ABJM Wilson loops in Fermi gas and topological string approach

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Klemm, Marino, Schiereck, Soroush arXiv/1207.0611

- Localisation in supersymmetric gauge theories
  - Wilson Loops in supersymmetric Chern-Simons theory with matter

- Topological-String on local Calabi-Yau
  - The Chern-Simons Gauge theory/Topological-String large N duality
  - The ABJM slice and the $N^{3/2}$ scaling law
  - Open topological string and Wilson line expectation values

- The Fermigas approach

- Reinterpretation of partition function and Wilson loop
- Fermigas techniques, Fermi surface and tropical geometry
- The all genus expansion

Localization in supersymmetric gauge theories

- Supersymmetry provides nilpotent operators $Q^2 = 0$ which localise the path integral to the fixpoints of the fermionic transformations given by the solutions to the e.o.m.

- Beside evaluating the instanton contributions it remains to calculate the one-loop determinant around the given vaccua

- This semiclassical approximation is exact and the integration over the fiels can be done by the Atiyah-Bott localisation formula

$$\int_{\mathcal{M}} \alpha = \int_{F \in \mathcal{M}} \frac{i * (\alpha)}{e(\mathcal{N})}$$

Localisation calculations in supersymmetric gauge theories on symmetric spaces with positive curvature have been done recently also for Wilson loop correlators Pestun 07

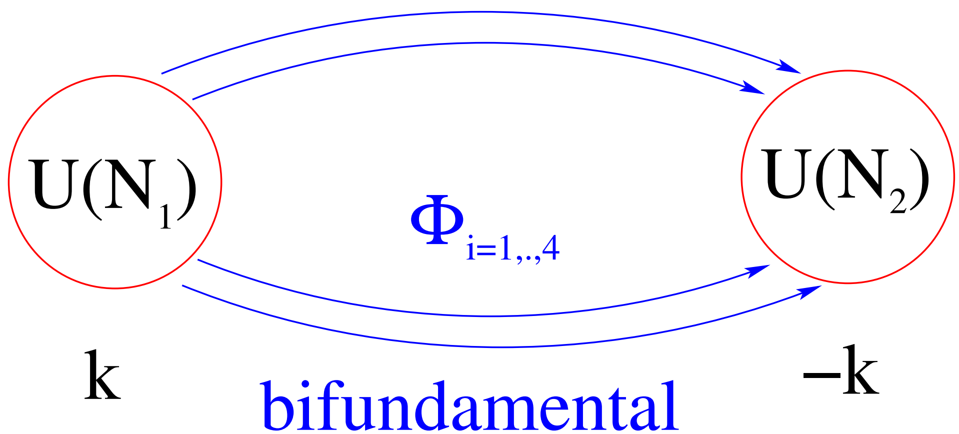
In 3d for sypersymmetric Chern-Simons quiver theories with matter Kapustin, Willet, Yakov: arXiv 0909.4559. Gauge action (in flat space) $g_{CS} = -\frac{k}{4\pi}$, $k \in \mathbb{Z}$

$$S_G = g_{CS} \int d^3x \mathrm{Tr} \left(\epsilon^{\alpha\beta\gamma} (A_\alpha \partial_\beta A_\gamma + \frac{2i}{3} A_\alpha A_\beta A_\gamma) - \lambda^\dagger \lambda 2D\sigma \right)$$

Coupled to chiral matter multiplet

$$\begin{aligned} S_M = & \int d^3x \big(D_\mu \phi^\dagger D^\mu \phi + \tfrac{3}{4} \phi^\dagger \phi + i\psi^\dagger D \psi + F^\dagger F - \\ \phi^\dagger \sigma \phi + & \phi^\dagger D \phi - \psi^\dagger \sigma \psi + i\phi^\dagger \lambda^\dagger \psi - i\psi^\dagger \lambda \phi \big) \end{aligned}$$

Groups and matter contents given by quiver diagrams. E.g.



