New features of scattering amplitudes

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Motivations and outline

Amazing progress in planar $\mathcal{N} = 4$ SYM in recent years

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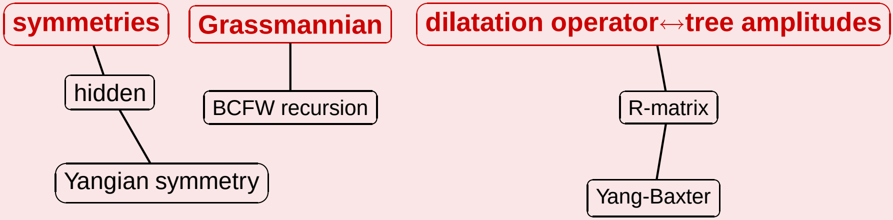


Scattering amplitudes and integrability

spectral parameter?

Scattering amplitudes

On-shell supermultiplet of $\mathcal{N} = 4$ SYM described by a superfield Φ

$$\Phi = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\overline{\Gamma}^{D} + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}$$

- $p^2 = 0 \Longleftrightarrow p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}, \quad q^{\alpha A} = \lambda^{\alpha}\eta^{A}$
- on-shell superspace: $(\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$

Expansion in powers of Grassmann parameters η:

$$\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}_n^{\mathrm{MHV}} + \mathcal{A}_n^{\mathrm{NMHV}} + \ldots + \mathcal{A}_n^{\overline{\mathrm{MHV}}}$$

MHV tree-level[Parke, Taylor]

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \quad \langle \mathrm{ij}\rangle = \lambda_i^{\alpha}\lambda_{j\alpha}$$

Symmetries

$\mathcal{N} = 4$ SYM symmetries:

- Superconformal symmetry: expected [Witten]

$$j_a \mathcal{A}_n^{\text{tree}} = 0, \qquad j_a \in \mathfrak{psu}(2,2|4)$$

- Dual superconformal symmetry: hidden [Drummond, Henn, Korchemsky, Sokatchev]
  - dual space: $x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^{A}$
  - not related to ordinary superconformal symmetry

$$J'_a \mathcal{A}_n^{\text{tree}} = 0, \qquad J'_a \in \mathfrak{psu}(2,2|4)^{\text{dual}}$$

Superconformal + dual superconformal algebras: [Drummond, Henn, Plefka]

Yangian structure

$\downarrow$

Hint of integrability

Yangian symmetry

- Level-zero generators: $[j_a, j_b] = f_{ab}{}^c j_c$

  $\rightarrow$ superconformal symmetry
- Level-one generators: $[j_a, j_b^{(1)}] = f_{ab}{}^c j_c^{(1)}$

  $\rightarrow$ dual superconformal symmetry
- Higher commutators constrained by the Serre relation

The Yangian $Y(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is generated by j and $j^{(1)}$

Full symmetry of the **tree-level** amplitudes

$$y\,\mathcal{A}_n^{\text{tree}} = 0$$

for any $y \in Y(\mathfrak{psu}(2,2|4))$. T-dual representation [Drummond, L.F.]

Broken beyond tree level

Integrability

- Another realization of $Y(\mathfrak{g})$ is given by the RTT definition
- Another approach to the Yangian algebra: monodromy matrices

$$\mathcal{M}(z) = R_1(z)...R_L(z)$$

are generating functions for all levels of the Yangian generators

$$\mathcal{M}(z)^{\mathcal{A}}_{\mathcal{B}} = \sum_{i=0}^{\infty} \frac{1}{z^i} (j^{(i)})^{\mathcal{A}}_{\mathcal{B}}$$

- R-matrices are the essential ingredients for all integrable models
- In order to find $R(z)$ we have to solve the Yang-Baxter equation

The Question:
is this something more than a hint?

Recent relation between the dilatation operator for the spin chain and tree-level scattering amplitudes [Zwiebel]

$$\langle \Lambda_1 \Lambda_2 | \mathcal{D}_{L\to 2} | \Lambda_3 \dots \Lambda_{L+2} \rangle = \mathcal{A}_{L+2}(\Lambda_1, \dots, \Lambda_{L+2})$$

Nearest-neighbor Hamiltonian $\mathcal{H}$ of spin chain $\equiv$ one-loop planar dilatation generator of $\mathcal{N} = 4$ SYM

Spin chain		Amplitudes
$\mathcal{H} = \mathcal{D}_{2\to 2}$	$\overset{Zwiebel}{\longleftrightarrow}$	$\mathcal{A}_4$
$\uparrow d \log$		$\uparrow$
$R(z)$	$\longleftrightarrow$	?

- How is this related to the Grassmannian formulation of amplitudes?

In twistor space $\mathcal{Z}_i^{\mathcal{A}} = (\tilde{\mu}_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^{A})$

$$\mathcal{A}_{n,k}^{\text{tree}} = \oint \frac{\prod_{a=1}^{k} \prod_{i=1}^{n} dc_{ai}}{\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4} \left(\sum_{i=1}^{n} c_{ai} \mathcal{Z}_i^{\mathcal{A}} \right)$$

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan, Postnikov, Trnka] [Mason, Skinner]

- c_{ai}: complex parameters forming a $(k \times n)$ matrix $\mathcal{C}$
- $\mathcal{M}_p = (p \ldots p+k-1)$: determinant of $(k \times k)$ submatrix of the c_{ai}'s
- Yangian invariant up to total derivatives [Drummond, L.F.]

