Algebraic curve for Wilson loops and correlation functions

Romuald A. Janik

Jagiellonian University
Kraków

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Outline

Introduction

The AdS/CFT dictionary:

Anomalous dimensions of operators in $\mathcal{N} = 4$ SYM	$\equiv$	Energies of corresponding string states in $AdS_5 \times S^5$

- Currently we have a very good understanding of these quantities for any value of the gauge theory coupling λ

  see talks by Frolov, Kazakov

- At strong coupling, $\lambda \to \infty$, many operators with large R-charges (spins) $\sim \sqrt{\lambda}$ correspond to *classical* spinning string solutions in $AdS_5 \times S^5$
- There is a very complete classification (more precisely a dense subset thereof) of spinning string solution:

  finite-gap solutions classified by algebraic curve constructions

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Finite gap spinning string solutions were important in the development of the field:

- Very general classification of solutions — more than a set of examples
- Formulas resemble the classical limit of Bethe equations
- Comparision with classical limit of *weak coupling* spin chain Bethe equations was helpful for extrapolation to all loops
- Classical solutions generate approximate solutions of the Y-system (transfer matrices $\equiv$ characters of the monodromy)

Chief motivation:
How much of this structure can be carried over to other physically interesting string configurations relevant for other gauge theory observables: *Wilson loops/correlation functions*

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Classical integrability of a string in AdS_3

- Write the AdS_3 σ-model as

$$S_{AdS_3} = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{tr}\, j\bar{j} d^2 w$$

where

$$j = g^{-1}\partial g \qquad \bar{j} = g^{-1}\bar{\partial} g \quad \text{and} \quad g = \begin{pmatrix} \frac{ix_1+x_2}{z} & \frac{1}{z} \\ -\frac{x_1^2+x_2^2+z^2}{z} & \frac{ix_1-x_2}{z} \end{pmatrix}$$

- Introduce the family of currents parametrized by the spectral parameter x

$$J = \frac{j}{1-x} \qquad \bar{J} = \frac{\bar{j}}{1+x}$$

- **Classical integrability** amounts to the flatness condition (for all x)

$$\partial \bar{J} - \bar{\partial} J + [J, \bar{J}] = 0$$

which is equivalent to the equations of motion of the AdS_3 σ-model

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is a **local** notion. i.e. it does not depend on the topology of the string worldsheet!

The (conventional) appearance of the algebraic curve

Kazakov, Marshakov, Minahan, Zarembo

- The monodromy is given by

$$\Omega = Pe^{\int J}$$

- When we move the reference point Ω changes as

$$\Omega \longrightarrow U\Omega U^{-1}$$

- Hence the eigenvalues $e^{ip(x)}, e^{-ip(x)}$ remain constants of motion

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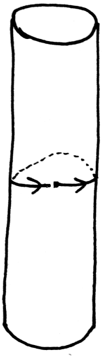

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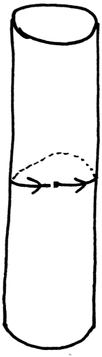

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- Define an operator associated to the monodromy

$$L_{monodromy}(w, \bar{w}; x) = -i \frac{\partial}{\partial x} \log \Omega(w, \bar{w}; x)$$

- $L_{monodromy}(w, \bar{w}; x)$ is a 2×2 matrix with entries being *rational* functions of the spectral parameter x
- The associated algebraic curve is obtained as

$$\det(\tilde{y} \cdot 1 - L_{monodromy}(w, \bar{w}; x)) = 0$$

- From the above algebraic curve, one can reconstruct the pseudomomentum $p(x)$, extract the energy/conserved charges and, if one wants, reconstruct the original target space solution

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Remarks and Questions

- The construction of the algebraic curve associated to a spinning string solution is based on the monodromy matrix
- Hence it makes crucial use of the existence of a noncontractible loop on a cylinder
- This is not the case for Wilson loops...

Q1: Is it possible to construct algebraic curves for Wilson loops? Are there any algebraic curves associated with classical examples (e.g. null cusp or the $q\bar{q}$ potential)

Q2: Can one construct an algebraic curve using only **local** notions?

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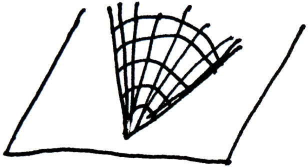

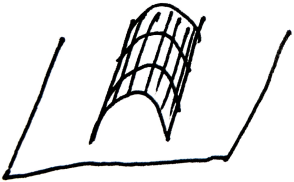

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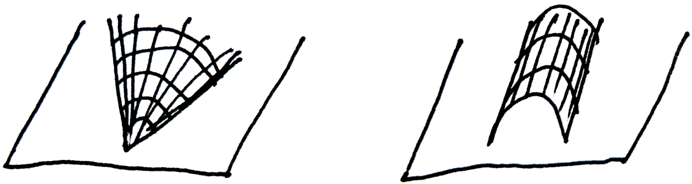

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Q3: Is it possible to treat closed string and open string solutions on the same footing?

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How do 3-point correlation functions of local operators fit into the picture???

