$



- Schematic form of gz^m vertex

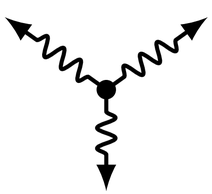
$$V_m^{\mu\nu} = \quad = m\sqrt{G}\, e^{ib\cdot k}\delta\left(v\cdot k + \sum_{i=1}^{m}\omega_i\right) \times \begin{pmatrix}\text{polynomial in } \omega_i, k \\ \text{of degree } 2+m\end{pmatrix}$$

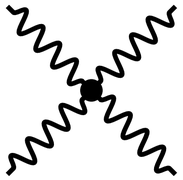


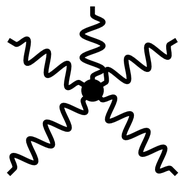
Energy conservation on worldline

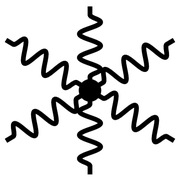
BULK GRAVITON VERTICES

$$\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R(g)$$


$\sim \sqrt{G}\,k^2$


$\sim \sqrt{G}^2 k^2,$

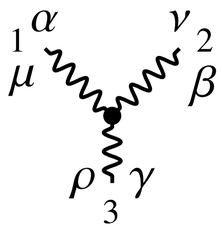

$\sim \sqrt{G}^3 k^2$


$\sim \sqrt{G}^4 k^2, \ldots$

Four-momentum conservation in bulk $\delta^4(\sum p)$

Highly involved structures! Emerge from Einstein-Hilbert action.

- E.g. three-point vertices:



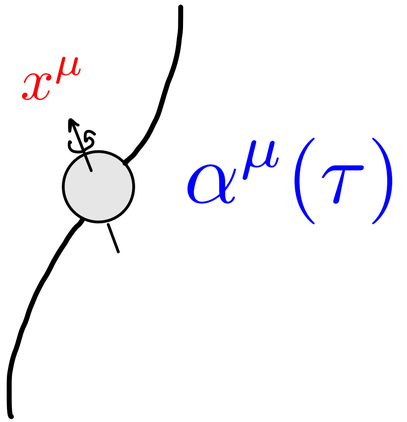
$$= \mathrm{i}\sqrt{G}\,\mathrm{sym}[$$
$$-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\rho\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\gamma})$$
$$+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\rho}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\rho}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\rho\gamma})$$
$$+ P_3(k_{1\rho}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\rho}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) - 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\rho})$$
$$+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\rho}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\nu\alpha}\eta_{\rho\beta}\eta_{\mu\gamma})],$$

n	# of terms
3	33
4	366
5	4630
6	51360

(in an optimised generalized De Donder gauge)

PUTTING SPIN ON THE WORLD-LINE

[Haddad,Jakobsen,Mogull,JP]



- Spinning-black holes: Capture spin d.o.f. by adding complex $\alpha^\mu(\tau)$:
  Inspired by bosonic string theory (tensionless $\alpha' \to \infty$ limit)

- Spin-orbit & spin-spin interactions

$$S_{\rm BH/NS} = -m \int d\tau \left[\tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \, \dot{x}^\nu + i \bar{\alpha}_\mu D_\tau \alpha^\mu + \tfrac{1}{2} R_{\mu\nu\rho\sigma} \bar{\alpha}^\mu \alpha^\nu \bar{\alpha}^\rho \alpha^\sigma + C_E R_{\mu\dot{x}\nu\dot{x}} \bar{\alpha}^\mu \alpha^\nu \, \bar{\alpha} \cdot \alpha \right]$$

spin degrees of freedom — neutron star term

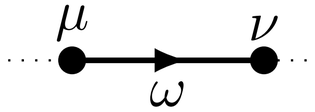
Scattering scenario:

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu \, \tau + z_i^\mu(\tau)$$
$$\alpha_i^\mu(\tau) = \alpha^\mu_{-\infty,i} + \alpha_i'^\mu(\tau)$$

Spin tensor of BHs/NSs

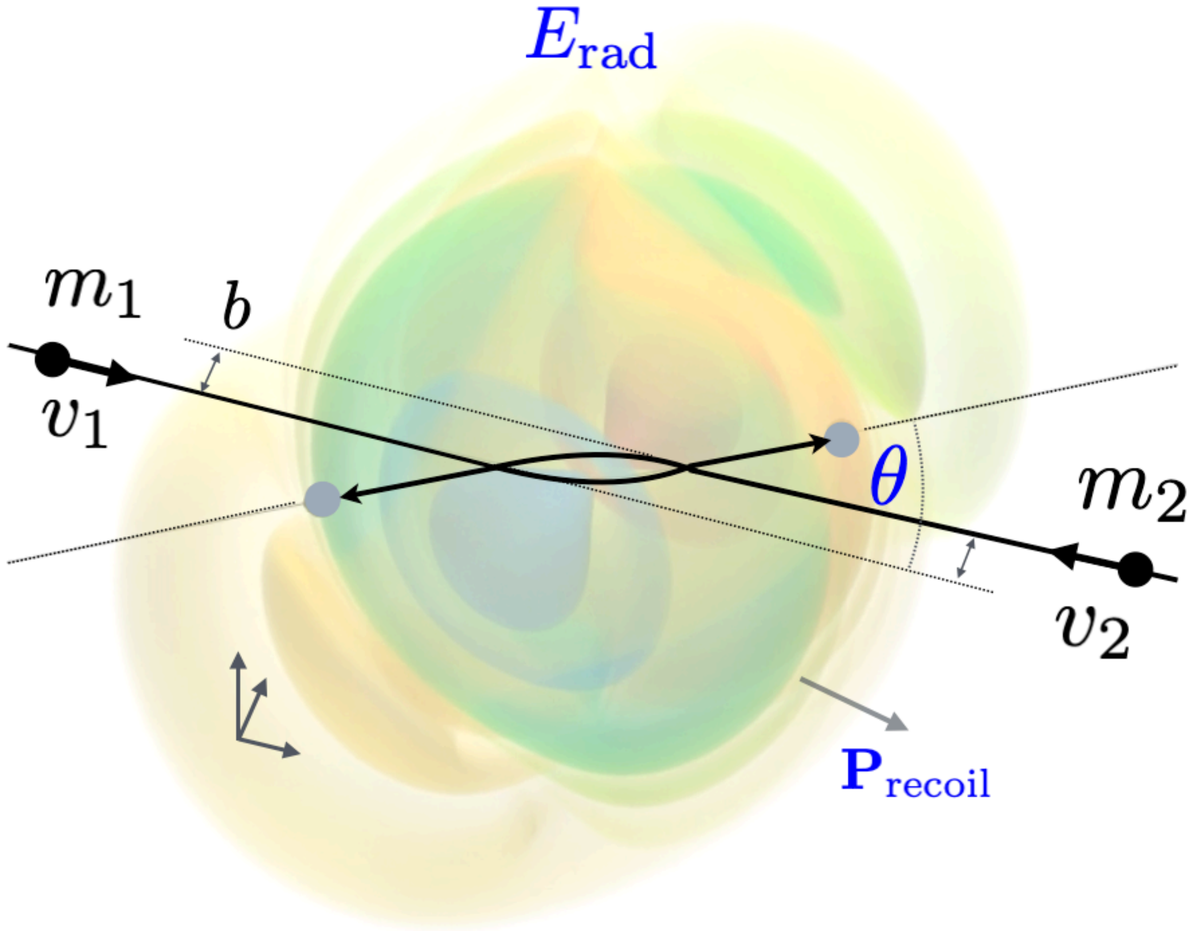
$$S_i^{\mu\nu} = 2mi \, \bar{\alpha}_i^{[\mu} \alpha_i^{\nu]}$$

Quantize $z_i^\mu \, , \alpha_i'^\mu \, , \bar{\alpha}_i'^\mu$



$$\langle \alpha^\mu(\omega) \bar{\alpha}^\nu(-\omega) \rangle = \frac{-i\eta^{\mu\nu}}{m(\omega + i0)}$$

WQFT OBSERVABLES



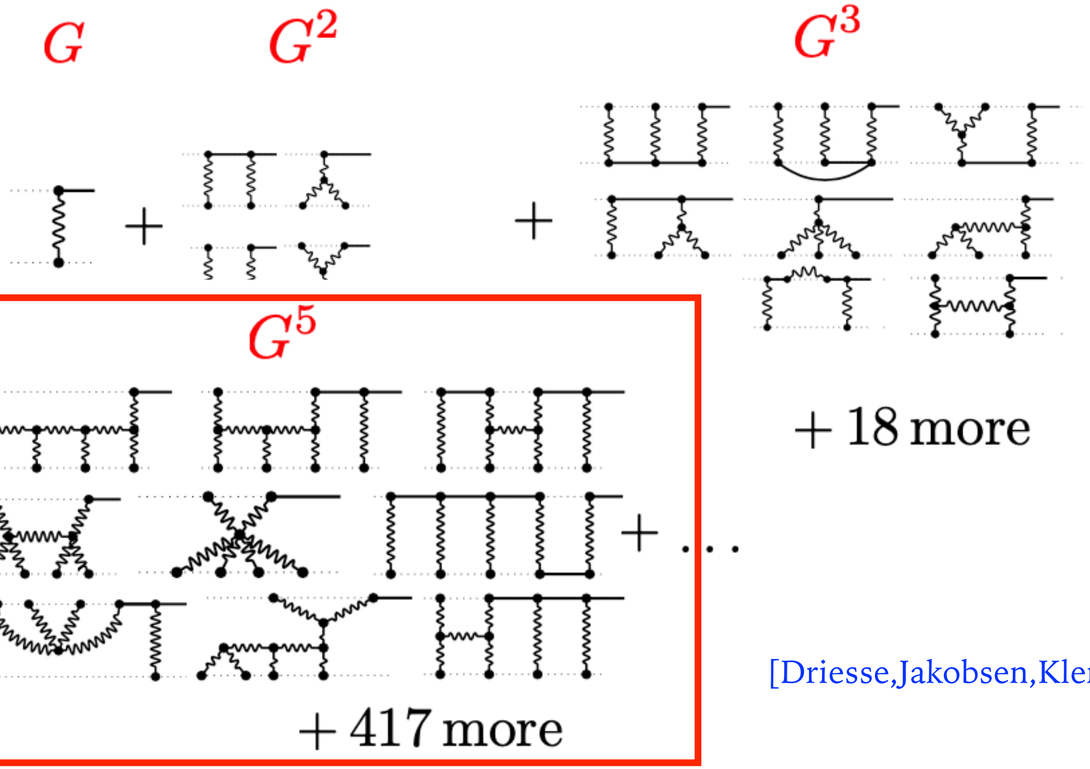
OBSERVABLES OF WQFT: ONE POINT FUNCTIONS

1) **Impulse** (change of momentum)

$$\Delta p_i^\mu = m_i \langle \dot{x}_i^\mu \rangle \Big|_{\tau=-\infty}^{\tau=+\infty} = m_i \int d\tau \langle \ddot{x}_i^\mu(\tau) \rangle = m_i \int d\tau \frac{d^2}{d\tau^2} \langle$$

Needs sum of all graphs with one outgoing z-line:

$$\Delta p_1 =$$





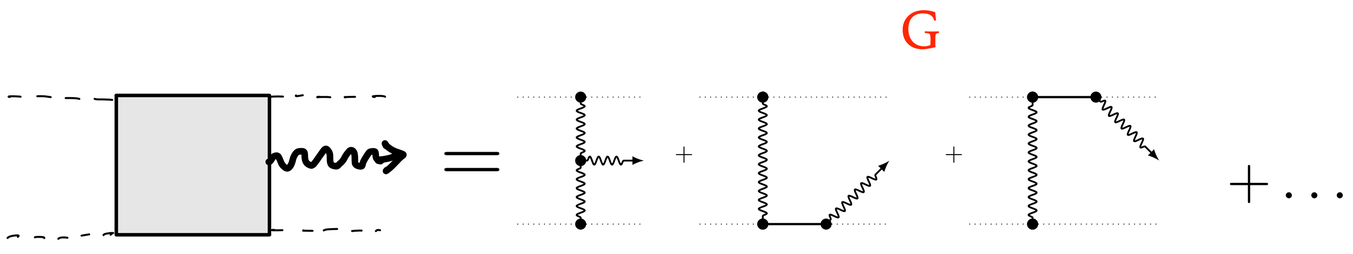
Cover Artwork: Matthias Driesse

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch, Nature 641 (2025) 603.]

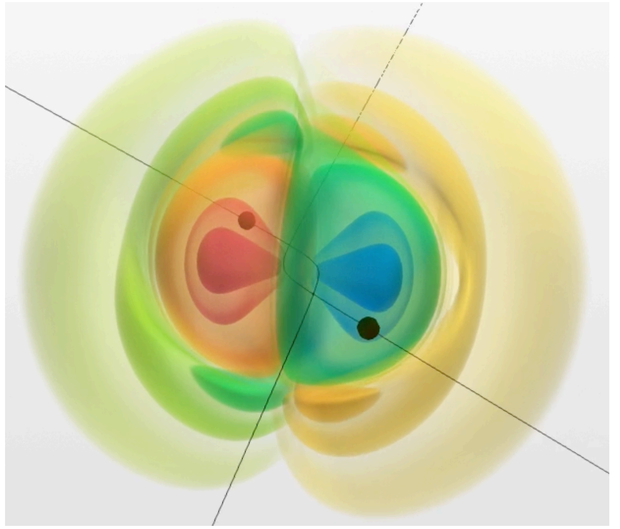
OBSERVABLES OF WQFT: ONE POINT FUNCTIONS

2) **Emitted Waveform** (Gravitational Bremsstrahlung)

$$\langle h_{\mu\nu} \rangle =$$



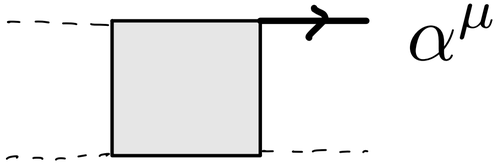
$$+ \ldots$$



3) **Spin-kick:** Change in spin-tensor from $\langle \Delta\psi_1^\mu \rangle$

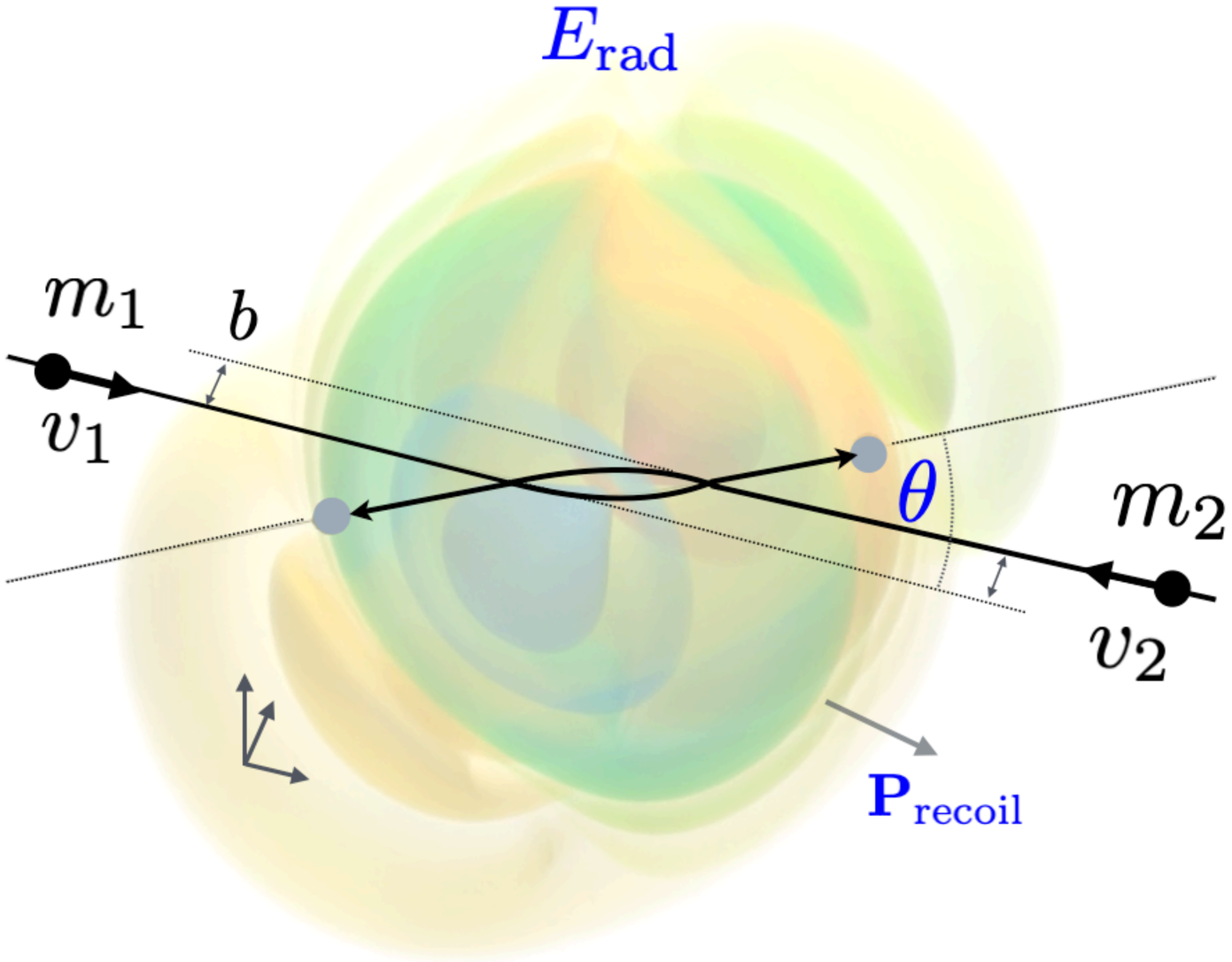
$$S_1^{\mu\nu} = -2i\bar{\alpha}_1^{[\mu}\alpha_1^{\nu]} \Rightarrow \Delta S_1^{\mu\nu} = -2i(\Delta\bar{\alpha}_1^{[\mu}\alpha^{\nu]} + \bar{\alpha}_1^{[\mu}\Delta\alpha^{\nu]} + \Delta\bar{\alpha}_1^{[\mu}\Delta\alpha^{\nu]})$$

$$\langle \Delta\alpha_1^\mu \rangle =$$



Spin-vector: $$S_1^\mu = \frac{1}{2}\epsilon^\mu{}_{\nu\rho\sigma} p_1^\nu S_1^{\rho\sigma}$$

IMPULSE

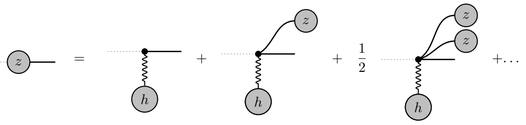


HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$ Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$ Fix boundary conditions in static limit (PN) $\gamma \to 1$

