AGT and the topological string

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AGT vs. the topological string

- AGT is a relation between CFT and $\mathcal{N} = 2$ gauge theory in an Ω background, (in its original form) exact in the ϵ-parameters, but in an instanton expansion in the gauge coupling.
- Topological string techniques (the holomorphic anomaly) yield results that are exact in the gauge coupling, but in an ϵ expansion.

What can we learn by comparing the CFT and the holomorphic anomaly approach?

Work in progress with Jan Troost.

Outline

Outline

The protagonist

$$Z_{top} = \exp \left[\sum_{n,g} F^{(n,g)} g_s^{2g-2} s^n \right]$$

Topological string

- $F^{(0,g)}(t)$ counts maps: $\Sigma_g \to X$.
- $F^{(n,g)}(t)$, $n > 0$ does not have a topological string interpretation yet.
- $g_s^2 = \epsilon_1 \epsilon_2$, $s = \epsilon_1 + \epsilon_2$.

The protagonist

$$Z_{top} = \exp\left[\sum_{n,g} F^{(n,g)} g_s^{2g-2} s^n\right]$$

Gauge theory

- $F^{(0,0)}(a)$ is the prepotential: governs special geometry of $\mathcal{N}=2$ theory, e.g. $\tau_{IR} = \frac{d^2F^{(0,0)}}{da^2}$.
- $F^{(0,g)}(a)$, $g > 0$ is graviphoton-curvature coupling when theory is coupled to a gravitational background:

$$F^{(0,g)} R_+^2 F_{gp}^{2g-2}$$

- $F^{(n,g)}(a)$, $n > 0$: field theory interpretation not yet settled.

Instanton calculus

Localization via rotations in spacetime with ϵ_1, ϵ_2 equivariant parameters computes Z_{inst} ($Z_{top} = Z_{pert} Z_{inst}$),

$$Z_{\text{inst}}(a,\epsilon_1,\epsilon_2) = \sum_{Y_1,Y_2} q_{inst}^{|Y_1|+|Y_2|}$$

$$\times \prod_{(i,j)\in Y_1} \frac{\prod_{k=1}^{N_f}(a+\epsilon_1(i-1)+\epsilon_2(j-1)+m_k)}{E_{i,j}^{Y_1,Y_1}(0)(\epsilon - E_{i,j}^{Y_1,Y_1}(0))E_{i,j}^{Y_1,Y_2}(2a)(\epsilon - E_{i,j}^{Y_1,Y_2}(2a))}$$

$$\times \prod_{(i,j)\in Y_2} \frac{\prod_{k=1}^{N_f}(-a+\epsilon_1(i-1)+\epsilon_2(j-1)+m_k)}{E_{i,j}^{Y_2,Y_2}(0)(\epsilon - E_{i,j}^{Y_2,Y_2}(0))E_{i,j}^{Y_2,Y_1}(-2a)(\epsilon - E_{i,j}^{Y_2,Y_1}(-2a))},$$

where

$$E_{i,j}^{Y_1,Y_2}(a) \equiv a + \epsilon_1(Y_{1,j}^T - i + 1) - \epsilon_2(Y_{2,i} - j)\,.$$

The holomorphic anomaly recursion

$F^{(0,1)}$, $F^{(1,0)}$ determined via geometry

$$F^{(0,1)} = -\frac{1}{2}\log(G_{u\bar{u}}|\Delta|^{\frac{1}{3}}) \,, \quad F^{(1,0)} = \frac{1}{24}\log(\Delta) \,.$$

$F^{(n,g)}$ for $n+g>1$ determined by recursion (supplemented by boundary condition at conifold points),

$$\bar{\partial}_{\bar{i}} F^{(n,g)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(D_j D_k F^{(n,g-1)} + {\sum_{m,h}}' D_j F^{(m,h)} D_k F^{(n-m,g-h)}\right) .$$

UV vs. IR

Seiberg-Witten curve for massless $N_f = 4$,

$$y^2 = (x - u\, e_1(\tau))(x - u\, e_2(\tau))(x - u\, e_3(\tau))$$

$$\Updownarrow$$

$$y^2 = x(x-u)(x - u\, q_{inst})\,,\; q_{inst} = \frac{e_3 - e_2}{e_1 - e_2}(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

Here, q_{inst} is the instanton expansion parameter (UV-parameter), τ is the effective coupling (IR-parameter).