ABJM showed that this theory has $N = 6$ susy and KWY localised the partition function on S^3 to a matrix

model integral

$$Z_{\mathrm{ABJM}} = \frac{1}{N!^2}\int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i \mathrm{d}\nu_j}{(2\pi)^2} \frac{\prod_{i<j} \sinh^2\left(\frac{\mu_i-\mu_j}{2}\right) \sinh^2\left(\frac{\nu_i-\nu_j}{2}\right)}{\prod_{i,j} \cosh^2\left(\frac{\mu_i-\nu_j}{2}\right)}$$

$$e^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 - \sum_j \nu_j^2\right)}$$

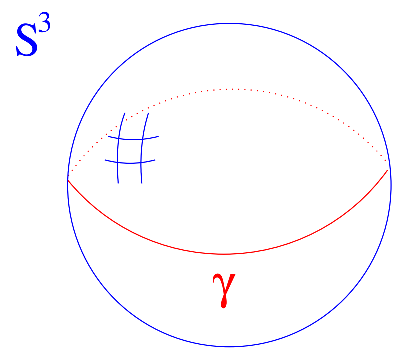


Figure 1: $1/6$ Wilson loop

$$W_R^{1/6} = \mathsf{tr}_R\, P\, \exp \int_\gamma \left(\mathrm{i} A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_J^I C_I \bar{C}^J \right) \mathrm{d}s \ ,$$

In ABJM theory one has three related WL's: In the first gauge group

$$\langle W_R^{1/6} \rangle = \langle \mathsf{tr}_R\, (e^{\mu_i}) \rangle_{\mathrm{ABJM}}$$

and in the second gauge group

$$\langle \widehat{W}_R^{1/6} \rangle = \langle \mathsf{tr}_R\, (e^{\nu_i}) \rangle_{\mathrm{ABJM}}$$

In addition there is a more symmetric combination, which breaks only half the super symmetry

$$\langle W^{1/2}_{\mathcal{R}} \rangle = \left\langle \mathrm{Str}_{\mathcal{R}} \begin{pmatrix} e^{\mu_i} & 0 \\ 0 & -e^{\nu_j} \end{pmatrix} \right\rangle_{\mathrm{ABJM}} .$$

The simplest insertion $\mathrm{Tr} e^{n\mu_i}$ in the Matrix model integral corresponds to a combination of Wilson loops over hook representations $R_{n,s}$

$$W^{1/6}_n = \sum_{s=1}^{n-1} (-1)^s W^{1/6}_{R_{n,s}}$$

and

$$\langle W_n^{1/2} \rangle = \langle W_n^{1/6} \rangle - (-1)^n \langle \widehat{W}_n^{1/6} \rangle \ .$$

Now that the relevant matrix integrals in 3d gauge theories are defined we decribe in the next section how to solve them in an $1/N$ expansion, which is exact in the t' Hooft parameters, using

- Large N-Duality between the topological string and matrix models Putrov and Marino arXiv/0912.3074, Drukker, Putrov and Marino arXiv/1007.3837,... .
  - Partition function maps to closed string partition

function

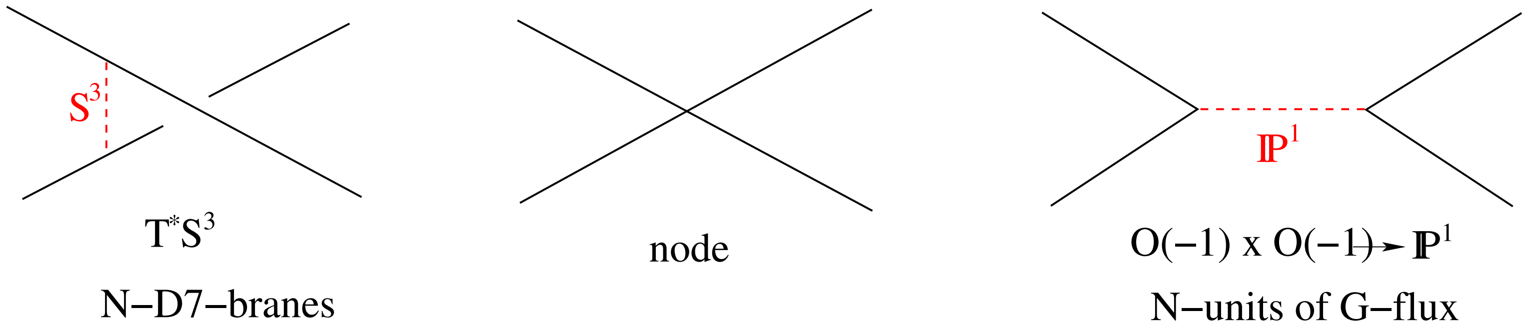
- The Wilson-Loop amplitudes maps to open string amplitudes
- A real slice in B-model moduli space allows to interpolate between strong and weak coupling.