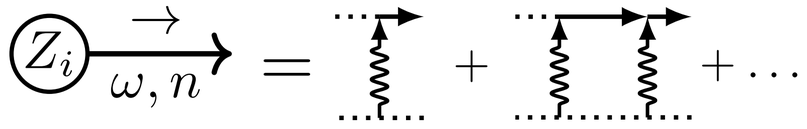
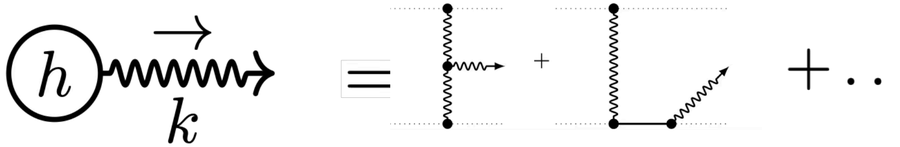
Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

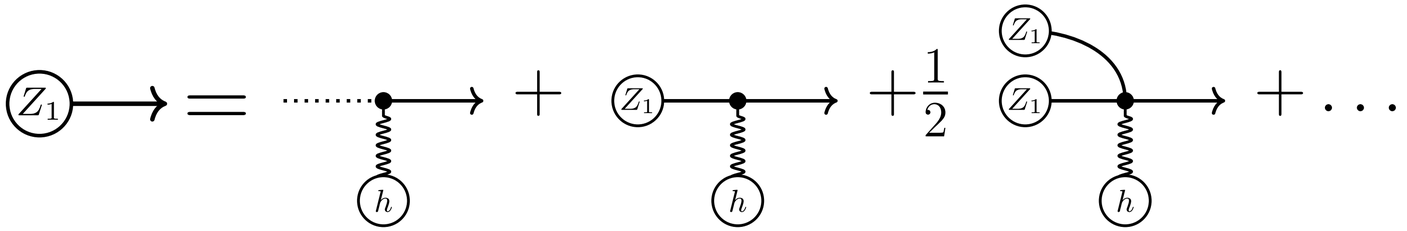
$$\Delta S_i^\mu$$

Generate graphs &
Insert Feynman rules

[Jakobsen,Mogull,JP,Sauer]

$\langle Z_i(\omega)\rangle =$  $\langle h_{\mu\nu}(k)\rangle =$ 

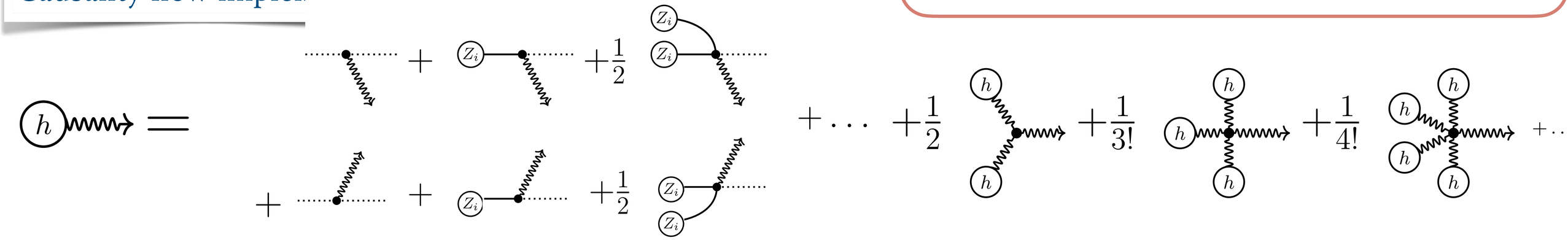
- All graphs are **trees**: Use **Berends-Giele recursion** to efficiently generate all contributions.



$$\langle \ddot{z}^\sigma \rangle = -\langle \Gamma^\sigma_{\mu\nu} \dot{z}^\mu \dot{z}^\nu \rangle = -\langle \Gamma^\sigma_{\mu\nu} \rangle \langle \dot{z}^\mu \rangle \langle \dot{z}^\nu \rangle$$

$$\langle R^{\mu\nu} \rangle - \tfrac{1}{2} \langle g^{\mu\nu} \rangle \langle R \rangle = 8\pi G_N \langle T^{\mu\nu} \rangle$$

Causality flow implemented

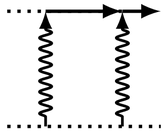
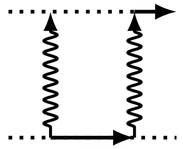
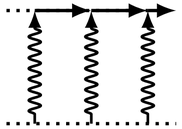
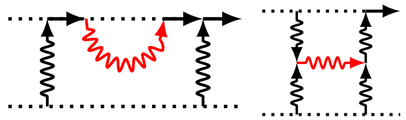
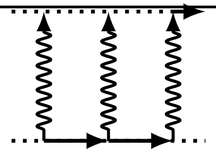
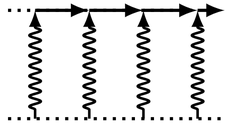
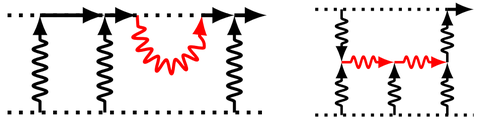
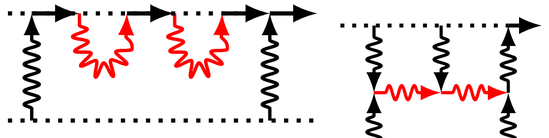
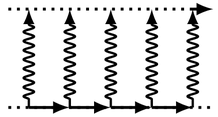
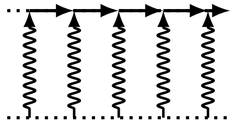
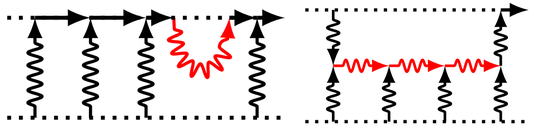
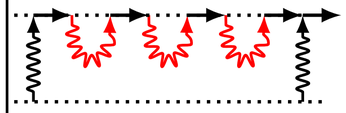
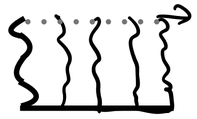


- Insertion of Feynman rules with **FORM** - fast on tensor algebra.

PM EXPANSION: LOOP AND SELF-FORCE (SF) ORDERS

Generate graphs &

Insert Feynman rules

	$m_1 m_2^{L+1}$	$m_1^2 m_2^L$	$m_1^3 m_2^{L-1}$	$m_1^4 m_2^{L-2}$	$m_1^5 m_2^{L-3}$
1PM tree-level	 0SF				
2PM 1-loop	 0SF	 $\overline{\text{0SF}}$ $m_1^2 m_2$			
3PM 2-loop	 0SF	 1SF	 $\overline{\text{0SF}}$ $m_1^3 m_2$		
4PM 3-loop	 0SF	 1SF	 $\overline{\text{1SF}}$ $m_1^3 m_2^2$	 $\overline{\text{0SF}}$ $m_1^4 m_2$	
5PM 4-loop	 0SF	 1SF	 ?	 $\overline{\text{1SF}}$ $m_1^4 m_2$	 $\overline{\text{0SF}}$ $m_1^5 m_2$

5PM MOMENTUM IMPULSE

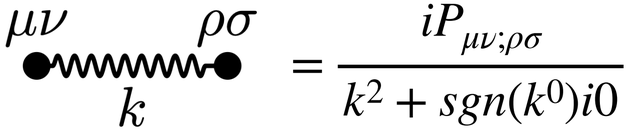
Generate graphs &
Insert Feynman rules

$$\Delta p_i^{(5)\mu} = m_1 m_2 \Big(m_1^4 \Delta p_{0SF}^{(5)\mu} + m_1^3 m_2 \Delta p_{1SF}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{2SF}^{(5)\mu} + m_1 m_2^3 \Delta \bar{p}_{1SF}^{(5)\mu} + m_2^4 \Delta \bar{p}_{0SF}^{(5)\mu} \Big)$$

0SF: 63 diagrams

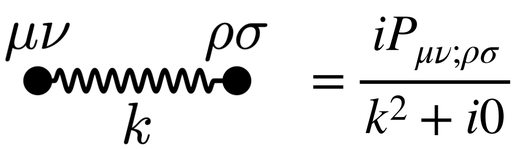
1SF: 363 diagrams

Radiative vs. conservative sectors:



$$= \frac{iP_{\mu\nu;\rho\sigma}}{k^2 + sgn(k^0) i0}$$

$$\Delta p_1 = \Delta p_{\rm diss} + \Delta p_{\rm cons}$$



$$= \frac{iP_{\mu\nu;\rho\sigma}}{k^2 + i0}$$

Disspative:

- retarded graviton propagators
- physical scattering!

$$\Delta p_1^\mu + \Delta p_2^\mu = P_{\rm rad}^\mu$$

Conservative:

- Feynman graviton propagator
- Definition (not unique)

$$\Delta p_1^\mu + \Delta p_2^\mu = 0$$

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch, Nature 641 (2025) 603.]

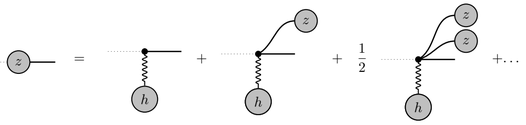
[Driesse,Jakobsen,Klemm,JP,Sauer,Usovitsch, PRL 132 (2024) 24]

HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$ Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$ Fix boundary conditions in static limit (PN) $\gamma \to 1$

Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

$$\Delta S_i^\mu$$

Reduction & planarization to scalar integral family

STRUCTURE OF WQFT INTEGRALS: IMPULSE & SPIN KICK

[Jakobsen,Mogull,JP,Sauer]

- Order **N-PM** : Single scale **(N-1)-loop** integrals:

$$I_{\rm nPM} = \int_q e^{-q\cdot b}\delta(q\cdot v_1)\delta(q\cdot v_2)\int_{l_1,l_2\ldots l_{n-1}} \frac{\text{num}[l_i]}{D_1^{a_1}\ldots D_j^{a_j}}\delta(l_1\cdot v_*)\delta(l_2\cdot v_*)\ldots\delta(l_{n-1}\cdot v_*)$$

Propagators $D_i(l_i,q,v_*)$ are linear or quadratic $v_*\in\{v_1,v_2\}$ (*'s determined by self-force order)

$$D_i(l_i,q,v_*) = \begin{cases} l_i\cdot v_* + i0 \\ (l_i^0+i0)^2 - \vec{l}_i^{\,2} \end{cases}$$

← For conservative sector use Feynman

Depend on single scale q —> left with **single parameter** integral $\gamma = v_1\cdot v_2 = \frac{1}{2}(x+x^{-1})$

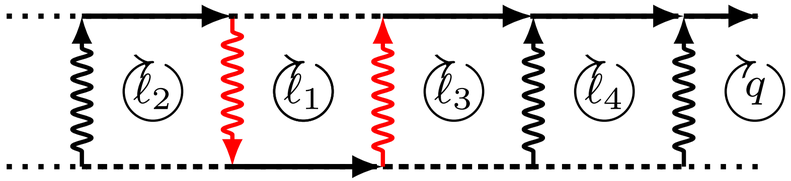
- Passarino-Veltman tensor reduction yields scalar integrals (spin-less case)

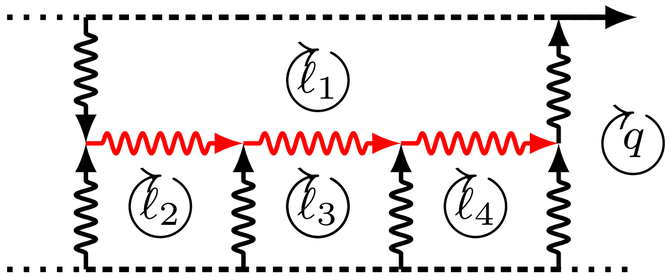
$$\ell_i^\mu \to \ell_i\cdot v_1\,\hat{v}_1^\mu + \ell_i\cdot v_2\,\hat{v}_2^\mu + \frac{\ell_i\cdot q}{q^2}q^\mu \qquad v_i\cdot\hat{v}_j = \delta_{ij}$$

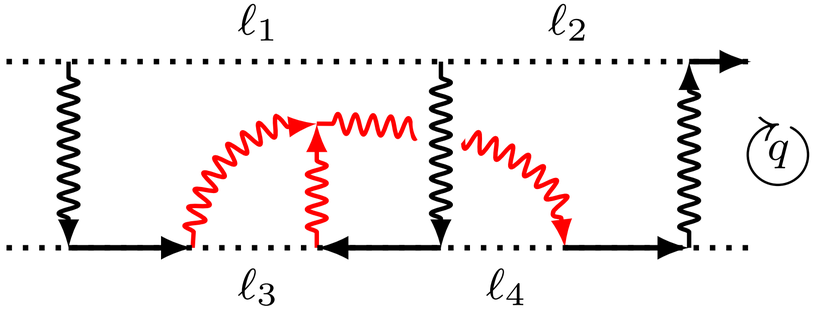
Reduction & planarization to scalar integral family

WQFT LOOP INTEGRALS:

Four loop example diagrams:







$$\xrightarrow{k^\mu} \sim\sim\sim\sim\sim\sim\rightarrow = \frac{1}{(k_0+i\epsilon)^2-\boldsymbol{k}^2}$$

$$\xrightarrow{k^\mu} \longrightarrow = \frac{1}{k\cdot v_j+i\epsilon}$$

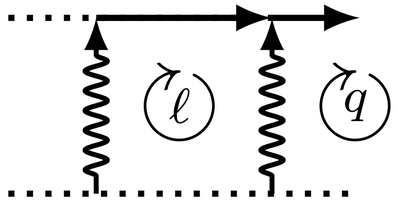
$$\xrightarrow{k^\mu} \cdots\cdots\cdots = \delta(k\cdot v_j)$$

External Kinematics:

$$v_1\cdot v_1 = v_2\cdot v_2 = 1$$

$$v_1\cdot q = v_2\cdot q = 0$$

$$v_1\cdot v_2 = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$= \int d^D\ell \frac{\delta(\ell\cdot v_2)}{\ell^2(q-\ell)^2\ell\cdot v_1}$$

$$= \int d^D\ell \frac{\delta(\ell\cdot v_2)}{((\ell_0+i\epsilon)^2-\boldsymbol{\ell}^2)((q_0-\ell_0+i\epsilon)-(\boldsymbol{q}-\boldsymbol{\ell})^2)(\ell\cdot v_1+i\epsilon)}$$

(Regulator: Dimensional Regularization)

FIRST SELF-FORCE ORDER INTEGRALS

Reduction & planarization to scalar integral family

- Conjecture **all loop** 1SF integral family:

$$I_{\{n_A\}} = \int_{\ell_1 \ldots \ell_L} \frac{\delta(\ell_1 \cdot v_1) \prod_{i=2}^{L} \delta(\ell_i \cdot v_2)}{\prod_A D_A^{n_A}}$$

$$A \in \{iv, ij, iq, i0\}$$

$$D_{iv} = \begin{cases} \ell_1 \cdot v_2 & i = 1 \\ \ell_i \cdot v_1 & i = 2, 3, \ldots, L \end{cases}$$

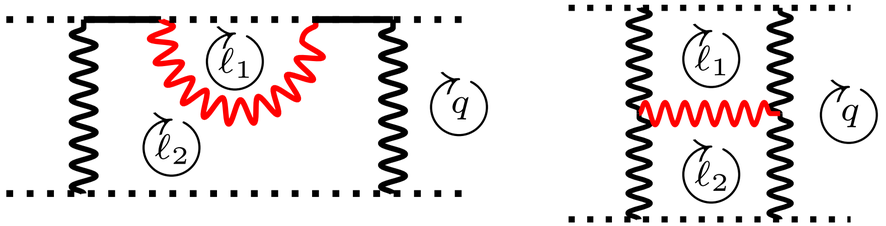
$$D_{ij} = (\ell_i - \ell_j)^2$$

$$D_{iq} = (\ell_i - q)^2$$

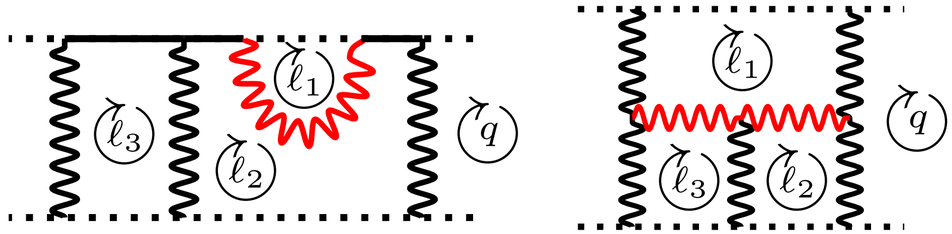
$$D_{i0} = \ell_i^2$$

[Driesse,Jakobsen,Mogull,JP,Sauer,Usovitsch]

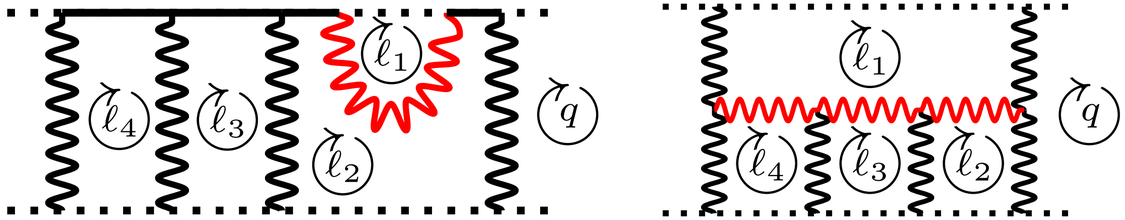
$L = 2$



$L = 3$



$L = 4$



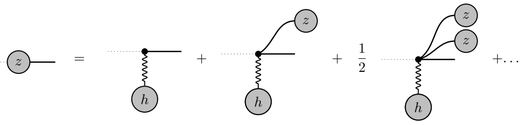
At L-loop order have L linear and L(L+3)/2 quadratic propagators & L delta-functions.

HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$

Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$

Fix boundary conditions in static limit (PN) $\gamma \to 1$

Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

$$\Delta S_i^\mu$$

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Integration by Parts (IBP) Relations: Yields vector space picture for Feynman integral family

$$0 = \int \prod_{i=1}^{4} \frac{d^D \ell_i}{(2\pi)^D} \frac{\partial}{\partial \ell_f^\mu} \left(\Lambda^\mu \, \frac{\delta(v_1 \cdot \ell_1)\,\delta(v_1 \cdot \ell_2)\,\delta(v_2 \cdot \ell_3)\,\delta(v_2 \cdot \ell_4)}{(v_2 \cdot \ell_1)^{a_1} (v_2 \cdot \ell_2)^{a_2} (v_1 \cdot \ell_3)^{a_3} (v_1 \cdot \ell_4)^{a_4} D_5^{a_5} \cdots D_J^{a_J}} \right)$$

Suitable choices of $\{\Lambda^\mu, a_1, \ldots, a_J\}$ yield huge sets of linear equations for $I_{a_1,a_2,\ldots,a_J}$ that can be expressed in basis of Master Integrals -> **Kira 3.0** [Lange,Usovitsch,Wu]

Computational bottleneck of project.

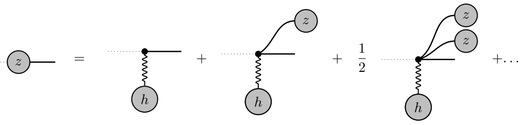
- Reduced 10^6 scalar integrals $I_{a_1,a_2,\ldots a_J}(x,\epsilon)$ to basis of **468 Master Integrals** $\vec{I}(x,\epsilon)$
- Crucial use of Integration by parts code **KIRA 3.0**
- Reconstruct rational functions from finite-field samples with **FireFly**
- Complete reduction took ~300k core hours on HPC cluster
- Masters split into 236+232 under parity ($v \to -v$)

HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$

Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$

Fix boundary conditions in static limit (PN) $\gamma \to 1$

Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

$$\Delta S_i^\mu$$

Solve masters by Diff. Eq. technique

METHOD OF DIFFERENTIAL EQUATIONS

[Kotikov][Gehrmann,Remiddi][Henn]

- Still needs to integrate the masters integrals: $\vec{I}(x,\epsilon)$

$$\frac{1}{2}(x+x^{-1}) = 1/\sqrt{1-v^2}$$

- Depend on x and dimensionsal regulator $D = 4 - 2\epsilon$

- Similarity transform to canonical Differential equations:

$$\frac{d}{dx}\vec{I}(x,\epsilon) = M(x,\epsilon)\,\vec{I}(x,\epsilon) \xrightarrow{\vec{I}(x,\epsilon) = T(\epsilon,x)\,\vec{I}'(x,\epsilon)} \frac{d}{dx}\vec{I}'(x,\epsilon) = \epsilon\, A(x)\,\vec{I}'(x,\epsilon)$$

Resulting solution:

$$\vec{I}'(x,\epsilon) = P\exp[\epsilon\int_C dA(x)]\,\vec{I}'_0(x_0,\epsilon)$$

boundary integrals

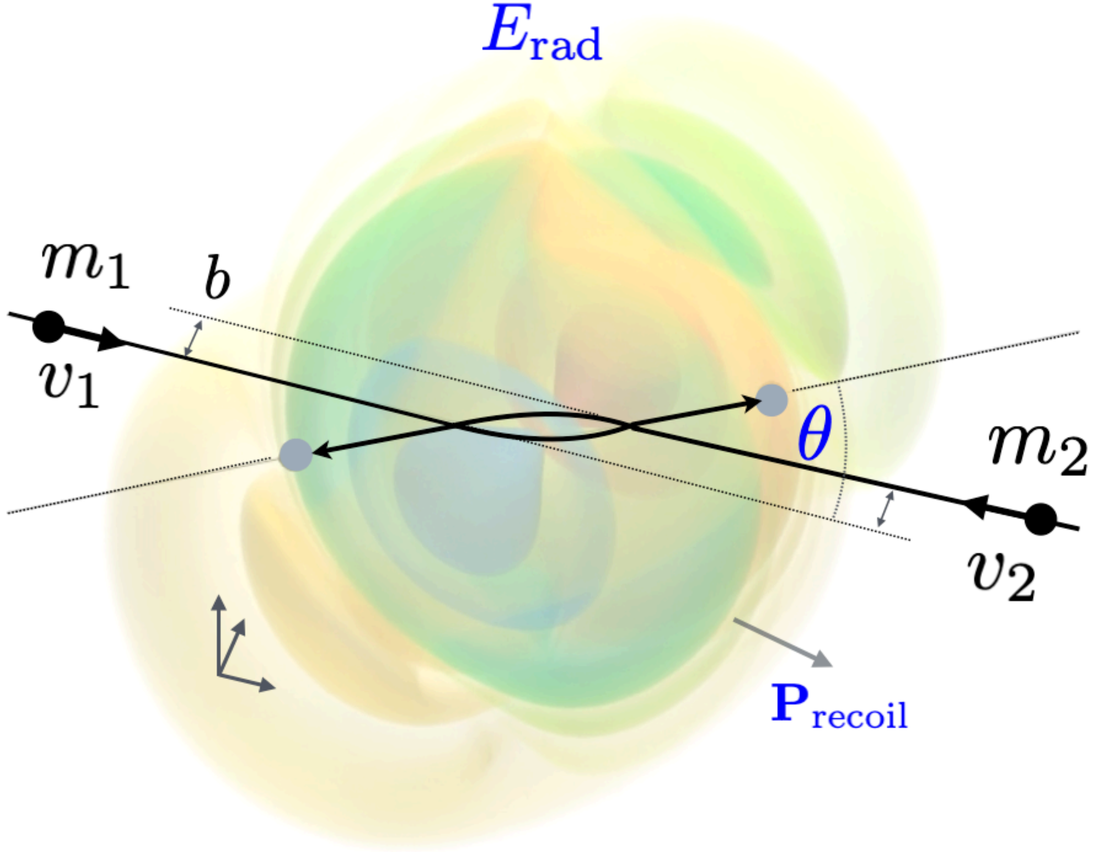
$$x_0 = 1 \Leftrightarrow v = 0$$

Solve masters by Diff. Eq. technique

PARITY OF INTEGRALS

[Driesse,Jakobsen,Mogull,JP,Sauer,Usovitsch]

- Split of impulse under parity ($v \to -v$)



$$\Delta p_1^\mu = c_b b^\mu + c_{v_1} v_1^\mu + c_{v_2} v_2^\mu$$

- Even:
  - Scattering angle θ
  - Integrals purely **real, even** number of linear propagators
  - **Conservative** & **dissipative** effects
- Odd:
  - Radiated energy $E_{\rm rad}$
  - Integrals purely **imaginary, odd** number of linear propagators
  - **Dissipative** effects & constrained by lower order

- Parity splits Master Integrals into 232 even & 234 odd members

Solve masters by Diff. Eq. technique

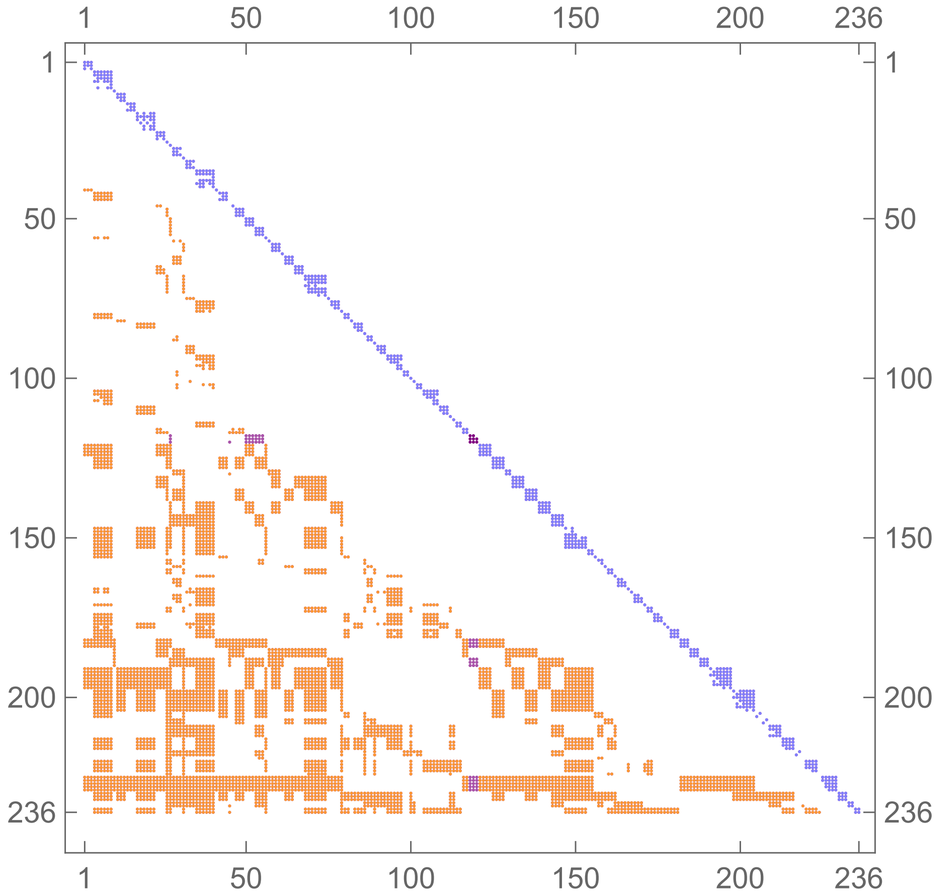
DIFFERENTIAL EQS @ 5PM-1SF

[Driesse,Jakobsen,Mogull,JP,Sauer,Usovitsch]

- After IBP reduction: Basis of Master Integrals $\vec{I}(x,\epsilon)$ for 5PM-1SF planar family

$$\frac{d}{dx}\vec{I} = M(x,\epsilon)\,\vec{I}$$

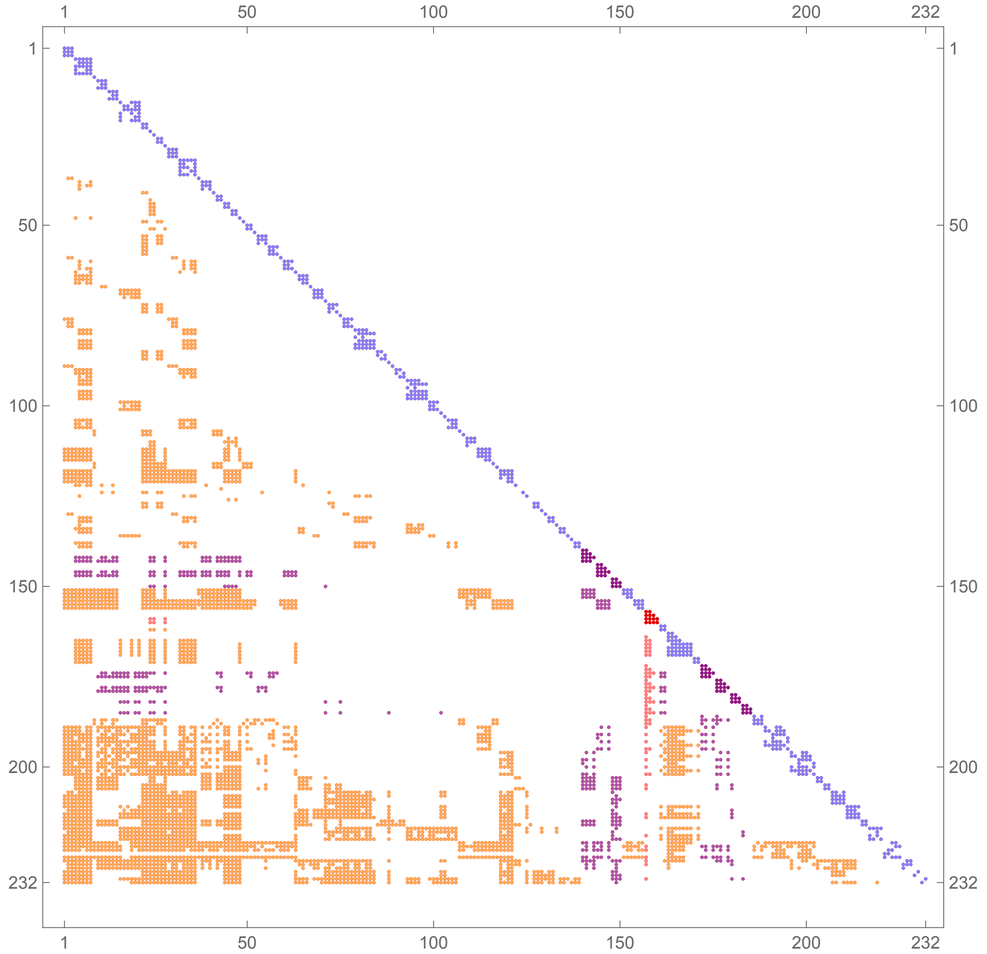
$$x = \gamma - \sqrt{\gamma^2-1} \qquad \gamma = \frac{1}{2}(x+x^{-1})$$



even

Heat plots of matrix

$$M(x,\epsilon)$$



odd

FUNCTION SPACES OF FEYNMAN INTEGRALS

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Solve masters by Diff. Eq. technique

- **Multiple Polylogarithms**: Iterated integrals of kernels $(t-a)^{-1}$

$$G(a_1,\ldots,a_n;y)=\int_0^y \frac{\mathrm{d}t}{t-a_1}G(a_2,\ldots,a_n;t) \qquad \text{with} \quad G(\vec{0}_n,y)=\log^n(y)/n!$$

- **Elliptic Integrals**: $K(y), E(y)$ Periods of torus: Integration of 1-form along cycles

  Iterated integral with kernels $E(y)$ & $K(y)$

- **Calabi-Yau periods**: CY n-fold: Complex (2n dimensional), closed, Ricci flat Kähler manifold.

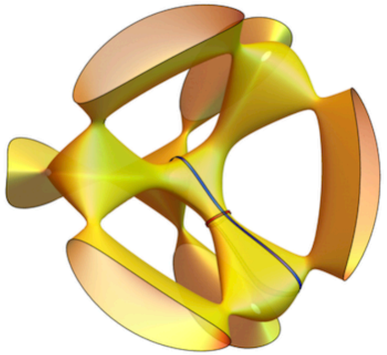
Period: Integration of holomorphic n-form ω along cycle C:

$$\varpi(x)=\int_C \omega$$

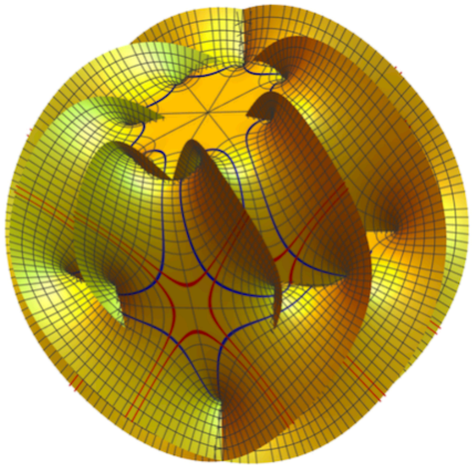
x: „shape“ of CY



Torus $(n=1)$



K3 $(n=2)$



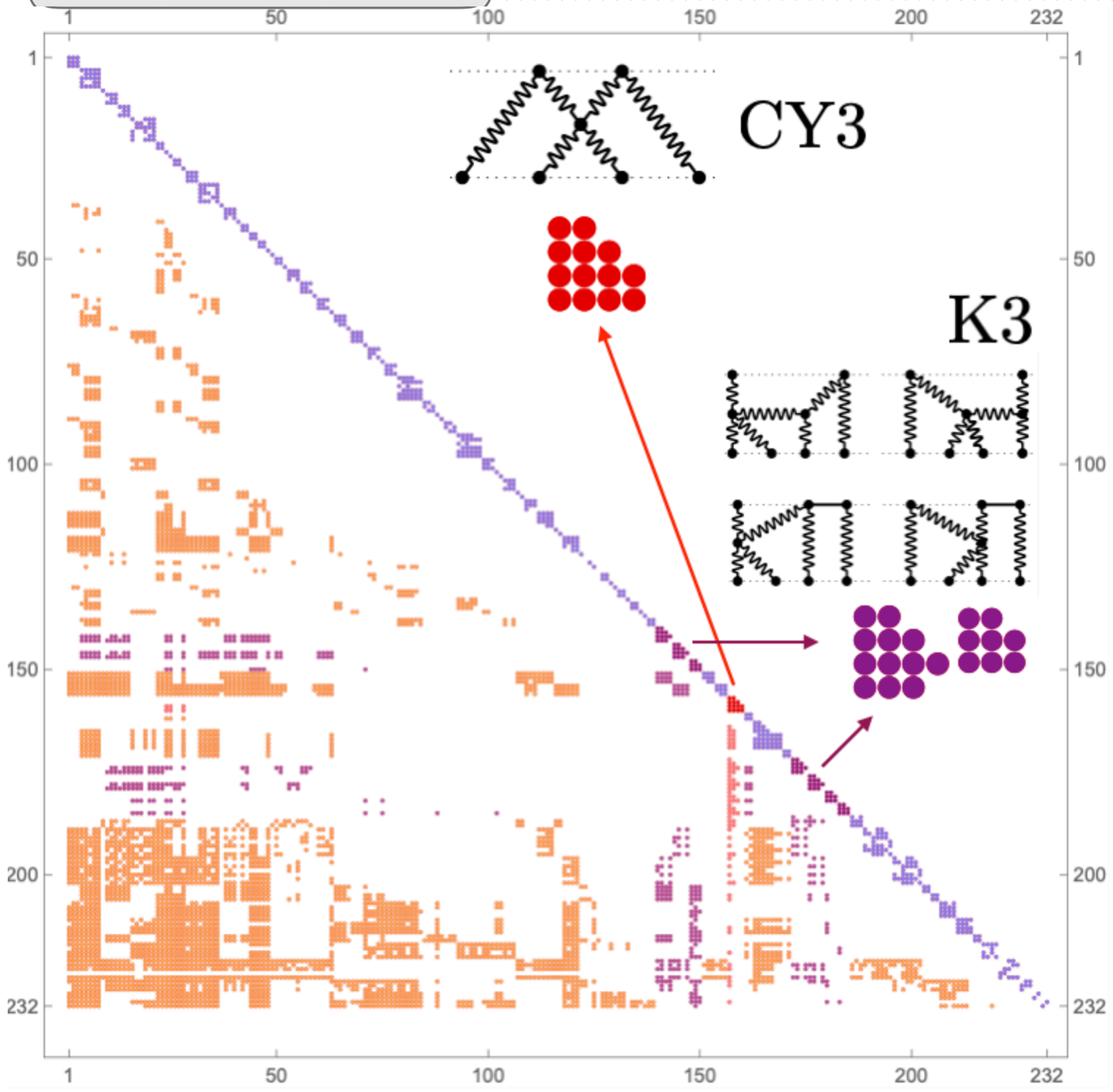
CY3 $(n=3)$

Solve masters by Diff. Eq. technique

CALABI-YAU IN THE SKY

[Klemm,Nega,Sauer,JP,Sauer]