Explicit results for massless $N_f = 4$

$$F^{(2,0)} = \frac{E_2(\tau)}{2^5 3 a^2}\,, \quad F^{(1,1)} = -\frac{E_2(\tau)}{2^3 3 a^2}\,, \quad F^{(0,2)} = \frac{E_2(\tau)}{2^5 a^2}\,,$$

$$\begin{aligned}
F^{(3,0)} &= -\frac{1}{2^8 3^2 5 a^4}(5E_2^2 + 13E_4)(\tau)\,, \\
F^{(2,1)} &= \frac{1}{2^6 3^2 5 a^4}(10E_2^2 + 17E_4)(\tau)\,, \\
F^{(1,2)} &= -\frac{1}{2^8 3^2 5 a^4}(95E_2^2 + 94E_4)(\tau)\,, \\
F^{(0,3)} &= \frac{1}{2^7 3 a^4}(2E_2^2 + E_4)(\tau)\,.
\end{aligned}$$

Explicit results for massless $N_f = 4$

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General structure

$$F^{(n,g)} = \frac{1}{a^{2(g+n)-2}} p_{2(g+n)-2}(E_2, E_4, E_6)(\tau)$$

Outline

1. Z_{top} and how to compute it via the holomorphic anomaly
2. Review of AGT
3. AGT and topological string

Degenerate weights

- Conformal primary: $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $L_k|\Delta\rangle = 0$ for $k > 0$
- Descendant of level n: $L_{-i_1} \cdots L_{-i_k}|\Delta\rangle$, $\sum_m i_m = n$
- Null vector at level n: a descendant that is also a primary

A descendant χ at level mn of the weight

$$\Delta_{mn} = \frac{c-1}{24} + \frac{1}{4}(m\,\alpha_+(c) + n\,\alpha_-(c))^2$$

is a null vector: $L_0|\chi\rangle = (\Delta_{mn} + mn)|\chi\rangle$, $L_k|\chi\rangle = 0$ for $k > 0$.

Δ_{mn} are called degenerate weights.

Conformal blocks I

ϕ_1, ϕ_2 primaries:

$$\phi_1(z)\phi_2(0) = \sum_{\Delta} z^{\Delta-\Delta_1-\Delta_2}\Psi_\Delta C_{12}^{\Delta}$$

with

$$\Psi_\Delta = \phi_\Delta(0) + z\,\beta_{12}^{\Delta,\{1\}} L_{-1}\phi_\Delta(0) + \bar{z}\,\beta_{12}^{\bar{\Delta},\{1\}} \bar{L}_{-1}\phi_\Delta(0) + \cdots$$

The coefficients $\beta_{12}^{\Delta,\{i_1,\ldots,i_n\}}$ are determined by conformal symmetry alone, and are the essential ingredients in conformal blocks.

Conformal blocks II

To define the 4-point conformal block, map $(z_1, z_2, z_3, z_4) \to (0, x, 1, \infty)$,

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \prod z_{ij}^{\gamma_{ij}} \bar{z}_{ij}^{\bar{\gamma}_{ij}} \langle \Delta_4|\phi_3(1)\phi_2(x)|\Delta_1\rangle$$

and substitute OPE

$$\begin{aligned} G_{43}^{21} &= \langle \Delta_4|\phi_3(1)\phi_2(x)|\Delta_1\rangle \\ &= \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} \langle \Psi_{\Delta}^{43}|\Psi_{\Delta}^{21}\rangle x^{\Delta-\Delta_1-\Delta_2} \bar{x}^{\Delta-\Delta_1-\Delta_2} \\ &= \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} \, |x^{\Delta-\Delta_1-\Delta_2+\sum k_i}|^2 \\ & \quad | \sum_{\{k\},\{k'\}} \beta_{43}^{\Delta,\{k\}} \beta_{21}^{\Delta,\{k'\}} \langle \Delta|L_{k_1} \cdots L_{k_n} L_{-k'_1} \cdots L_{-k'_m}|\Delta\rangle|^2 \end{aligned}$$