Topological-String on local Calabi-Yau

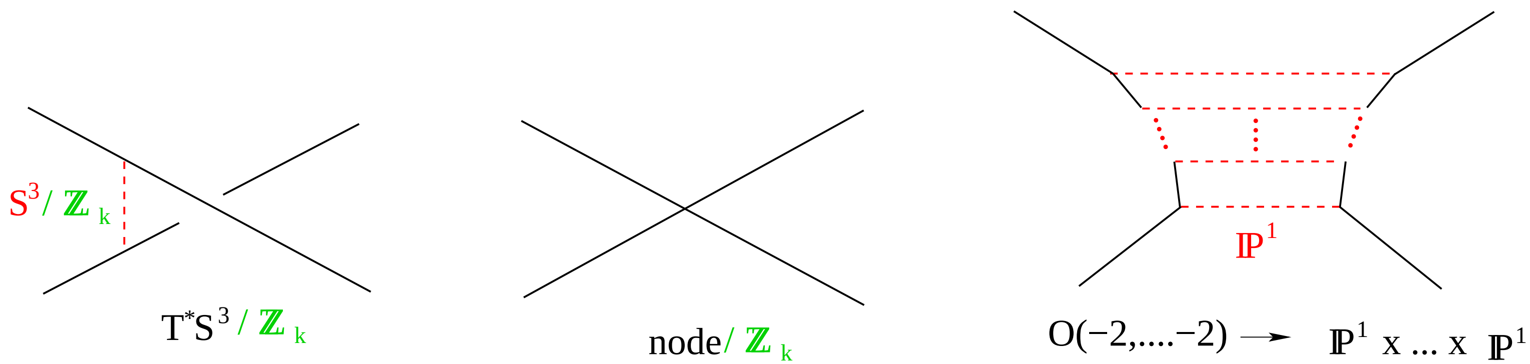
Chain of Chern-Simons Gauge theory/Topological-String large N dualities

- Chern-Simons theory on the real three manifold M_3 is dual to open topological String on the non-compact Calabi-Yau manifold T^*M_3. Witten hep-th/9207094

- Large N Conifold transition Gopakumar Vafa hep-th/9207094



• Lense space transition Aganagic,AK,Marino Vafa hep th/02211098



Toric description of the geometry as moduli space of $(N,N)=(2,2)$ gauged linear σ model by r chiral fields X_i with charge

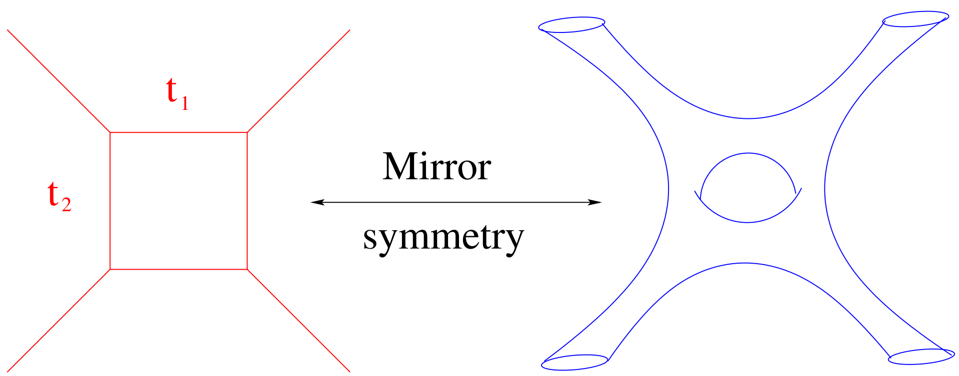
$$Q_j^{(k)}, \quad j=1,\ldots,r, \quad k=1,\ldots,3-r$$