[Driesse,Jakobsen,Klemm,Mogull,Nega, JP,Sauer,Usovitsch]



- CY3:

4x4 subsector represented as MI and its 3 derivatives yields DE

$$z = x^4$$

$$\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^4 - z\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\right)^4\right]\varpi(z) = 0$$

underlying CY3 manifold:

$$y_i^2 + w_j^2 - z^8 y_{j+1} w_{j+1} = 0 \quad j = 0, 1, 2, 3$$

- K3s:

$$z = x^2$$

$$Y^2 = X(X-1)(X-z)Z(Z-1)(Z-1/z)$$

Exhibits $x \to x^{-1}$ symmetry

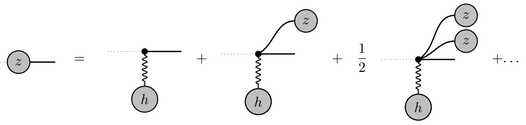
Solution of DE: $K(1-x)^2$

HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$

Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$

Fix boundary conditions in static limit (PN) $\gamma \to 1$

Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

$$\Delta S_i^\mu$$

Fix boundary conditions in static limit (PN)

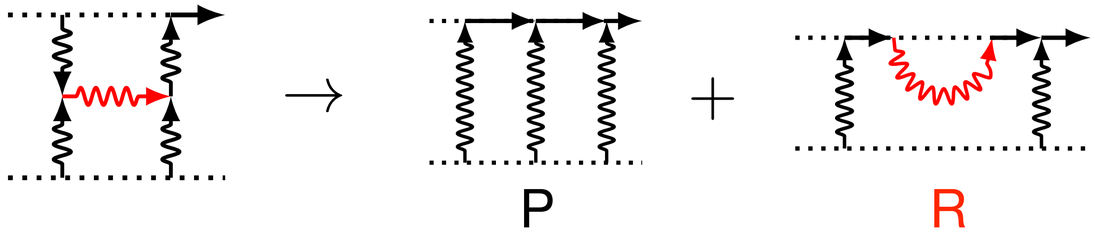
Compute boundary integrals $\vec{I}'_0(x,\epsilon)$ by **Method of Regions** [Beneke,Smirnov] in static limit $v \to 0$

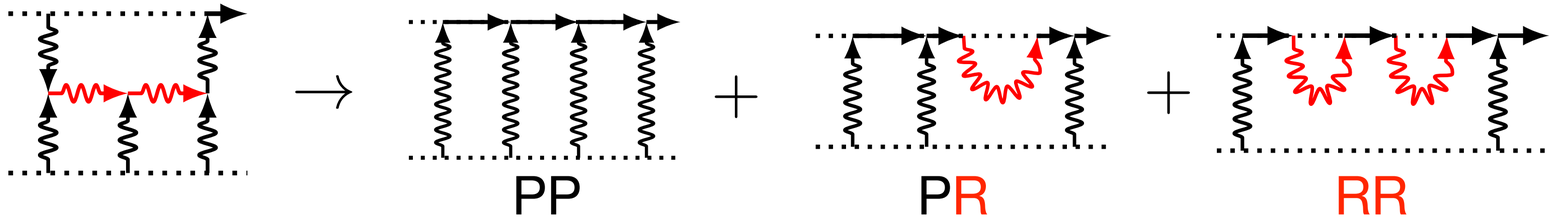
- Regions determined by scalings graviton momenta:

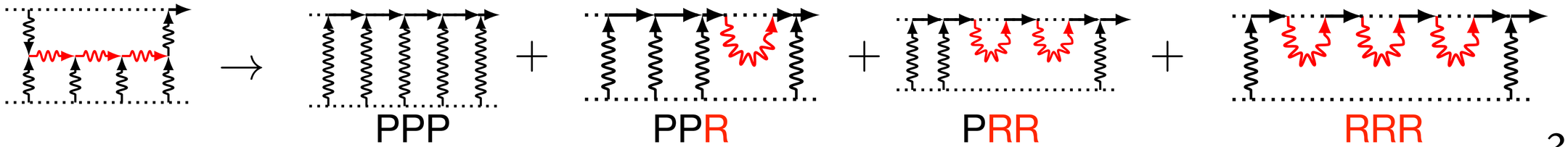
$$\ell_i^{\rm P} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v,1)\,, \qquad \ell_i^{\rm R} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v,v)\,.$$

potential (P) radiation (R)

- **Active:** Propagator can go on-shell, $i\epsilon$ prescription matters:

2 loops 

3 loops 

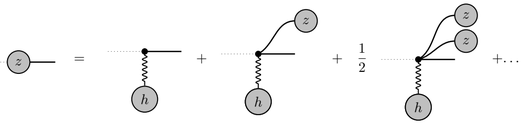
4 loops 

HIGH PRECISION WQFT COMPUTATIONS: WORKFLOW

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch]

Berends-Giele recursion

Generate graphs & Insert Feynman rules



(5PM: 6g vertex)

Reduction & planarization to scalar integral family

$$\int \prod_i d^D l_i \frac{l_{i_1}^{\mu_1} \dots l_{i_n}^{\mu_n}}{\prod_i D(l_i)}$$

$i0^+$

Integration by parts & Master Integrals

Kira 3.0

Reduced symmetries due to retarded prop.

Solve masters by Diff. Eq. technique

Single scale integrals

$$x = \gamma - \sqrt{\gamma^2 - 1}$$

$$\gamma = v_1 \cdot v_2$$

$i0^+$

Fix boundary conditions in static limit (PN) $\gamma \to 1$

Method of regions (potential & radiation gravitons)

$$\Delta p_i^\mu \quad f_{\mu\nu}(u, \hat{x})$$

$$\Delta S_i^\mu$$

RESULTS: SCATTERING ANGLE@ 1PM, 2PM & 3PM PRECISION:

Δp_i^μ

- Scattering angle in center of mass frame:

$$p_1^\mu = (E_1, \boldsymbol{p}) \qquad p_2^\mu = (E_2, -\boldsymbol{p})$$

$$\cos\theta = \frac{\boldsymbol{p}\cdot(\boldsymbol{p}+\Delta\boldsymbol{p})}{|\boldsymbol{p}||\boldsymbol{p}+\Delta\boldsymbol{p}|}$$

Including recoil: $\Delta\boldsymbol{p} = \Delta\boldsymbol{p}_1 + E_1\boldsymbol{P}_{\rm rad}/E$

$$\frac{\theta}{\Gamma} = \frac{GM}{|b|}\frac{2(2\gamma^2-1)}{\gamma^2-1} + \left(\frac{GM}{|b|}\right)^2\frac{3\pi(5\gamma^2-1)}{4(\gamma^2-1)}$$

1PM 2PM

$$\gamma = v_1\cdot v_2$$

$$\Gamma = E/M = \sqrt{1+2\nu(\gamma-1)}$$

$$\nu = \frac{m_1 m_2}{M^2}$$

$$+\left(\frac{GM}{|b|}\right)^3\left(2\frac{64\gamma^6-120\gamma^4+60\gamma^2-5}{3(\gamma^2-1)^3}\Gamma^2 - \frac{8\nu\gamma(14\gamma^2+25)}{3(\gamma^2-1)} - 8\nu\frac{(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\frac{\operatorname{arccosh}\gamma}{\sqrt{\gamma^2-1}}\right)$$

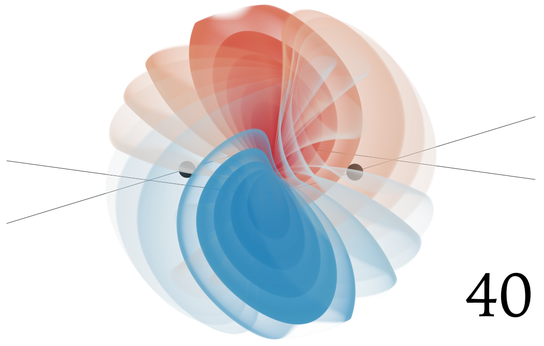
3PM conservative

[Bern,Cheung,Roiban,Shen, Solon,Zeng][Kälin, Liu, Porto][Di Vecchia, Heissenberg, Russo,Veneziano]
[Bjerrum-Bohr,Vanhove,Damgaard][Brandhuber,Chen,Travaglini,Wen][Jakobsen,Mogull,JP,Sauer]

$$+\left(\frac{GM}{|b|}\right)^3\frac{4\nu(2\gamma^2-1)^2}{(\gamma^2-1)^{3/2}}\left(-\frac{8}{3}+\frac{1}{v^2}+\frac{(3v^2-1)}{v^3}\operatorname{arccosh}\gamma\right)$$

1SF piece

3PM disspative



RESULT: SCATTERING ANGLE@ 4PM PRECISION:

$$\frac{\theta}{\Gamma} = \quad \cdots \quad + \left(\frac{GM}{|b|}\right)^4 \left(\frac{105\pi\left(33\gamma^4-18\gamma^2+1\right)}{64\left(\gamma^2-1\right)^2}\right.$$

4PM conservative

[Bern,Roiban,Shen,Parra-Martinez,Ruf,Zeng..]
[Dlapa, Kälin, Liu, Neef,Porto]

$$+\nu\Bigg[-\frac{3h_3\mathrm{K}^2\left(\frac{\gamma-1}{\gamma+1}\right)}{32\left(\gamma^2-1\right)^2}+\frac{3h_4\mathrm{E}\left(\frac{\gamma-1}{\gamma+1}\right)\mathrm{K}\left(\frac{\gamma-1}{\gamma+1}\right)}{32\left(\gamma^2-1\right)^2}+\frac{\pi^2h_5}{16(1-\gamma^2)}+\frac{3h_{27}\log^2\left(\frac{\gamma+1}{2}\right)}{4(1-\gamma^2)}-\frac{h_6\log\left(\frac{\gamma-1}{2}\right)}{32\left(\gamma^2-1\right)^2}+\frac{3h_{15}\log\left(\frac{\gamma-1}{2}\right)\log\left(\frac{\gamma+1}{2}\right)}{16\left(\gamma^2-1\right)}$$

$$-\frac{h_{22}\log\left(\frac{\gamma+1}{2}\right)}{32\left(\gamma^2-1\right)^2}-\frac{h_{23}\log(\gamma)}{4\left(\gamma^2-1\right)^2}+\frac{3h_{26}\operatorname{arccosh}^2(\gamma)}{64\left(\gamma^2-1\right)^4}+\frac{h_{24}\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{7/2}}-\frac{3h_{16}\log\left(\frac{\gamma-1}{2}\right)\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{5/2}}-\frac{3h_{28}\log\left(\frac{\gamma+1}{2}\right)\operatorname{arccosh}(\gamma)}{32\left(\gamma^2-1\right)^{5/2}}$$

$$-\frac{h_{62}}{384\gamma^7\left(\gamma^2-1\right)^3}-\frac{21h_2\mathrm{E}^2\left(\frac{\gamma-1}{\gamma+1}\right)}{64(\gamma-1)^2(\gamma+1)}-\frac{3\sqrt{\gamma^2-1}h_7\mathrm{Li}_2\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)}{2(\gamma-1)^2(\gamma+1)^3}+\frac{h_{29}\mathrm{Li}_2\left(\frac{1-\gamma}{\gamma+1}\right)}{8(1-\gamma^2)}+\left(\frac{3\sqrt{\gamma^2-1}h_7}{8(\gamma-1)^2(\gamma+1)^3}+\frac{3h_{30}}{16-16\gamma^2}\right)\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\Bigg]$$

$$+\left(\frac{GM}{|b|}\right)^4\left(\ldots+\nu\Big[\ldots\Big]\right)$$

4PM disspative

[Dlapa, Kälin, Liu, Neef,Porto]
[Damgaard,Plante,Vanhove]
[Jakobsen,Mogull,JP,Sauer]

- Function space: Logs, Dilogs & Elliptic Integrals of 1st & 2nd kind: E(x) & K(x)

$$h_3 = 1200\gamma^2+2095\gamma+834$$
$$h_4 = 1200\gamma^3+2660\gamma^2+2929\gamma+1183$$
$$h_5 = -25\gamma^6+30\gamma^4+60\gamma^3-129\gamma^2+76\gamma-12$$
$$h_6 = 210\gamma^6-552\gamma^5+339\gamma^4-912\gamma^3+3148\gamma^2-3336\gamma+1151$$
$$h_7 = -\gamma\left(2\gamma^2-3\right)\left(15\gamma^2-15\gamma+4\right)$$
$$\vdots$$

RESULT: SCATTERING ANGLE@ 5PM-1SF PRECISION:

[Driesse,Jakobsen,Klemm,Mogull,Nega,JP,Sauer,Usovitsch, Nature 641 (2025) 603.]

$$\frac{\theta}{\Gamma} = \cdots + \left(\frac{GM}{|b|}\right)^5 \Bigg(\underbrace{\frac{2\left(1792\gamma^{10} - 5760\gamma^8 + 6720\gamma^6 - 3360\gamma^4 + 630\gamma^2 - 21\right)}{5\left(\gamma^2-1\right)^5}}_{\text{0SF}} \underbrace{+\nu\,\theta_{5,1}}_{\text{1SF}} \underbrace{+\nu^2\theta_{5,2}}_{\text{2SF}} + \underbrace{\frac{\nu^3}{\Gamma^2}\,\theta_{5,3}}_{\text{from 1SF}} \Bigg)$$

- 5PM-1SF function space: **Multiple polylogarithms** up to weight n=3!

$$G(a_1,\ldots,a_n;1-x) = \int_0^{1-x} \frac{dy}{y'-a_1} G(a_2,\ldots,a_n;y)$$

$$\gamma = (x + x^{-1})/2$$

relativistic γ-factor

Alphabet: $a_i \in \{0, 1, 2, 1 \pm i\}$

$$\nu = \frac{m_1 m_2}{M^2}$$

Symmetric mass ratio

- General structure:

5PM-1SF conservative + dissipative

$$\theta_{5,1} = \sum_k c_k(\gamma)\, f_k(\gamma)$$

Polynomial in γ

Functions of Multiple Polylogs

$G(a_1,\ldots,a_j;1-x) \quad j = 1,2,3$

RESULT: SCATTERING ANGLE@ 5PM-1SF PRECISION:

[Driesse,Jakobsen,Klemm, Mogull, Nega,JP,Sauer,Usovitsch]

theta[5, 1]

$$\frac{\pi^2\left(282+4983\gamma-17040\gamma^2+15211\gamma^3+4116\gamma^4-21355\gamma^5+16992\gamma^6+1545\gamma^7-4350\gamma^8\right)}{64\left(-1+\gamma^2\right)^{7/2}}-\frac{\pi^2\gamma\left(-7817455-1872978\gamma^2+1564672\gamma^4-90112\gamma^6+32768\gamma^8\right)}{336\left(-1+\gamma^2\right)^2}+\frac{1}{20160\gamma^8\left(-1+\gamma^2\right)^{9/2}}\left(182070+74424\gamma^2+4360020\gamma^4-454207691\gamma^6+1129907465\gamma^8+109023783\gamma^{10}-1735934159\gamma^{12}+1193573100\gamma^{14}-210635716\gamma^{16}+589824\gamma^{18}\right)+\frac{1}{7560\left(-1+\gamma\right)^4\gamma^7\left(1+\gamma\right)^5}$$

$$\left(10395+10395\gamma+393120\gamma^2+393120\gamma^3-536235\gamma^4-536235\gamma^5+16955505\gamma^6+16701489\gamma^7+62849808\gamma^8+70470288\gamma^9-404608691\gamma^{10}-445251251\gamma^{11}+616714446\gamma^{12}+697999566\gamma^{13}-296527410\gamma^{14}-366200370\gamma^{15}+32117126\gamma^{16}+53793158\gamma^{17}+331776\gamma^{18}+331776\gamma^{19}\right)-\frac{\pi^2\left(-199207-1240416\gamma^2+189180\gamma^4+81920\gamma^6\right)G[1,1-x]}{48\left(-1+\gamma^2\right)^{5/2}}+$$

$$\frac{1}{64\left(-1+\gamma^2\right)^4}3\pi^2\gamma\left(-144+279\gamma+1680\gamma^2-1791\gamma^3-6096\gamma^4+4355\gamma^5+6960\gamma^6-3765\gamma^7-2400\gamma^8+1050\gamma^9\right)G[1,1-x]-\frac{1}{2520\gamma^8\left(-1+\gamma^2\right)^{9/2}}\left(-18585+240905\gamma^2-988015\gamma^4+6510035\gamma^6-135199638\gamma^8+298351698\gamma^{10}-328443582\gamma^{12}+142748328\gamma^{14}-1122408\gamma^{16}-10389522\gamma^{18}+651264\gamma^{20}\right)G[1,1-x]+\frac{1}{20160\gamma^7\left(-1+\gamma^2\right)^5}$$

$$\left(-182070+651756\gamma^2-6189260\gamma^4-112925680\gamma^6-933951939\gamma^8+4374190165\gamma^{10}-4768470415\gamma^{12}+1163580195\gamma^{14}+523789372\gamma^{16}-169736284\gamma^{18}+2949120\gamma^{20}\right)G[1,1-x]-\frac{1}{42\gamma^8\left(-1+\gamma^2\right)^{11/2}}\left(1260-4788\gamma^2+55524\gamma^4-2047276\gamma^6+10029063\gamma^8-17437932\gamma^{10}+8628424\gamma^{12}+5754570\gamma^{14}-6704127\gamma^{16}+1842210\gamma^{18}-230400\gamma^{20}+36864\gamma^{22}\right)G[1,1-x]^2-\frac{1}{840\gamma^7\left(-1+\gamma^2\right)^5}$$

$$\left(-11025+72450\gamma^2-330890\gamma^4+1352120\gamma^6-5191767\gamma^8-54902470\gamma^{10}+55184542\gamma^{12}+16762956\gamma^{14}-30376644\gamma^{16}+10224984\gamma^{18}-2482176\gamma^{20}+491520\gamma^{22}\right)G[1,1-x]^2+\frac{256\gamma\left(18+96\gamma^2-63\gamma^4-18\gamma^6+8\gamma^8\right)G[1,1-x]^5}{3\left(-1+\gamma^2\right)^4}-\frac{64\left(32-72\gamma^2-1665\gamma^4+2058\gamma^6-84\gamma^8-600\gamma^{10}+160\gamma^{12}\right)G[1,1-x]^5}{3\left(-1+\gamma^2\right)^{11/2}}+\frac{1536\left(-19-116\gamma^2+14\gamma^4+8\gamma^6\right)G[1,1-x]^2\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)}{\left(-1+\gamma^2\right)^{5/2}}-\frac{1}{\left(-1+\gamma^2\right)^4}$$