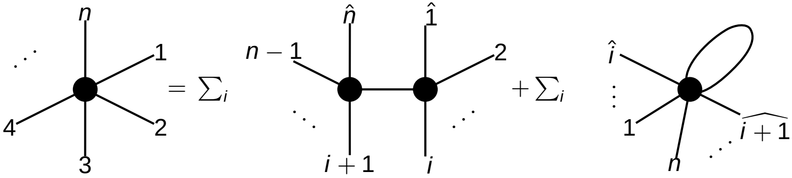
Example: **MHV four points**

$$\mathcal{A}_{4,2} = \int \frac{dc_{13} dc_{14} dc_{23} dc_{24}}{c_{13} c_{24} (c_{13} c_{24} - c_{14} c_{23})} \delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3 + c_{14}\mathcal{Z}_4) \delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3 + c_{24}\mathcal{Z}_4)$$

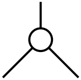
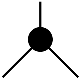
$$\mathcal{C} = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}$$

Building blocks for amplitudes

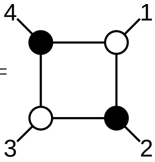
All-loop recursion relation [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]



Using BCFW recursion relations, all amplitudes can be written employing only two on-shell diagrams:

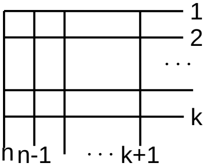
$\mathcal{A}_{3,1} =$  $\mathcal{A}_{3,2} =$ 

Example: **MHV four points tree level**

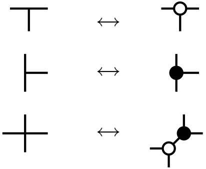
$\mathcal{A}_{4,2} =$ 

Amplitudes construction

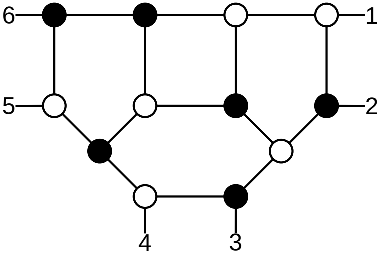
For tree level:



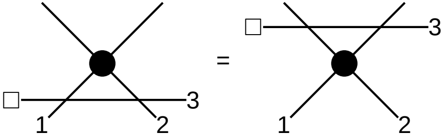
Dictionary[Postnikov]



$\mathcal{A}_{6,3} =$



Yang-Baxter equation:



$$R_{12}(z_1 - z_2)R_{13}(z_1)R_{23}(z_2) = R_{23}(z_2)R_{13}(z_1)R_{12}(z_1 - z_2)$$

with $R^{A}_{13,B}(z_1) = z_1 \delta^A_B + J^A_{1\,B}$ with ansatz

$$\mathcal{R}(z) = \oint \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24} - c_{14}c_{23})} F(C;z) \delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3 + c_{14}\mathcal{Z}_4)\delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3 + c_{24}\mathcal{Z}_4)$$

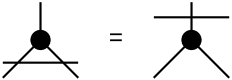
and the fact that all particles have vanishing central charge leads to the answer

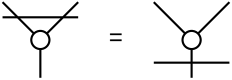
$$F(C;z) = \left(\frac{c_{13}c_{24}}{c_{13}c_{24} - c_{14}c_{23}}\right)^z$$

Spectral deformation of $\mathcal{A}_{4,2}$

Building blocks for R(z)

MHV R-matrix – $R_\bullet(z_1, z_2)$ and $\overline{\text{MHV}}$ R-matrix – $R_\circ(z_1, z_2)$





They satisfy equations similar to Yang-Baxter:

$$\begin{aligned} z_1 (J_1)^A_C R_\bullet(z_1,z_2) &= R_\bullet(z_1,z_2)(J_1)^A_B(z_1\delta^B_C + (J_2)^B_C) \\ (J_1)^A_B(z_1\delta^B_C + (J_2)^B_C) R_\circ(z_1,z_2) &= z_1 R_\circ(z_1,z_2)(J_1)^A_C \end{aligned}$$

A second set of equations with $(1 \leftrightarrow 2)$ leads to a second spectral parameter z_2

$$\begin{aligned} \mathcal{R}_\bullet(z_1,z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_1^{\mathcal{A}} + c_1 \mathcal{Z}_3^{\mathcal{A}}) \delta^{4|4}(\mathcal{Z}_2^{\mathcal{A}} + c_2 \mathcal{Z}_3^{\mathcal{A}}) \\ \mathcal{R}_\circ(z_1,z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_3^{\mathcal{A}} + c_1 \mathcal{Z}_1^{\mathcal{A}} + c_2 \mathcal{Z}_2^{\mathcal{A}}) \end{aligned}$$