under $U(1)^{(3-p)}$. With $\vec{x} \mapsto \vec{x}\mu_{\vec{(k)}}^{\vec{Q^{(k)}}}$

$$M=\{\mathbb{C}[x_1,\ldots,x_r]\setminus SR\}/(\mathbb{C}^*)^{(3-r)}$$

Mirror geometry: $\vec{Y} \sim \mu \vec{Y}$, $\mu \in \mathbb{C}^*$

$$W = uv + H = uv + \sum_{i=1}^{r} Y_i = 0, \qquad \prod_{i=1}^{r} Y_i^{Q_i^{(k)}} = z_k$$



The ABJM theory: $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$
$Q^{(1)} = (-2,1,1,0,0)$, $Q^{(2)} = (-2,0,0,1,1)$ The mirror

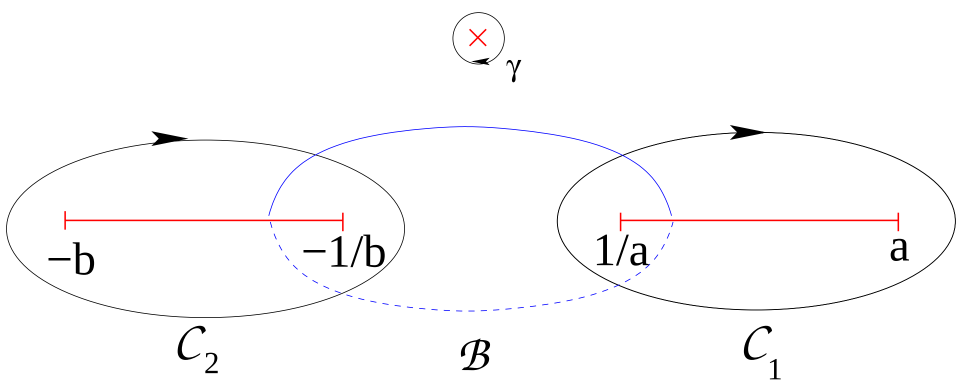
geometry is the genus 1 curve

$$H = 1 + x + y + \frac{z_1}{x} + \frac{z_2}{y} = 0$$

with the meromorphic differentials

$$\mu_k = log(y) X^{(k-1)} dx \ .$$

The Wilson loops integrals:



$$\langle W_n^{\frac{1}{6}} \rangle_{g=0} = \frac{k}{2\pi^2} \int_{\mathcal{C}_1} \mu_n , \qquad \langle \hat{W}_n^{\frac{1}{6}} \rangle_{g=0} = (-1)^n \frac{k}{2\pi^2} \int_{\mathcal{C}_2} \mu_n \ .$$

$$\langle W_n^{\frac{1}{2}} \rangle_{g=0} = \frac{k}{2\pi^2} \oint_{\gamma} \mu_n \ .$$

Note that

$$\langle W_n^{1/2} \rangle = \langle W_n^{1/6} \rangle - (-1)^n \langle \widehat{W}_n^{1/6} \rangle \ .$$

is a relation in homology.

The $1/N^2 \sim g$ expansion:

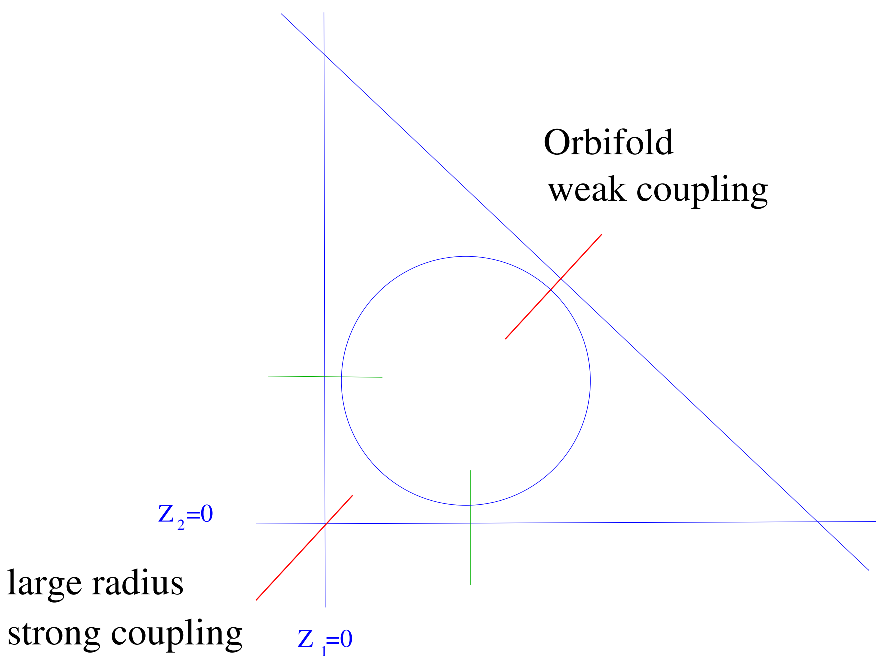
$$\mu_0(x) = \sum_{g=0}^{\infty} g_s^{2g} \mu_0^{(g)}(x) \ , \quad \mu_k^{(g)}(x) = x^k \mu_0^{(g)}(x) \ .$$

Now $\mu_0^{(g)}(x) = W_g(x)dx$ and the $W_g(\vec{p})$ fullfill the Eynard-Oratin recursion

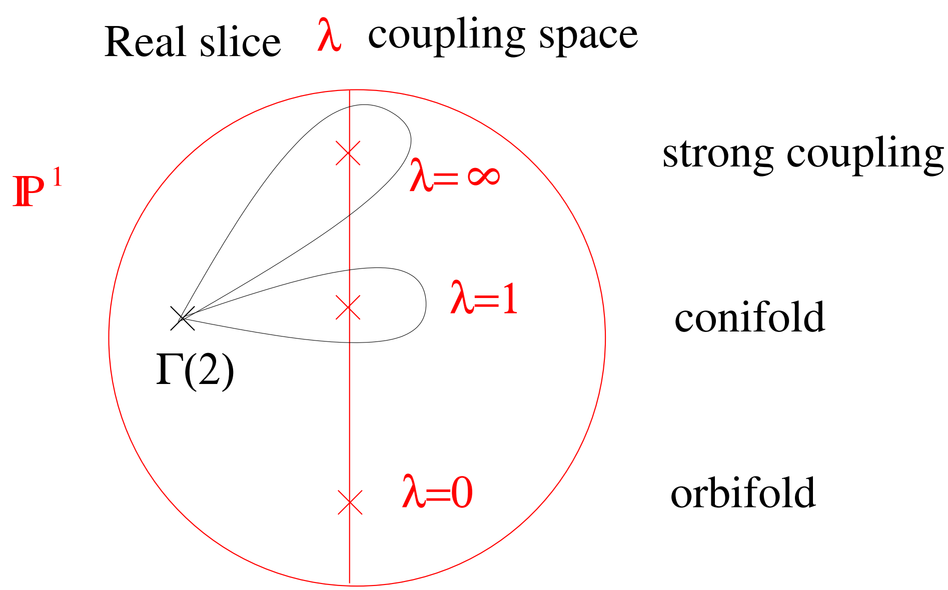
$$\begin{aligned} W_g(p, p_K) = \ & \textstyle\sum_i \mathrm{Res}_{q=x_i} \frac{dS(p,q)}{\tilde{y}(q)} \big(\sum_{h=0}^{g} \sum_{J \subset K} \\ & W_h(q, p_J) W_{g-h}(q, p_{K \backslash J}) + W_{g-1}(q, q, p_K) \big) \ . \end{aligned}$$

The ABJM slice is defined by $\lambda_1 = \lambda_2$ and because of the homological relation between $\lambda_i = \frac{N_i}{k} = \frac{1}{4\pi i} \int_{\mathcal{C}_i} \mu_0^{(0)}$

$$\exp(4\pi i(\lambda_1 - \lambda_2)) = \frac{z_2}{z_1} \ .$$



The ABJM coupling λ lives on the real slice



- The correlations functions are analytically continued without changing the polarisation.