$$\gamma\left(6117+35387\gamma^2-22647\gamma^4-7911\gamma^6+3646\gamma^8\right)G[1,1-x]^2\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)-\frac{3\pi^2\left(124-285\gamma-1032\gamma^2+2415\gamma^3+420\gamma^4-1695\gamma^5-600\gamma^6+525\gamma^7\right)\left(G[1,1-x]-2G[2,1-x]\right)}{32\left(-1+\gamma^2\right)^{5/2}}+\frac{1}{840\gamma^8\left(-1+\gamma^2\right)^{7/2}}\left(3675-9100\gamma^2+73150\gamma^4+110460\gamma^6-22565445\gamma^8+45089712\gamma^{10}+16553580\gamma^{12}-44617784\gamma^{14}+13319000\gamma^{16}-1916928\gamma^{18}+491520\gamma^{20}\right)G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)$$

$$\frac{1}{4\left(-1+\gamma\right)^5\left(1+\gamma\right)^2}\gamma\sqrt{-1+\gamma^2}\left(-987-3621\gamma-3911\gamma^2+839\gamma^3+3046\gamma^4+1050\gamma^5\right)G[1,1-x]^2\left(G[1,1-x]-2G[2,1-x]\right)+\frac{1}{4\left(-1+\gamma^2\right)^4}\gamma\left(54+333\gamma+6282\gamma^2-1887\gamma^3+2286\gamma^4+4305\gamma^5-11826\gamma^6-3705\gamma^7+4996\gamma^8+1050\gamma^9\right)G[1,1-x]^2\left(G[1,1-x]-2G[2,1-x]\right)-\frac{3\left(13-9\gamma-155\gamma^2-415\gamma^3-505\gamma^4-55\gamma^5+375\gamma^6+175\gamma^7\right)G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)^2}{8\left(-1+\gamma^2\right)^{5/2}}-$$

$$\frac{1}{8\left(-1+\gamma^2\right)^{3/2}}\left(-90927-1974\gamma-575634\gamma^2-15064\gamma^3+91860\gamma^4+7770\gamma^5+39160\gamma^6+2100\gamma^7\right)G[1,1-x]\left(2G[0,1,1-x]-G[1,1,1-x]\right)+\frac{1}{4\left(-1+\gamma^2\right)^4}\gamma\left(12180+333\gamma+64492\gamma^2-1887\gamma^3-47580\gamma^4+4305\gamma^5-3996\gamma^6-3705\gamma^7+2296\gamma^8+1050\gamma^9\right)G[1,1-x]\left(2G[0,1,1-x]-G[1,1,1-x]\right)-\frac{1}{8\left(-1+\gamma^2\right)^{3/2}}$$

$$\left(258-1935\gamma-7518\gamma^2+5488\gamma^3+198\gamma^4-5775\gamma^5+1846\gamma^6+3150\gamma^7\right)\left(G[1,1-x]-2G[2,1-x]\right)\left(2G[0,1,1-x]-G[1,1,1-x]\right)+\frac{1}{12\gamma^8\left(-1+\gamma^2\right)^{5/2}}\left(-105+155\gamma^2-1935\gamma^4-5091\gamma^6+666260\gamma^8-154716\gamma^{10}-689664\gamma^{12}+198856\gamma^{14}\right)\left(G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)+G[1,1,1-x]-G[1,1-i,1-x]-G[1,1+i,1-x]\right)+$$

$$\frac{1}{4\left(-1+\gamma^2\right)^{5/2}}\left(343249+2093934\gamma^2-258348\gamma^4-149256\gamma^6\right)G[1,1-x]\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)-\frac{27\left(1-5\gamma^2\right)^2\left(-G[1,1-x]+2G[2,1-x]\right)\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)}{\left(-1+\gamma^2\right)^{5/2}}+\frac{1}{28\left(-1+\gamma^2\right)^4}$$

$$\gamma\left(-4374328+6896463\gamma^2-937538\gamma^4-2912259\gamma^6+1098178\gamma^8-155648\gamma^{10}+32768\gamma^{12}\right)\left(-G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)+\frac{1}{8\left(-1+\gamma^2\right)^{3/2}}\left(-90927-1974\gamma-594066\gamma^2-15064\gamma^3+79572\gamma^4+7770\gamma^5+55544\gamma^6+2100\gamma^7\right)G[1,1-x]\left(G[1,1,1-x]-2G[1,2,1-x]\right)-$$

$$\frac{1}{4\left(-1+\gamma^2\right)^4}\gamma\left(-12288+333\gamma-63232\gamma^2-1887\gamma^3+43008\gamma^4+4305\gamma^5+9216\gamma^6-3705\gamma^7-4096\gamma^8+1050\gamma^9\right)G[1,1-x]\left(G[1,1,1-x]-2G[1,2,1-x]\right)-\frac{1}{8\left(-1+\gamma\right)^5\left(1+\gamma\right)^2}\sqrt{-1+\gamma^2}\left(-114-1041\gamma-2241\gamma^2-1691\gamma^3+389\gamma^4+1696\gamma^5+1050\gamma^6\right)\left(G[1,1-x]-2G[2,1-x]\right)\left(G[1,1,1-x]-2G[1,2,1-x]\right)+\frac{1}{8\left(-1+\gamma^2\right)^{7/2}}$$

$$\gamma\left(-5369-12598\gamma^2+17700\gamma^4-450\gamma^6+525\gamma^8\right)\left(-G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)-2G[0,1,1-x]+2G[1,1,1-x]-2G[1,2,1-x]\right)+\frac{1}{84\gamma^8\left(-1+\gamma^2\right)^{7/2}}\left(1260-2268\gamma^2+49728\gamma^4-2828161\gamma^6+1882567\gamma^8+3040161\gamma^{10}-3741239\gamma^{12}+1182464\gamma^{14}-208896\gamma^{16}+49152\gamma^{18}\right)\left(-2G[0,1,1-x]+2G[1,2,1-x]\right)+\frac{1}{56\left(-1+\gamma^2\right)^4}$$

$$\gamma\left(-3534792+6736536\gamma^2-1166271\gamma^4-2657925\gamma^6+917418\gamma^8-77824\gamma^{10}+16384\gamma^{12}\right)\left(G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)-2G[0,1,1-x]+2G[1,2,1-x]\right)+\frac{1}{896\left(-1+\gamma^2\right)^4}3\left(16359+1836171\gamma^2+189841\gamma^4-564667\gamma^6-264740\gamma^8+34300\gamma^{10}\right)\left(G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)+2G[0,1,1-x]-2G[1,1,1-x]+2G[1,2,1-x]\right)-\frac{1}{4\left(-1+\gamma^2\right)^{5/2}}$$

$$\left(75-1182\gamma+3747\gamma^2-5177\gamma^3+2817\gamma^4+1920\gamma^5-3871\gamma^6+1575\gamma^7\right)\left(4G[0,0,1,1-x]-2G[0,1,1,1-x]-2G[1,0,1,1-x]+G[1,1,1,1-x]\right)+\frac{1}{2\left(-1+\gamma^2\right)^{3/2}}\left(-226513-1381230\gamma^2+172332\gamma^4+100104\gamma^6\right)\left(-G[1,1,1,1-x]+G[1,1,1-i,1-x]+G[1,1,1+i,1-x]\right)-\frac{1}{\left(-1+\gamma^2\right)^4}2\gamma\left(6171+34757\gamma^2-20361\gamma^4-10521\gamma^6+4546\gamma^8\right)\left(-G[1,1,1,1-x]+G[1,1,1-i,1-x]+G[1,1,1+i,1-x]\right)-$$

$$\frac{1}{4\left(-1+\gamma^2\right)^{3/2}}\left(111-5865\gamma^2-2067\gamma^4+3421\gamma^6\right)\left(-2G[0,1,1,1-x]+4G[0,1,2,1-x]+G[1,1,1,1-x]-2G[1,1,2,1-x]\right)+\frac{9\gamma\left(1-5\gamma^2\right)^2\left(-3+2\gamma^2\right)\left(2G[0,1,1,1-x]-2G[1,1,1,1-x]+2G[1,1,2,1-x]\right)}{\left(-1+\gamma^2\right)^5}+\frac{1}{4\left(-1+\gamma^2\right)^{5/2}}3\left(-30309-194950\gamma^2+28572\gamma^4+15784\gamma^6\right)\left(2G[0,1,1,1-x]-2G[1,1,1,1-x]+2G[1,1,2,1-x]\right)-\frac{1}{\left(-1+\gamma^2\right)^{5/2}}$$

$$2\left(-6263-38115\gamma^2+6459\gamma^4+1823\gamma^6\right)\left(G[1,1-x]\left(G[1,1,1-x]-G[1,1-i,1-x]-G[1,1+i,1-x]+2\left(-G[2,1,1-x]+G[2,1-i,1-x]+G[2,1+i,1-x]\right)\right)+2\left(-G[0,1,1,1-x]+G[0,1-i,1,1-x]+G[0,1+i,1,1-x]\right)+G[1,1,1,1-x]-G[1,1-i,1,1-x]-G[1,1+i,1,1-x]\right)+\frac{1}{\left(-1+\gamma^2\right)^{5/2}}\left(-12553-75933\gamma^2+11973\gamma^4+4321\gamma^6\right)\left(-G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\right.$$

$$\left.G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\left(-G[1,1-x]+2G[2,1-x]\right)+2\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)-2\left(G[1,1,1,1-x]-G[1,1,1-i,1-x]-G[1,1,1+i,1-x]-G[1,1-i,1,1-x]+G[1,1-i,1-i,1-x]-G[1,1-i,1+i,1-x]-G[1,1+i,1,1-x]+G[1,1+i,1-i,1-x]-G[1,1+i,1+i,1-x]\right)\right)$$

$$-\frac{128\gamma\left(24+137\gamma^2-84\gamma^4-36\gamma^6+16\gamma^8\right)\left(-4G[1,1-x]\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)+3G[1,1-x]^2\operatorname{Log}[2]\right)}{\left(-1+\gamma^2\right)^4}$$

$$\frac{1}{8\left(-1+\gamma^2\right)^4}$$

$$2G[1,1,2,1-x]-2\left(2\left(-G[0,1,1,1-x]+G[0,1-i,1,1-x]+G[0,1+i,1,1-x]\right)+G[1,1,1,1-x]-G[1,1-i,1,1-x]-G[1,1+i,1,1-x]\right)+2\left(G[1,1,1,1-x]-G[1,1-i,1,1-x]-G[1,1+i,1,1-x]+2\left(-G[1,1,2,1-x]+G[1,1-i,2,1-x]+G[1,1+i,2,1-x]\right)\right)+2\left(-2G[0,1,1,1-x]+4G[0,2,1,1-x]+G[1,1,1,1-x]-2G[1,2,1,1-x]\right)+$$

$$\frac{1}{\left(-1+\gamma^2\right)^{3/2}}768\left(-19-116\gamma^2+14\gamma^4+8\gamma^6\right)\left(\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\left(2G[0,1,1-x]-G[1,1,1-x]\right)-\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\left(-G[1,1,1-x]+2G[1,2,1-x]\right)-2\left(-G[0,1,1,1-x]+G[0,1,1-i,1-x]+G[0,1,1+i,1-x]\right)+2\left(-G[1,2,1,1-x]+G[1,2,1-i,1-x]+G[1,2,1+i,1-x]\right)\right)+$$

$$\frac{1}{4\left(-1+\gamma^2\right)^{3/2}}\left(-75-1182\gamma-3747\gamma^2-5177\gamma^3-2817\gamma^4+1920\gamma^5+3871\gamma^6+1575\gamma^7\right)\left(G[1,1,1,1-x]-2\left(G[1,1,2,1-x]+G[1,2,1,1-x]-2G[1,2,2,1-x]\right)\right)-\frac{1}{40\gamma^8\left(-1+\gamma^2\right)^{7/2}}\left(525-3100\gamma^2+13690\gamma^4-55260\gamma^6+816595\gamma^8+3752006\gamma^{10}-1978290\gamma^{12}-1029342\gamma^{14}+213480\gamma^{16}+24576\gamma^{18}\right)G[1,1-x]\operatorname{Log}[2]+$$

$$\frac{1}{360\gamma^7\left(-1+\gamma^2\right)^5}\left(1575-8250\gamma^2+35710\gamma^4-142640\gamma^6-5560073\gamma^8-417302\gamma^{10}+4034092\gamma^{12}-587336\gamma^{14}+6144\gamma^{16}\right)\left(-G[1,1-x]+2G[2,1-x]+3\operatorname{Log}[2]\right)-\frac{1}{\left(-1+\gamma^2\right)^4}128\gamma\left(24+137\gamma^2-84\gamma^4-36\gamma^6+16\gamma^8\right)\left(-4G[1,1-x]\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)+3G[1,1-x]^2\operatorname{Log}[2]\right)+$$

$$\frac{1}{\left(-1+\gamma^2\right)^{3/2}}\left(-2039-13155\gamma^2-1221\gamma^4+1823\gamma^6\right)\left(-3G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)\operatorname{Log}[2]+3\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)\operatorname{Log}[2]\right)+\frac{1}{2\left(-1+\gamma^2\right)^{3/2}}$$

$$\left(-9-1053\gamma^2-1083\gamma^4+1249\gamma^6\right)\left(3G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)\operatorname{Log}[2]-3\left(2G[0,1,1-x]-G[1,1,1-x]\right)\operatorname{Log}[2]-3\left(G[1,1,1-x]-2G[1,2,1-x]\right)\operatorname{Log}[2]\right)+\frac{1}{8\left(-1+\gamma^2\right)^{5/2}}\gamma\left(-1209-6422\gamma^2+1755\gamma^4+2100\gamma^6\right)\left(3G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)\operatorname{Log}[2]+3\left(2G[0,1,1-x]-G[1,1,1-x]\right)\operatorname{Log}[2]-3\left(G[1,1,1-x]-2G[1,2,1-x]\right)\operatorname{Log}[2]\right)+$$

$$\left(\frac{1}{360\gamma^7\left(-1+\gamma^2\right)^5}\left(1575-8250\gamma^2+35710\gamma^4-142640\gamma^6-5560073\gamma^8-417302\gamma^{10}+4034092\gamma^{12}-587336\gamma^{14}+6144\gamma^{16}\right)-\frac{1}{120\gamma^8\left(-1+\gamma^2\right)^{7/2}}\left(525-3100\gamma^2+13690\gamma^4-55260\gamma^6+816595\gamma^8+3752006\gamma^{10}-1978290\gamma^{12}-1029342\gamma^{14}+213480\gamma^{16}+24576\gamma^{18}\right)G[1,1-x]-\frac{128\gamma\left(24+137\gamma^2-84\gamma^4-36\gamma^6+16\gamma^8\right)G[1,1-x]^2}{\left(-1+\gamma^2\right)^4}-\right.$$

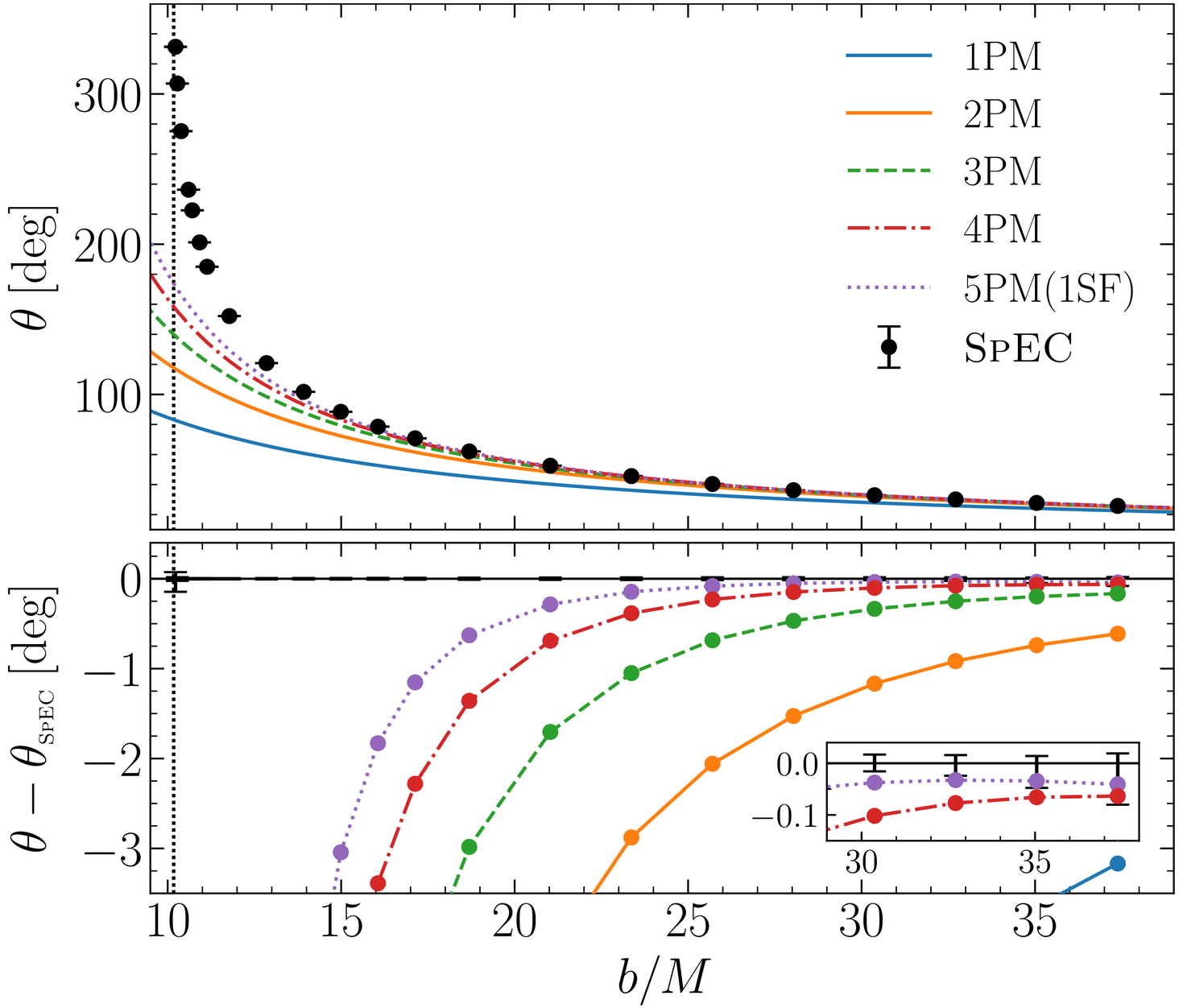
$$\frac{\left(-2039-13155\gamma^2-1221\gamma^4+1823\gamma^6\right)G[1,1-x]\left(-G[1,1-x]+G[1-i,1-x]+G[1+i,1-x]\right)}{\left(-1+\gamma^2\right)^{5/2}}+\frac{\left(-9-1053\gamma^2-1083\gamma^4+1249\gamma^6\right)G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)}{2\left(-1+\gamma^2\right)^{5/2}}+\frac{\gamma\left(-1209-6422\gamma^2+1755\gamma^4+2100\gamma^6\right)G[1,1-x]\left(G[1,1-x]-2G[2,1-x]\right)}{8\left(-1+\gamma^2\right)^{5/2}}-\frac{\left(-9-1053\gamma^2-1083\gamma^4+1249\gamma^6\right)\left(2G[0,1,1-x]-G[1,1,1-x]\right)}{2\left(-1+\gamma^2\right)^{5/2}}+$$

$$\frac{\gamma\left(-1209-6422\gamma^2+1755\gamma^4+2100\gamma^6\right)\left(2G[0,1,1-x]-G[1,1,1-x]\right)}{8\left(-1+\gamma^2\right)^{5/2}}+\frac{\left(-2039-13155\gamma^2-1221\gamma^4+1823\gamma^6\right)\left(-G[1,1,1-x]+G[1,1-i,1-x]+G[1,1+i,1-x]\right)}{\left(-1+\gamma^2\right)^{5/2}}-\frac{\left(-9-1053\gamma^2-1083\gamma^4+1249\gamma^6\right)\left(G[1,1,1-x]-2G[1,2,1-x]\right)}{2\left(-1+\gamma^2\right)^{5/2}}-\frac{\gamma\left(-1209-6422\gamma^2+1755\gamma^4+2100\gamma^6\right)\left(G[1,1,1-x]-2G[1,2,1-x]\right)}{8\left(-1+\gamma^2\right)^{5/2}}\Bigg)\operatorname{Log}[-1+\gamma]$$