Folklore: For a genus g algebraic curve, the solutions are essentially parametrized by the Jacobian $\equiv$ a g-dimensional torus

$\longrightarrow$ We expect a much richer structure!

AdS/CFT: solution with fixed monodromy $\longrightarrow$ particular operator in gauge theory $\longrightarrow$ can appear in arbitrary correlation functions!

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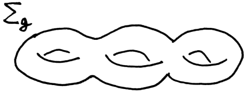

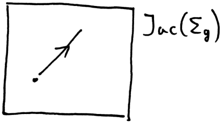

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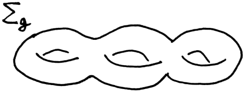

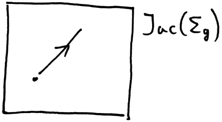

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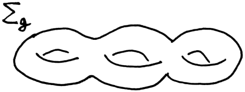

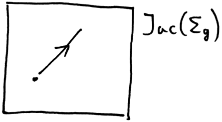

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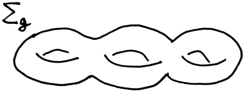

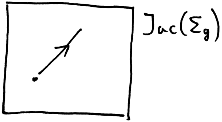

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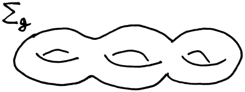

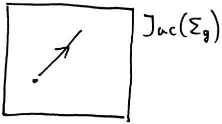

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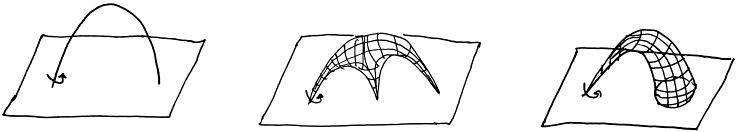

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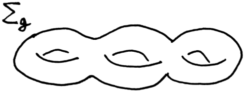

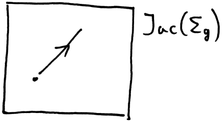

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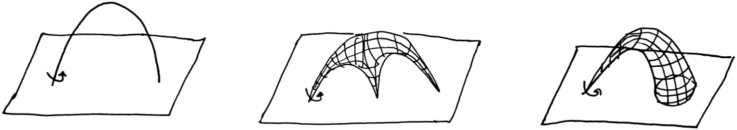

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Compare $\langle \mathrm{tr}\, Z^J \mathrm{tr}\, \bar{Z}^J \rangle$ and $\langle \mathrm{tr}\, Z^J W(C) \rangle$:

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Open problem: 3-point correlation function in $\mathcal{N} = 4$ SYM

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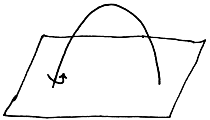

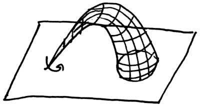

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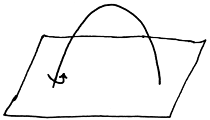

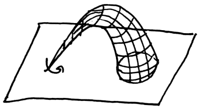

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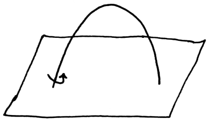

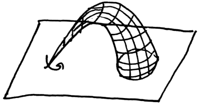

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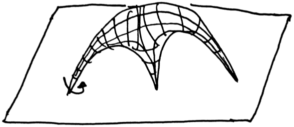

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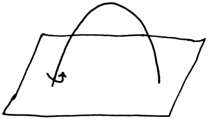

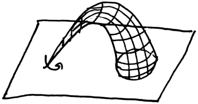

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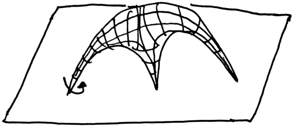

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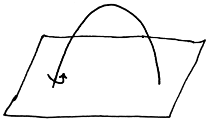

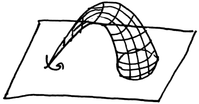

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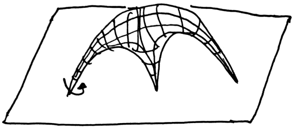

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How to construct an algebraic curve from local data

- Recall the construction for spinning strings based on the monodromy Ω

$$L_{monodromy}(w, \bar{w}; x) = -i \frac{\partial}{\partial x} \log \Omega(w, \bar{w}; x)$$

- $L_{monodromy}(w, \bar{w}; x)$ had *rational* dependence on x
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Interlude

- Recall that the string solution defines for us the family of flat currents $J(w, \bar{w}; x)$ and $\bar{J}(w, \bar{w}; x)$
- It is very useful to construct an *associated linear system* by the linear differential equations for a 2-component vector Ψ:

$$\partial \Psi + J \Psi = 0 \qquad \bar{\partial} \Psi + \bar{J} \Psi = 0$$

- The compatibility condition for the local existence of a solution is just the flatness condition
- We generically have 2 independent solution $\longrightarrow$ put them into columns of a 2×2 matrix $\hat{\Psi}(w, \bar{w}; x)$ which satisfies the matrix differential equations

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- The knowledge of $\hat{\Psi}(w, \bar{w}; x)$ encodes full knowledge about the original string solution
- The original currents can be reconstructed from

$$j = -\partial \hat{\Psi} \cdot \hat{\Psi}^{-1}|_{x=0}$$

- The original string solution can be also obtained Dorey, Vicedo

$$g(w, \bar{w}) = \sqrt{\det \hat{\Psi}} \cdot \hat{\Psi}^{-1}|_{x=0}$$

- This is really useful since there exists a procedure of reconstructing $\hat{\Psi}(w, \bar{w}; x)$ just from the algebraic curve (+ some mild assumptions) without solving *any* differential equations
- We will use $\hat{\Psi}(w, \bar{w}; x)$ to construct Lax matrices...