- In particular the $g = 0$ free energy at $\hat{\lambda} \to \infty$

$$F = g_s^{-2} F^{(0)} = \frac{\pi\sqrt{2}}{3} k^2 \hat{\lambda}^{\frac{3}{2}} + \mathcal{O}(\hat{\lambda}^0, e^{-2\pi\sqrt{2\hat{\lambda}}})$$

exhibits the $\hat{\lambda}^{\frac{3}{2}} \sim N^{\frac{3}{2}}$ scaling law.

- The Wilson-loops can be similarly evaluated in low genus at at points in the coupling space.

The Fermigas approach

In contrast the Fermigas approach Marino Putrov, arXiv:1110.4066 sums up the genus expansion. In particular the leading terms in the λ can be calculated in various regions in the coupling space.

The rewritting of the matrix sum in terms of correlators in a free Fermi gas has been explained in Marcos lecture. In particular the Hamiltonian for the ABJM theory is given by

$$H(p,q,\hbar) = \log\left(2\cosh\frac{p}{2}\right) + \log(\left(2\cosh\frac{q}{2}\right) + \mathcal{O}(\hbar^2)$$

and the insertion of the $1/6$ Wilson line with winding n is given by

$$\mathcal{O}_n = \exp\left(\frac{n(p+q)}{k}\right)$$

So one needs to calculate to one body operator

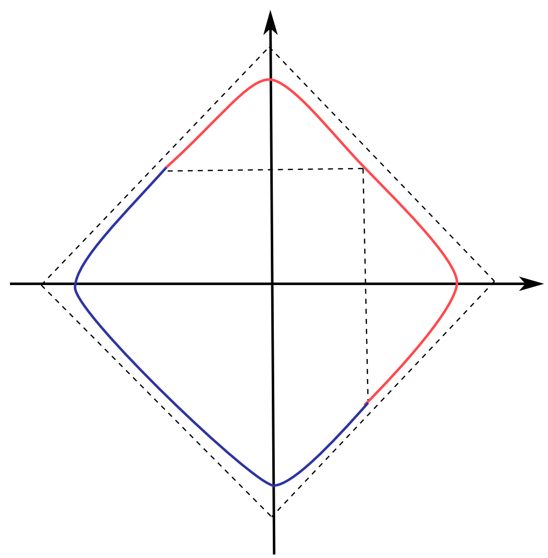
$$\langle \mathcal{O}_n \rangle = \operatorname{tr}\left(\mathcal{O}_n \hat{\rho}_{12} \hat{\rho}_{21}\right)$$

in an ideal Fermi gas of N particles.

The calculation is done in the semiclassical limit, which is very complicated as the Hamiltonian has explicite $\hbar$ corrections to all orders, which however can be resummed up.

One does a low temperature expansion in the grand canonical ensemble and transforms back to the original correlator

$$\langle W_n^{1/6} \rangle = \frac{1}{2\pi i Z} \int d\mu \, e^{-\mu N} \langle \mathcal{O}_n \rangle^{\mathsf{GC}} \; ,$$



The results are of the type

$$\langle W_{\square}^{1/2}\rangle = \frac{1}{4}\csc\left(\frac{2\pi}{k}\right)\frac{\mathsf{Ai}\left[C^{-1/3}\left(N - \frac{k}{24} - \frac{7}{3k}\right)\right]}{\mathsf{Ai}\left[C^{-1/3}\left(N - \frac{k}{24} - \frac{1}{3k}\right)\right]}$$

with

$$C = \frac{2}{\pi^2 k}.$$

Conclusions

- Extension of the Fermigas formalism to Wilson Loops, which leads to exact expressions in $1/N$

- We expand them and compared with B-model approach, thereby checking the Fermigas approach to Wislon loops

- Some properties have been confirmed by numerical studies Hatsuda,Moriyama,Okuyama arXiv1207.4283