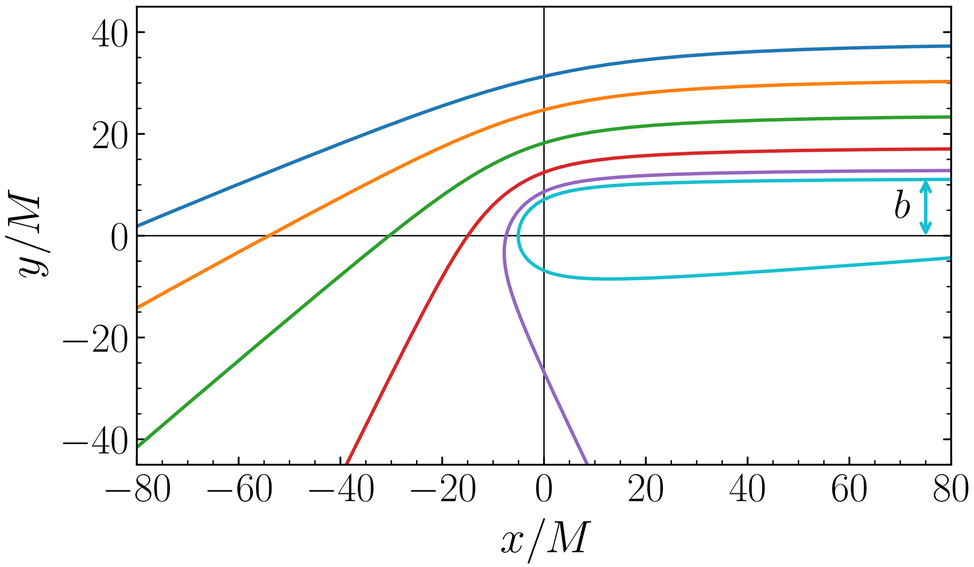
SCATTERING ANGLE: COMPARISON TO NUMERICAL RELATIVITY

[Long, Pfeiffer, Kidder, Scheel, 2511.10196]

Scattering angle



Trajectories:



Parameters:

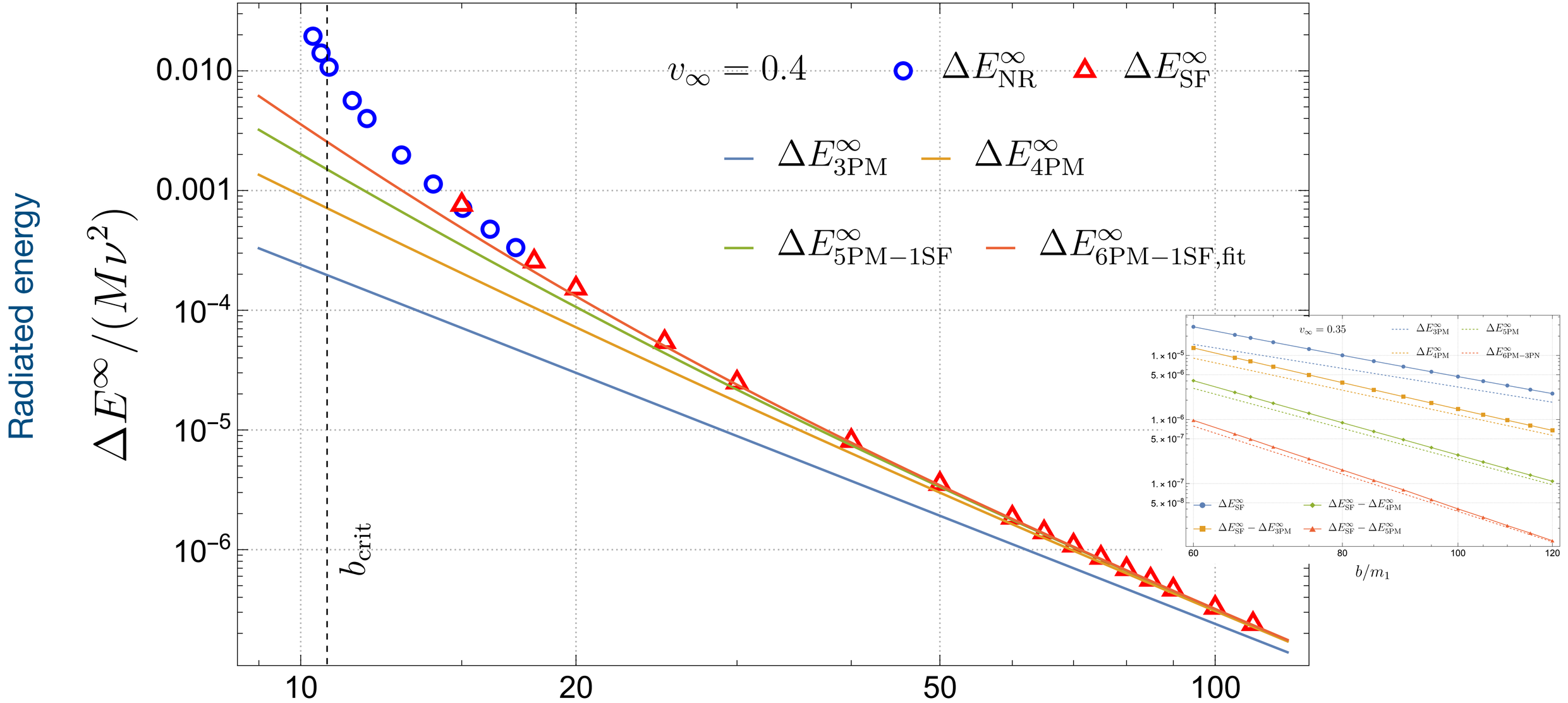
$$m_1 = m_2$$

$$v = 0.296c$$

$$M = m_1 + m_2$$

RADIATED ENERGY: COMPARISON TO SEMI-NUMERICAL SELF-FORCE

[Warbuton, 2512.02274]

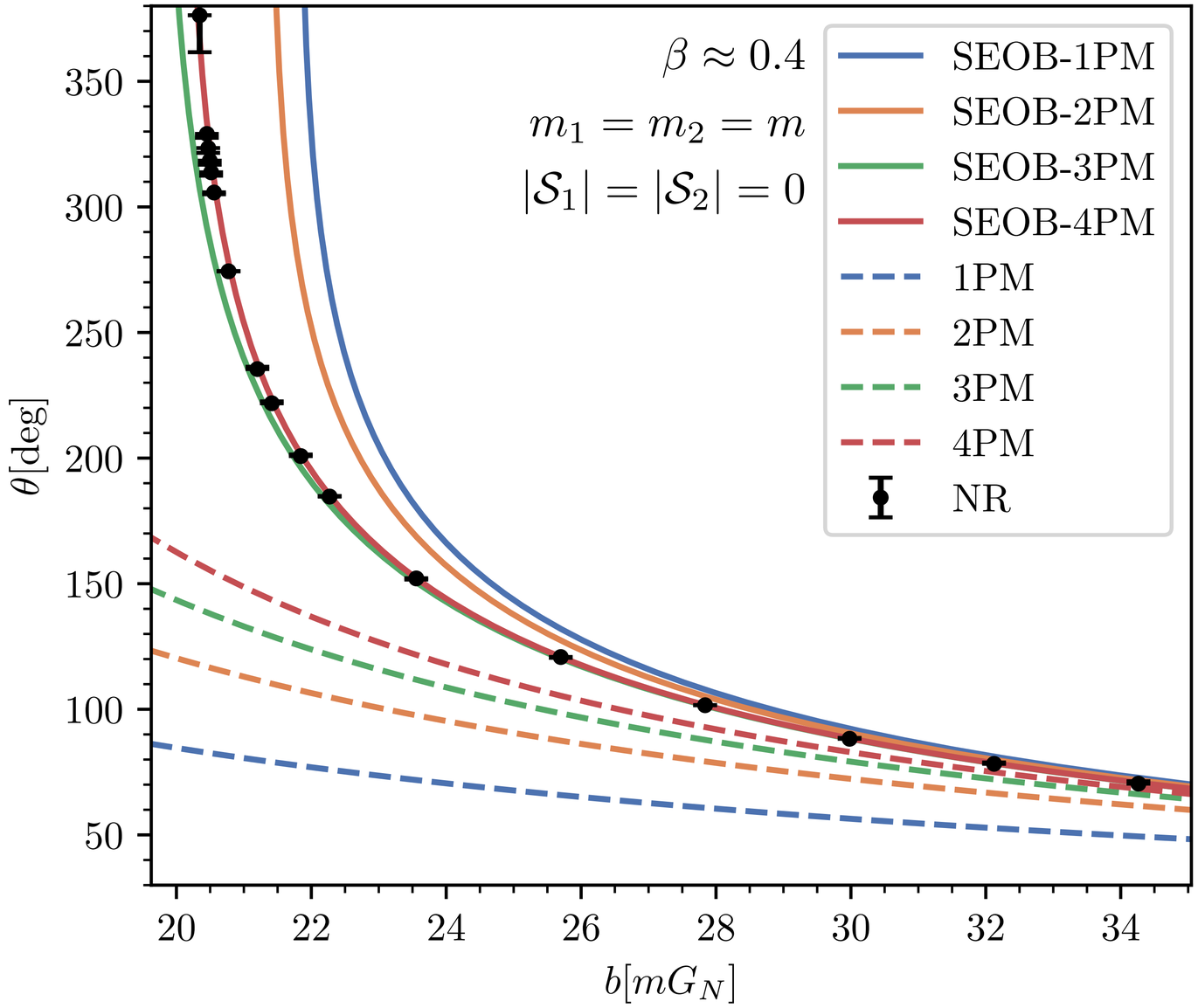


Checks the CY3's!

COMPARISON TO NUMERICAL RELATIVITY USING RESUMMATION

[Buonanno,Jakobsen,Mogull]

- **Effective One Body** model: resummation technique



Not done for 5PM-1SF yet!

See also

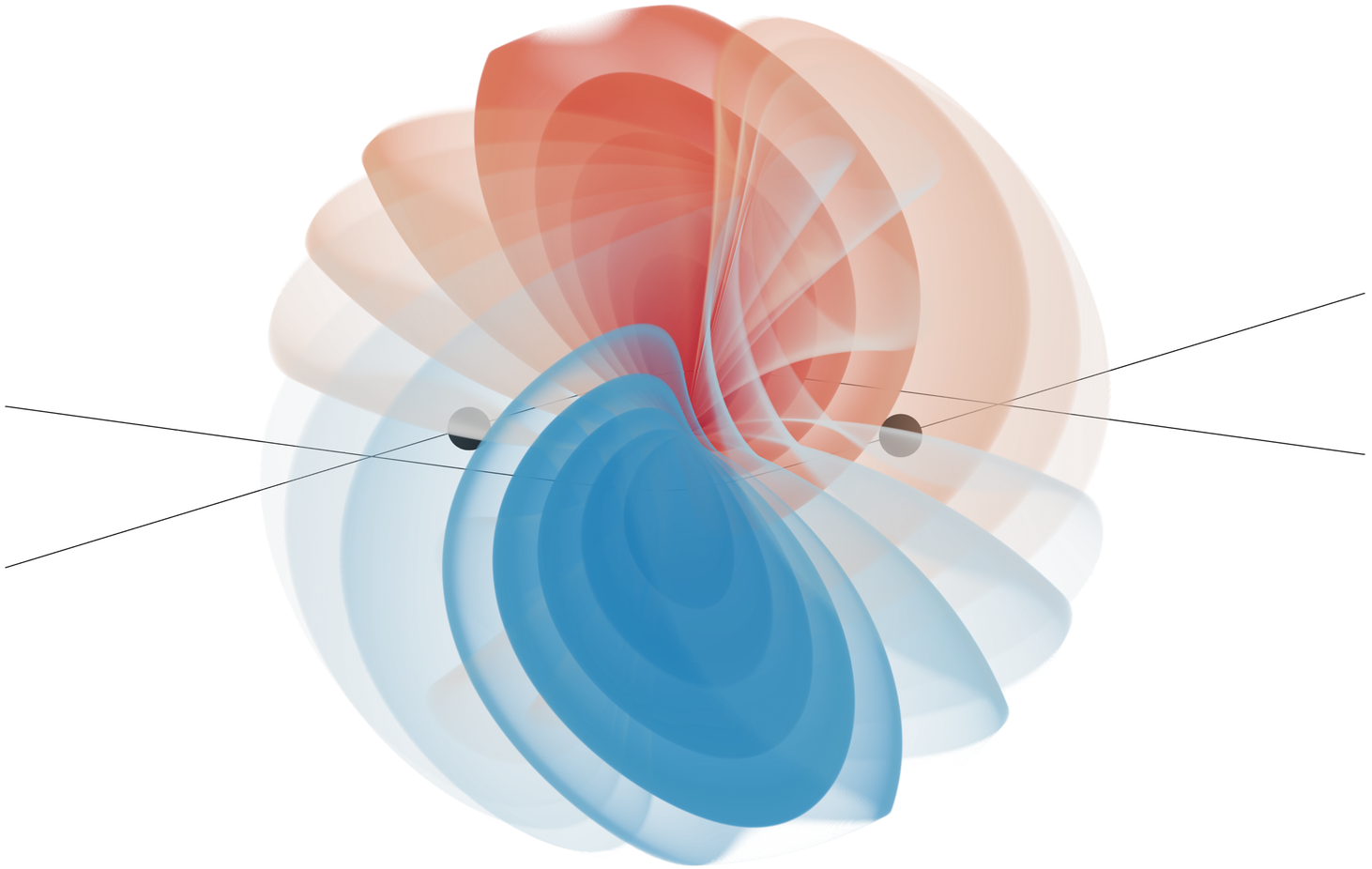
- Damour and Rettegno w_{EOB} model in [2211.01399]
- Pratten, Schmidt, Swain [2411.09652]

SCATTERING SPINNING BLACK HOLES: STATE-OF-THE-ART

Higher orders in spin are **suppressed by physical PM counting**: $a_i^\mu = G m_i \chi_i^\mu \,, \qquad |\chi_i| < 1$

	S^0	S^1	S^2	S^3	S^4	S^5
1PM (tree level)	G	G^2	G^3	G^4	Hoogeveen, Jakobsen, JP `25	G^6
2PM (1 loop)	G^2	G^3	G^4	G^5	Guevara, Ochirov, Vines '18 Chen, Chung,Huang, Kim '21 Haddad, Jakobsen, **Mogull** JP '24	Bern, Kosmopoulos, Luna, Roiban, Teng '22; Aoude, Haddad, Helset '23; Bautista '23; Chen, Wang '24; Bohnenblust, Cangemi, Johansson, Pichini '24
3PM (2 loops)	Bern, Cheung, Parra-Martinez, Ruf, Herrmann, Roiban, Shen, Solon, Zeng, Kälin, Liu, Porto, Di Vecchia, Heissenberg, Russo, Veneziano, Travaglini, Brandhuber, Damgaard, Planté, Vanhove, Bjerrum-Bohr	**Jakobsen, Mogull '22** Akpinar, Cordero, Kraus, Ruf, Zeng '24	**Jakobsen, Mogull '22** Akpinar, Cordero, Kraus, Ruf, Zeng '24	Akpinar, Cordero, Kraus, Smirnov, Zeng '25 Haddad, Jakobsen, **Mogull,** JP `25	Akpinar, Cordero, Kraus, Smirnov, Zeng '25 Haddad, Jakobsen, **Mogull,** JP `25	G^8
4PM (3 loops)	Dlapa, Kälin, Liu, Neef, Porto, Damgaard, Hansen, Planté, Vanhove, Bern, Parra-Martinez, Roiban, Ruf, Shen Solon, Zeng	**Jakobsen, Mogull, JP, Sauer, Xu '23**	G^6	G^7	G^8	G^9
5PM (4 loops)	**Driesse, Jakobsen, Klemm, Mogull, Nega, JP, Sauer, Usovitsch '24**	G^6	G^7	G^8	G^9	G^{10}

THE FARFIELD GRAVITATIONAL WAVEFORM



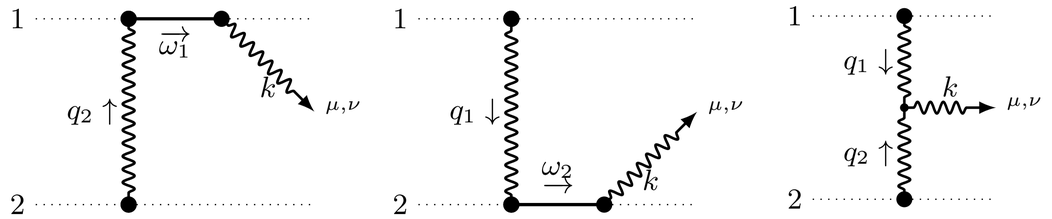
[Jakobsen,Mogull,JP,Sauer]

FAR FIELD WAVEFORM @ LEADING ORDER

[Jakobsen,Mogull,JP,Steinhoff]