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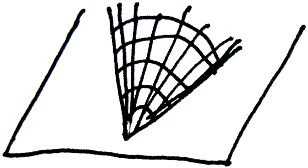

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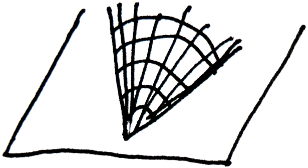

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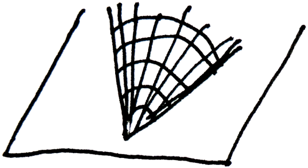

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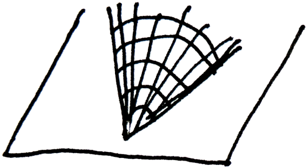

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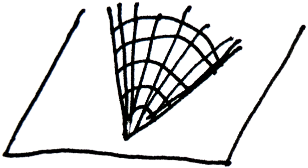

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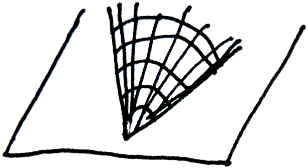

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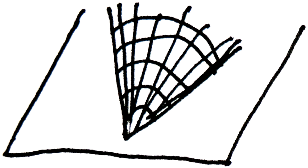

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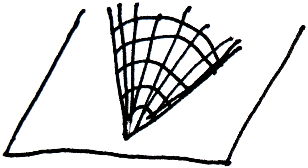

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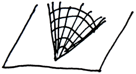

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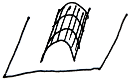

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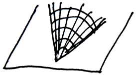
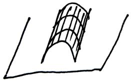
 $\longrightarrow$ $y^2 = 1 - x^2$

 $\longrightarrow$ $y^2 = x(1 - x^2)$

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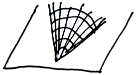
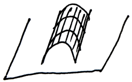
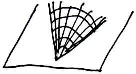
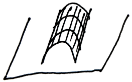
 $\longrightarrow$ $y^2 = 1 - x^2$

 $\longrightarrow$ $y^2 = x(1 - x^2)$

Conversely, we can reconstruct the original solutions just from these algebraic curves

$y^2 = 1 - x^2$ $\longrightarrow$ 

$y^2 = x(1 - x^2)$ $\longrightarrow$ 

Open and closed string solutions

- Our method of constructing algebraic curves is purely local — it does not depend on the topology of the worldsheet
- In particular it applies equally well to open and closed string solutions (Wilson loops and spinning strings)
- How does the relation between the GKP folded string and null cusp appear in this context?

$$\underbrace{y^2 = (x^2-1)(x^2-a^2)}_{\text{GKP folded string}} \quad \stackrel{a\to\infty}{\longrightarrow} \quad \underbrace{y^2 = (x^2-1)}_{\text{null cusp}}$$

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$$\langle \mathrm{tr}\, \bar{Z}^J \mathrm{tr}\, Z^J \rangle \qquad \text{and} \qquad \langle W(C) \mathrm{tr}\, Z^J \rangle$$

- These are both closed string solution with *identical* monodromy (pseduomomentum) — genus $g = 0$ case

- Simpler to make a conformal transformation

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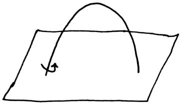

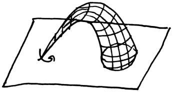

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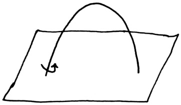

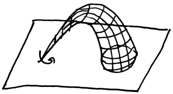

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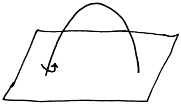

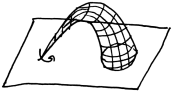

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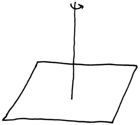

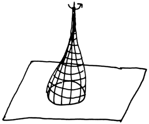

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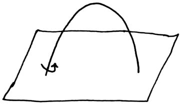

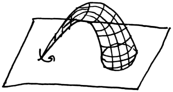

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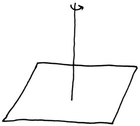

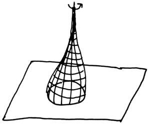

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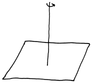

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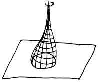

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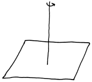

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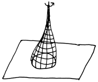

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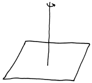

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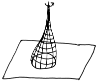

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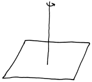

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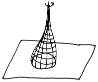

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- The construction treated open and closed string solutions on the same footing
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