Interpretation of the spectral parameters

$$\mathcal{R}_\bullet(z_1,z_2) = \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_1^{\mathcal{A}} + c_1 \mathcal{Z}_3^{\mathcal{A}}) \delta^{4|4}(\mathcal{Z}_2^{\mathcal{A}} + c_2 \mathcal{Z}_3^{\mathcal{A}})$$

$$\mathcal{R}_\circ(z_1,z_2) = \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_3^{\mathcal{A}} + c_1 \mathcal{Z}_1^{\mathcal{A}} + c_2 \mathcal{Z}_2^{\mathcal{A}})$$

- 3-point amplitudes are very singular objects: they vanish when demanding all legs to be on-shell, with vanishing central charges, total momentum conserved and real momenta
- non-trivial 3-point amplitudes: complexify momenta
- non-trivial 3-point R-matrices: relax also the central charge constraints

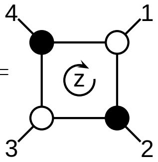
$$\begin{aligned}
(J_1)^A_A \mathcal{R}_\circ(z_1,z_2) &= z_1\, \mathcal{R}_\circ(z_1,z_2) \\
(J_2)^A_A \mathcal{R}_\circ(z_1,z_2) &= z_2\, \mathcal{R}_\circ(z_1,z_2) \\
\mathcal{R}_\circ(z_1,z_2)(J_3)^A_A &= (z_1+z_2)\mathcal{R}_\circ(z_1,z_2)
\end{aligned}$$

Interpretation of the spectral parameters

The spectral parameters have the interpretation of unphysical-particle helicities

- For 4 points we get one non-vanishing spectral parameter after demanding that outer particles have vanishing central charge

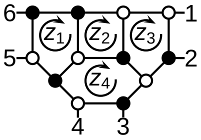
$\mathcal{R}_{4,2} =$



$$= \oint \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24}-c_{14}c_{23})} \left(\frac{c_{13}c_{24}}{c_{13}c_{24}-c_{14}c_{23}}\right)^{z} \delta^{4|4}(\mathcal{C}\cdot\mathcal{Z})$$

- For higher-point R-matrices the number of parameters grows and is equal to the number of loops in the on-shell diagram

$\mathcal{R}_{6,3} =$



$$= \oint \frac{d^9C}{\mathcal{M}_1\mathcal{M}_2\ldots\mathcal{M}_6} \delta^{4|4}(\mathcal{C}\cdot\mathcal{Z}) \left(\frac{c_{36}(c_{16}c_{25}-c_{15}c_{26})}{c_{16}(c_{26}c_{35}-c_{25}c_{36})}\right)^{z_1} \left(\frac{c_{15}c_{26}\det C}{c_{16}(c_{14}c_{25}-c_{15}c_{24})(c_{25}c_{36}-c_{26}c_{35})}\right)^{z_2} \left(\frac{c_{14}(c_{15}c_{26}-c_{16}c_{25})}{c_{16}(c_{14}c_{25}-c_{15}c_{24})}\right)^{z_3} \left(\frac{c_{15}c_{26}}{(c_{16}c_{25}-c_{15}c_{26})}\right)^{z_4}$$

Loop amplitudes

- One can write any loop amplitude using on-shell diagrams

$$\mathcal{A}_{n,k}^{\ell} \longrightarrow \mathcal{A}_{n-2,k-1}^{\ell+1}$$

- The simplest example is $\mathcal{A}_{6,3}^{\text{tree}} \longrightarrow \mathcal{A}_{4,2}^{\text{1-loop}}$. One gets:

$$\mathcal{A}_{4,2}^{\text{1-loop}} = \mathcal{A}_{4,2}^{\text{tree}} \int \frac{d^4 q}{q^2 (q+p_1)^2 (q+p_1+p_2)^2 (q+p_1+p_2+p_3)^2}$$

- Tree-level amplitudes factorize, and we are left with the box integral (B)
- The box integral is known to be divergent $\rightarrow$ one uses dimensional or mass regularization to calculate it

$$B \sim \frac{2}{\epsilon^2}\left(\left(\frac{s}{\mu^2}\right)^{-\epsilon} + \left(\frac{t}{\mu^2}\right)^{-\epsilon}\right) - \log^2\left(\frac{s}{t}\right) - \frac{4\pi^2}{3}$$

Exciting possibility: spectral regularization

- Use the spectral parameter(s) to regulate loop amplitudes in a novel way
- Should provide a symmetry-preserving regularization scheme
- In particular, we stay in exactly four dimensions: replace dimensional regularization by spectral regularization
- Currently under active investigation

Summary and conclusions

- We constructed a new class of objects which are deformations of the $\mathcal{N} = 4$ amplitudes
- We introduced spectral parameters into the scattering amplitude problem

What is it good for?

- Analyticity requirements in the spectral parameter plane should fix contours of integrations
- Possibility to provide a novel symmetry-preserving regulator
- Should allow to establish the exact link between the amplitude and the spectral problem

Outlook:

- Introduction of the spectral parameter was the key to solve the AdS/CFT all-loop spectral problem (ABA, TBA, Y-system). Hopefully history will repeat itself!