Sum of diagrams with outgoing graviton:

$$\langle h_{\mu\nu}(k)\rangle =$$



1.5 PM

For time-domain waveform needs to integrate over outgoing energy : Ω

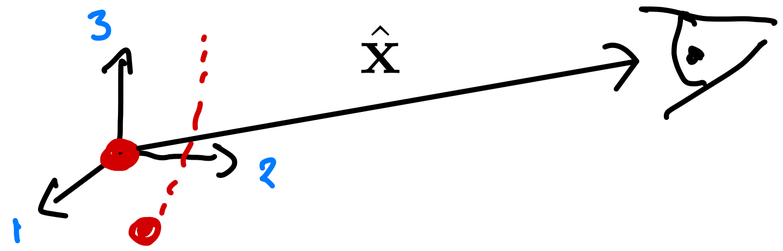
$$\frac{f_{+,\times}(t-r,\hat{\mathbf{x}})}{r} = \frac{4G}{r}\int d\Omega\, e^{-i\Omega(t-r)}\varepsilon^{\mu\nu}_{+,\times}\langle h_{\mu\nu}(k=\Omega(1,\hat{\mathbf{x}})\rangle$$

where unit vector $\hat{\mathbf{x}}$ points towards the observer

The waveform has two polarizations

$$f_{+,\times}(\underbrace{t-r}_{u},\underbrace{\theta,\phi}_{\hat{\mathbf{x}}};v,|b|,m_1,m_2)$$

retarded time



INTEGRATED WAVEFORM

[Jakobsen,Mogull,JP,Steinhoff]

- LO is non-radiating:

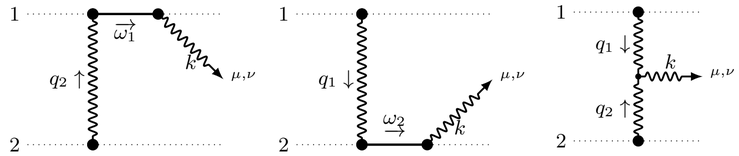


$$f^{(1)}(\hat{\mathbf{x}}) = \frac{2m_1}{\rho\cdot v_1}(\epsilon\cdot v_1)^2 + \frac{2m_2}{\rho\cdot v_2}(\epsilon\cdot v_2)^2$$

0.5 PM

- Our NLO result reproduces [Kovacs,Thorne '75] obtained with traditional GR techniques in 4 long papers

$$f^{(2)}(u,\hat{\mathbf{x}}) = \frac{1}{|\tilde{\mathbf{b}}|_1}\left[\alpha_1(u_1,\rho) + \frac{\beta_1(u_1,\rho)}{\tilde{b}^2}\right] + (1\leftrightarrow 2)$$

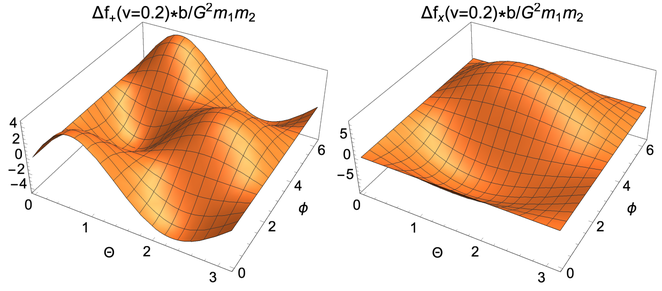


$$|\tilde{\mathbf{b}}|_{1,2} := \sqrt{|b|^2 + (\gamma^2-1)u_{2,1}^2} \qquad u_i = \frac{\rho\cdot(x-b_i)}{\rho\cdot v_i} \qquad \rho = (1,\hat{\mathbf{x}})$$

retarded time in ith rest frame

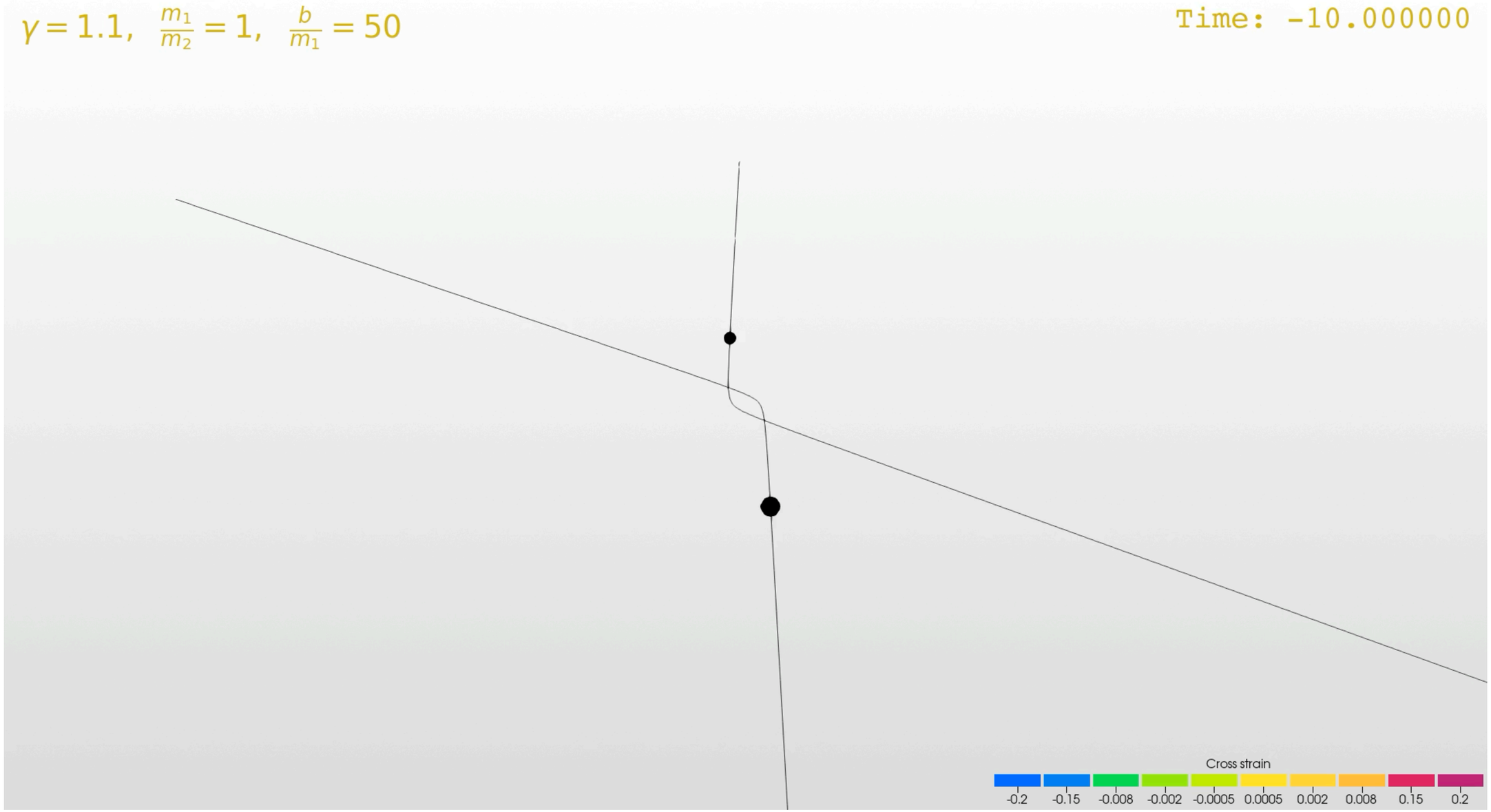
- The wave memory effect: $\Delta f^{2)} = f^{(2)}(u=+\infty) - f^{(2)}(u=-\infty)$

$$\frac{\Delta f^{(2)}_{\mathcal{S}=0}}{m_1 m_2} = \frac{4(2\gamma^2-1)\epsilon\cdot v_1(2b\cdot\epsilon\,\rho\cdot v_1 - b\cdot\rho\,\epsilon\cdot v_1)}{|b|^2\sqrt{\gamma^2-1}(\rho\cdot v_1)^2}$$



$$\gamma = v_1\cdot v_2$$

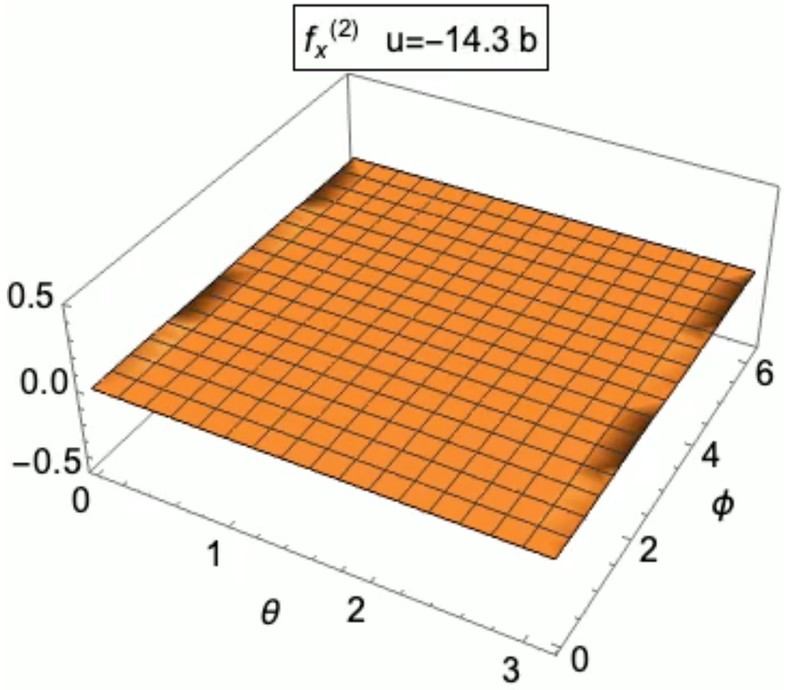
Visualization: Cross-Polarization $f_+^{(2)}$

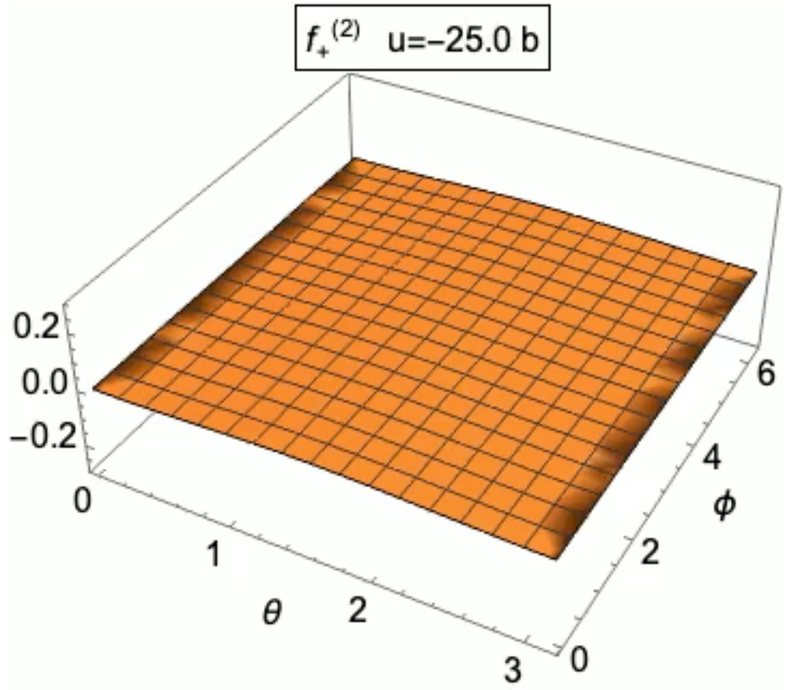


$v = 0.4$

[HU BSc thesis Kleinmond]

Memory effect





- wave memory:

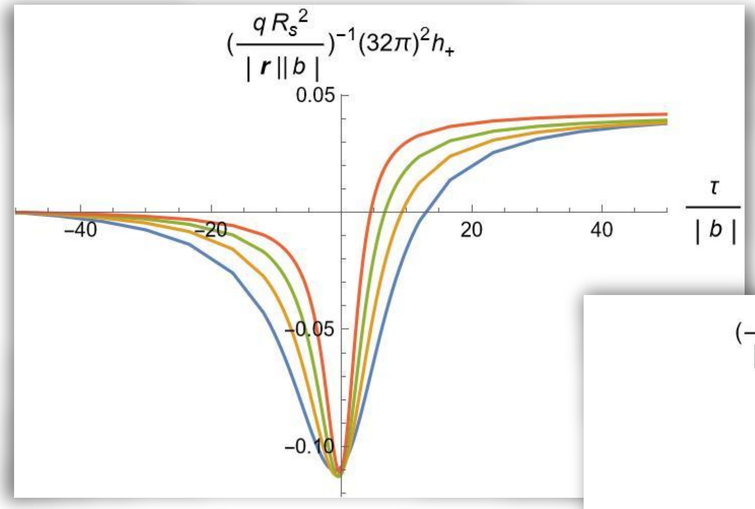
$$\frac{\Delta f^{(2)}_{\mathcal{S}=0}}{m_1 m_2} = \frac{4(2\gamma^2-1)\epsilon\cdot v_1(2b\cdot\epsilon\,\rho\cdot v_1 - b\cdot\rho\,\epsilon\cdot v_1)}{|b|^2\sqrt{\gamma^2-1}(\rho\cdot v_1)^2}$$

$$\rho = (1, \hat{\mathbf{x}})$$

$$\gamma = v_1 \cdot v_2$$

DETECTABILITY OF SCATTERING EVENT WITH LISA

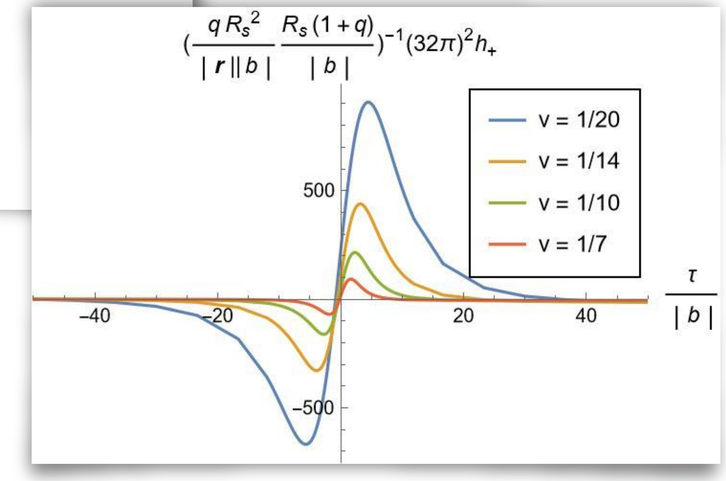
[Slide from L. Bohnenblust]

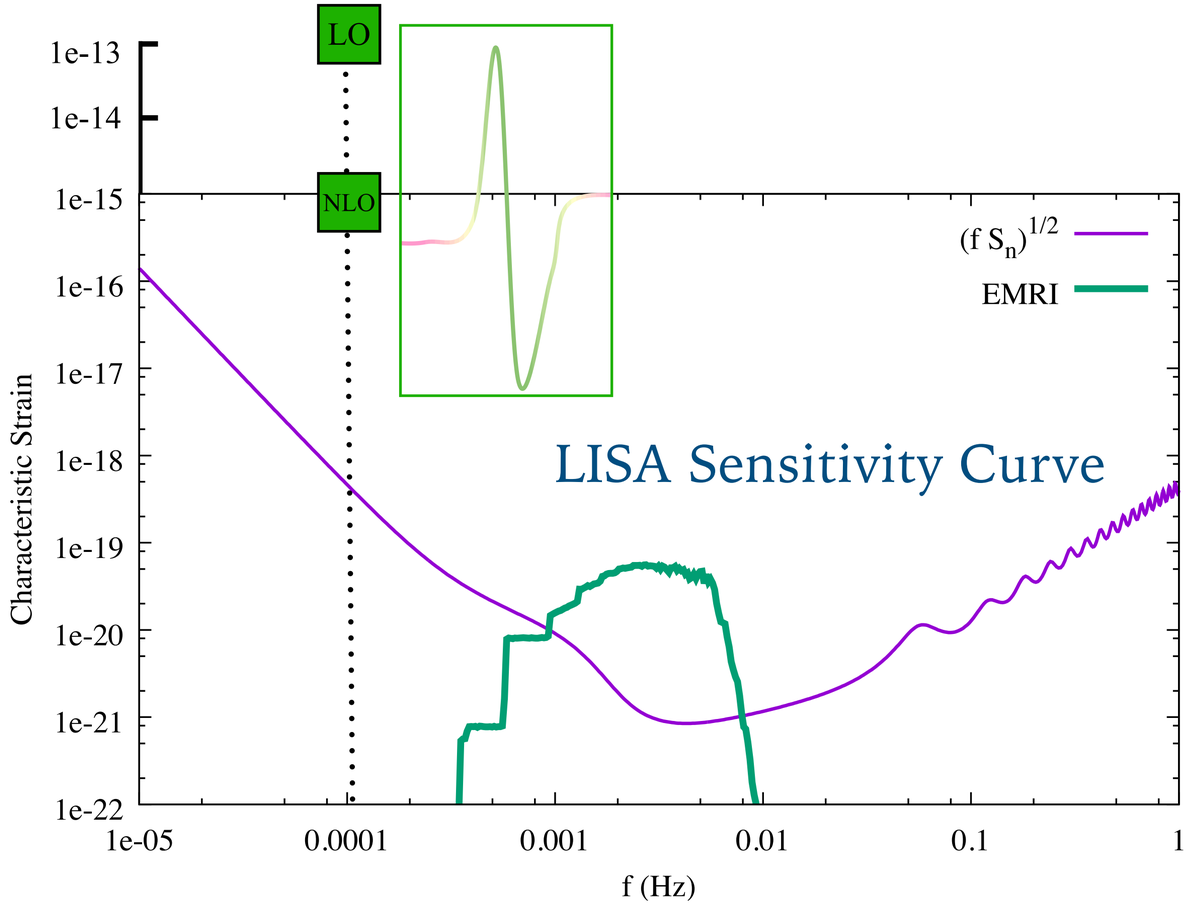


Leading order

Time domain waveforms

Next-to-leading order





[Robson,Cornish,Liu 1803.01944]

- Example event:

$$|b| = 1000\, r_S^1\,,$$
$$m_1 = 5 \cdot 10^4\, M_\odot, m_2 = 50\, M_\odot\,, \qquad T \sim 10^4\,\mathrm{s}\,,$$
$$|\boldsymbol{r}| \approx 10^3\,\mathrm{kpc}\text{, distance to Andromeda,} \qquad f \sim 10^{-4}\,\mathrm{Hz}\,,$$

$$h_{\mathrm{LO}} \sim 10^{-13},$$
$$h_{\mathrm{NLO-corr.}} \sim 10^{-15}$$

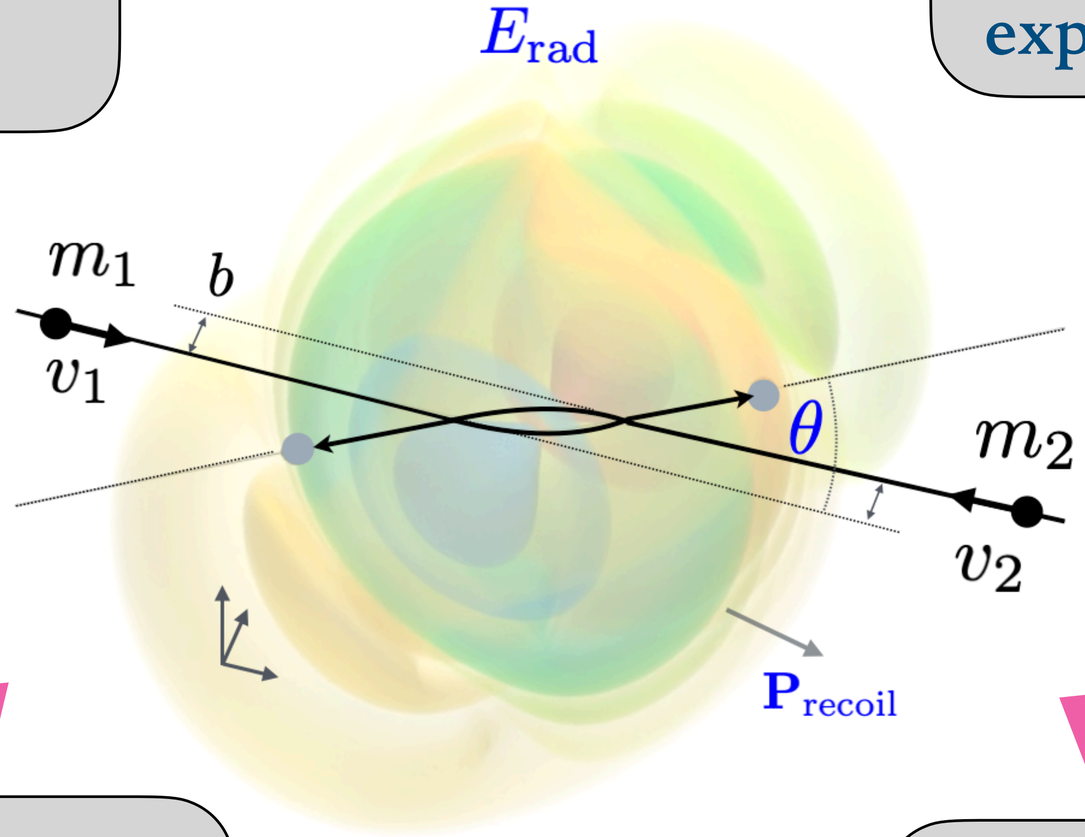
Collider physics:

State-of-the-art Feynman integration technology

Diff Eqs & IBPs

Classical Gravity:

Resummation techniques, geodesics, self-force expansion



String Theory: Point particle limit & spin from oscillators

Mathematics:

Calabi-Yau 3-fold periods

SUMMARY

WQFT: Highly efficient quantum field theory technology for classical scattering in classical gravity

- „Quantize“ world-line degrees of freedom
- One-point functions = observables, Classical theory = tree-level diagrams
- Rich function space @ 5PM: First time CY3 appears in nature!

Challenges ahead:

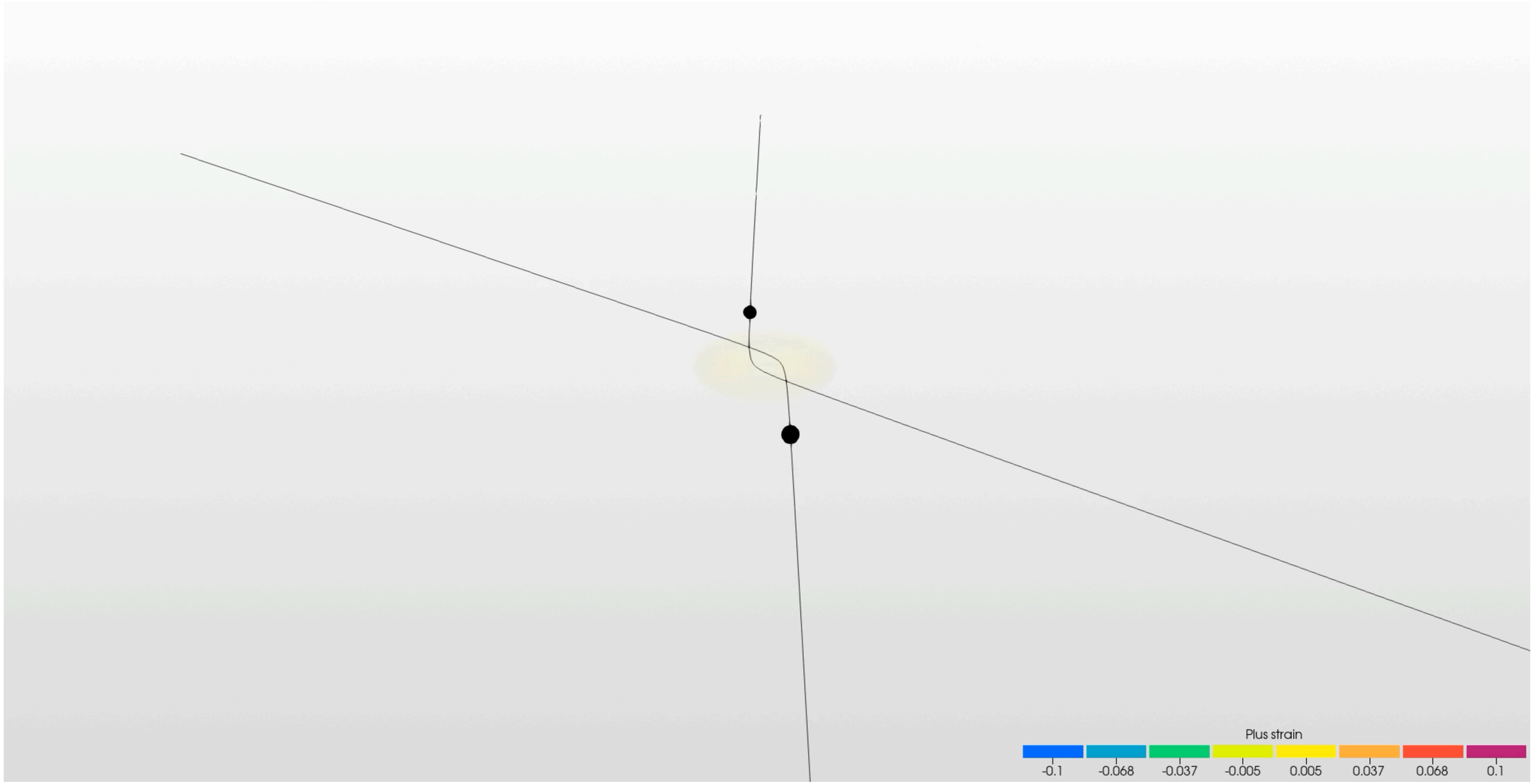
- Ongoing: **5PM-2SF**. Non-planar integrals, extended CY3 sector.
- Make contact to bound state problem
- Resummation strategies, EOB & contact to gravitational self-force expansion

BIG THANKS TO THE GREAT 5PM TEAM!





THANK YOU!



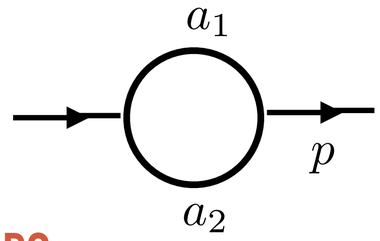
BACKUP

IBPS, MIS & DES: THE GENERAL IDEA

$$\frac{\mathrm{d}\vec{I}(x)}{\mathrm{d}x} = \epsilon A(x)\,\vec{I}(x)$$

- Bubble family:

$$G_{a_1,a_2}(s,m^2;D) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{[-k^2+m^2]^{a_1}[-(k+p)^2+m^2]^{a_2}}$$



- IBP relations:

$$\int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial k^\mu} \left\{ v^\mu \frac{1}{[-k^2+m^2]^{a_1}[-(k+p)^2+m^2]^{a_2}} \right\} = 0$$

IBPS

$$0 = (D-2a_1-a_2)\, G_{a_1,a_2} - a_2\, G_{a_1-1,a_2+1} + 2m^2 a_1\, G_{a_1+1,a_2} + (2m^2-s)\, a_2\, G_{a_1,a_2+1}$$

$$G_{a_1,a_2} = G_{a_2,a_1}$$

- 2 Master Integrals: $\{G_{1,1}, G_{0,1}\}$

- Basis: $G_{a_1,a_2} = c_1\, G_{1,1} + c_2\, G_{0,1}$

eg $G_{2,0} = \frac{\epsilon}{m^2} G_{0,1}$ $G_{2,1} = \frac{\epsilon}{m^2(4m^2-s)} G_{0,1} + \frac{1+2\epsilon}{4m^2-s} G_{1,1}$

- Differential eqs:

$$\partial_{m^2} \begin{pmatrix} G_{0,1} \\ G_{1,1} \end{pmatrix} = \begin{pmatrix} -G_{2,0} \\ -2\, G_{2,1} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{-2}{4m^2-s} \end{pmatrix} + \epsilon \begin{pmatrix} \frac{-1}{m^2} & 0 \\ \frac{-2}{m^2(4m^2-s)} & \frac{-4}{4m^2-s} \end{pmatrix} \right] \cdot \begin{pmatrix} G_{0,1} \\ G_{1,1} \end{pmatrix}$$

$$\partial_s \begin{pmatrix} G_{0,1} \\ G_{1,1} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{s-2m^2}{s(4m^2-s)} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ \frac{2}{s(4m^2-s)} & \frac{1}{4m^2-s} \end{pmatrix} \right] \cdot \begin{pmatrix} G_{0,1} \\ G_{1,1} \end{pmatrix}$$

CANONICAL DIFFERENTIAL EQUATIONS

- Change of basis for MIs: $\mathbf{f} = T \cdot \begin{pmatrix} G_{0,1} \\ G_{1,1} \end{pmatrix}$ $\qquad T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(-s)(4m^2 - s)} \end{pmatrix}$

- To canonical DE: $\partial_{m^2} \mathbf{f} = \epsilon A_{m^2} \cdot \mathbf{f}\,, \qquad \partial_s \mathbf{f} = \epsilon A_s \cdot \mathbf{f}\,.$

  or $\mathrm{d}\mathbf{f} = \epsilon\, (\mathrm{d}\tilde{A}) \cdot \mathbf{f}\,,$ with $\mathrm{d} = \mathrm{d}s\, \partial_s + \mathrm{d}m^2\, \partial_{m^2}$ and $\tilde{A} = \begin{pmatrix} -\log m^2 & 0 \\ -2\log\left(\frac{\sqrt{1-4m^2/s}-1}{\sqrt{1-4m^2/s}+1}\right) & -\log(4m^2 - s) \end{pmatrix}$

- Solution: $\mathbf{f}(\mathbf{x};\epsilon) = \mathbb{P}\exp\left[\epsilon \int_C \mathrm{d}\tilde{A}(\mathbf{x}')\right] \cdot \mathbf{f}(\mathbf{x}_0;\epsilon)$

  boundary integral $\mathbf{f}(s = 0, m^2; D = 2 - 2\epsilon) = \begin{pmatrix} \Gamma(\epsilon)(m^2)^{-\epsilon} \\ 0 \end{pmatrix}$ as $G_{0,1} = \Gamma(\epsilon)\left(m^2\right)^{-\epsilon}$

  and T=diag[1,0]

SOLVING CANONICAL DIFFERENTIAL EQUATIONS

- Normalization:

$$\mathbf{f}(x;\epsilon) := \frac{1}{(m^2)^{-\epsilon}\,\Gamma(\epsilon)} \left(\frac{G_{0,1}}{\sqrt{(-s)(4m^2-s)}\,G_{1,1}} \right)$$

- DEs in new variable x:

$$s = -m^2(1-x)^2/x$$

$$\mathrm{d}\,\mathbf{f}(x;\epsilon) = \epsilon\,\mathrm{d} \begin{pmatrix} 0 & 0 \\ -2\log x & \log\frac{x}{(1+x)^2} \end{pmatrix} \cdot \mathbf{f}(x;\epsilon)$$

$$\mathbf{f}(1;\epsilon) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Boundary condition

$$\partial_x \mathbf{f}(x;\epsilon) = \epsilon \left[\frac{1}{x} \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} + \frac{1}{1+x} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \right] \cdot \mathbf{f}(x;\epsilon)$$

Solution in ϵ-expansion

$$\mathbf{f}(x;\epsilon) = \sum_{k\geq 0} \epsilon^k \mathbf{f}^{(k)}(x)$$

- First three orders:

$$\mathbf{f}^{(0)}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{f}^{(1)}(x) = \begin{pmatrix} 0 \\ -2\log x \end{pmatrix} \qquad \mathbf{f}^{(2)}(x) = \begin{pmatrix} 0 \\ 4\,\mathrm{Li}_2(-x) + 4\log x \log(1+x) - \log^2 x + \pi^2/3 \end{pmatrix}$$

SPINNING WQFT AS TENSIONLESS LIMIT OF THE BOSONIC STRING

[Haddad,Jakobsen,Mogull,JP]

- Open bosonic string mode exansion:

$$X^\mu(\sigma,\tau) = q^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left\{ \frac{1}{n} \left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau-\sigma)} \right) \right\} \qquad \alpha' \text{ inv. string tension}$$

- Commutators: $[\alpha_k^\mu, \alpha_l^\nu] = k\delta_{k+l,0}\eta^{\mu\nu}$ $\quad (\alpha_k^\mu)^\dagger = \alpha_{-k}^\mu$ $\quad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$ $\quad p^\mu = -i\partial_{q^\mu}$

- Spectrum: $\alpha_{-n_1}^{\mu_1} \ldots \alpha_{-n_l}^{\mu_l} |k^\mu\rangle$ carries spin $\sum n_i$ and mass $M^2 = \dfrac{\sum n_i - 2}{\alpha'}$

- Virasoro algebra: $L_n = \dfrac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^3-m)\delta_{m+n}$$

- Tensionless ($\alpha' \to \infty$) limit: [Bonezzi; Bouatta, Campere, Sagnotti]

Rescale $L_0 \to \dfrac{1}{\alpha'} L_0$

$L_{\pm 1} \to \dfrac{1}{\sqrt{2\alpha'}} L_{\pm 1}$

in $\alpha' \to \infty$ limit has reduced generators $l_0 = p^2 \quad l_{\pm 1} = p\cdot\alpha_{\pm 1}$

that obey SL(2,R) algebra. Phase space $\{q^\mu, p^\mu, \alpha_{-1}, \alpha_{+1}\}$ is as in our spinning worldline model.

CLAIM:

$$\langle \mathcal{O} \rangle_{\mathrm{WQFT}} = \int D[h, z] \mathcal{O}\, e^{-\frac{i}{\hbar} S[z,h]} \xrightarrow{\hbar \to 0}$$

Tree-level one-point functions $\langle h_{\mu\nu} \rangle$ and $\langle z^{\mu} \rangle$ solve classical equations of motion

CLASSICAL DYNAMICS FROM ONE-POINT FUNCTIONS

[Boulware,Brown,'68]

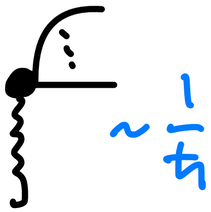
- Action: $S[\Phi_A]$ with fields $\Phi_A(x_A) = \{h_{\mu\nu}(x), z^\mu(\tau)\}$ and coordinates $x_A = \{x^\mu, \tau\}$

- Partition function in the presence of sources

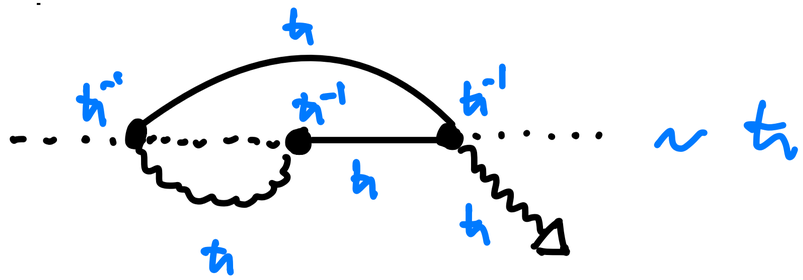
$$Z[J_A] = \int D[\Phi_A] \exp\left[\frac{i}{\hbar}\left(S[\Phi_A] + \sum_A \int dx_A J_A(x_A)\, \Phi_A(x_A)\right)\right]$$

- $\hbar$ counting:





E.g:



⇒ Loops are quantum effects!

- Scalings of **connected n-point functions:**

$$\langle \Phi_{A_1} \ldots \Phi_{A_n} \rangle_{\text{conn}} \sim \sum_L \hbar^{-1+n+L}$$ (L-loop connected n-point diagrams)

Well defined classical limit only for n=1 and L=0: **Tree-level one-point functions**

CLASSICAL DYNAMICS FROM ONE-POINT FUNCTIONS

[Boulware,Brown,'68]

- Factorization

$$\lim_{\hbar\to 0} \langle \Phi_{A_1}\Phi_{A_2}\dots\Phi_{A_n}\rangle_{\text{discon}} = \langle \Phi_{A_1}\rangle^{\text{tree}}_{\text{con}} \langle \Phi_{A_2}\rangle^{\text{tree}}_{\text{con}} \dots \langle \Phi_{A_n}\rangle^{\text{tree}}_{\text{con}}$$

- Consequence for Schwinger-Dyson equations: ($\hat{=}$ Ehrenfest Thm in QM)

$$\left\langle \frac{\delta S[\Phi_A]}{\delta \Phi_A} \right\rangle = 0 \qquad \xrightarrow{\hbar \to 0} \qquad \boxed{\frac{\delta S[\langle \Phi_A\rangle_{\text{tree}}]}{\delta \Phi_A} = 0}$$

Tree-level one-point functions solve classical equations of motion

- Importantly $S[\Phi_A]$ must be independent of $\hbar$ (not the case in amplitudes approach - massive field!) $\Rightarrow$ Advantage of WQFT approach (no „super classical“ terms)
- WQFT approach yields clean separation between $\hbar$ and **G** expansion!
- Need non-trivial background field configurations for non-vanishing one-point functions