Conformal blocks III

$\Rightarrow$ We have split the four point function

$$G_{43}^{21} = \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} \, |F_{\Delta}(c, \Delta_i|x)|^2$$

into a CFT specific contribution $C_{43}^{\Delta} C_{21}^{\Delta}$ and a generic contribution

$$F_{\Delta}(c, \Delta_i|x) = \sum_{\{k\},\{k'\}} \beta_{43}^{\Delta,\{k\}} \beta_{21}^{\Delta,\{k'\}} \langle \Delta | L_{k_1} \cdots L_{k_n} L_{-k'_1} \cdots L_{-k'_m} | \Delta \rangle \times x^{\Delta - \Delta_1 - \Delta_2 + \sum k_i} \, .$$

The AGT dictionary

AGT

The four-point conformal block is equal to the instanton part of the partition function of $N_f = 4$ Seiberg-Witten theory:

$$F_{\Delta}(c, \Delta_i | x) = Z_{inst}(\epsilon_1, \epsilon_2, a, m_i, q_{inst})(1-x)^{-\nu}$$

With $Q = b + \frac{1}{b}$ such that $c = 1 + 6Q^2$,

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}, \quad \Delta = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1 \epsilon_2}, \quad \Delta_i = \frac{Q^2}{4} - \frac{m_{1/4} \pm m_{2/3}}{\epsilon_1 \epsilon_2},$$

$$x = q_{inst}\,.$$

How to compute conformal blocks

The equations determining $\beta_{12}^{\Delta,\{k\}}$ lead to poles when Δ is a degenerate weight.

Given c, Δ is degenerate at

$$\Delta_{mn} = \frac{c-1}{24} + \frac{1}{4}(m\,\alpha_+(c) + n\,\alpha_-(c))^2 \,.$$

To render a fixed Δ degenerate, c must be chosen as

$$c_{mn} = 13 - 6(\alpha_+^2(\Delta) + \alpha_-^2(\Delta)) \,.$$

Recursion relations for $F_\Delta(c, \Delta_i|x)$ exist which make either the poles in c or those in Δ manifest.

The large c-recursion for conformal blocks

$$F_\Delta(c,\Delta_i|x) = f_\Delta(\Delta_i|x) + \sum_{m,n} \frac{R'_{mn}(\Delta,\Delta_i)}{c - c_{mn}(\Delta)} F_{\Delta+mn}(c_{mn},\Delta_i|x)$$

where

$$\begin{aligned} f_\Delta(\Delta_i|x) &= \lim_{c\to\infty} F_\Delta(c,\Delta_i|x) \\ &= x^{\Delta-\Delta_1-\Delta_2}\,{}_2F_1(\Delta+\Delta_2-\Delta_1,\Delta+\Delta_3-\Delta_4,2\Delta,x) \\ &= x^{\Delta-\Delta_1-\Delta_2}(1+\mathcal{O}(x)) \end{aligned}$$

$\Rightarrow$ yields recursion relation in x.

AGT use this recursion to compare to gauge theory results calculated via Nekrasov formula, which yields expansion in q_{inst}.

Outline

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With $Q = b + \frac{1}{b}$ such that $c = 1 + 6Q^2$,

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,, \quad \Delta = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1\epsilon_2}\,, \quad \Delta_i = \frac{Q^2}{4} - \frac{m_{1/4} \pm m_{2/3}}{\epsilon_1\epsilon_2}\,,$$

$$x = q_{inst}\,.$$

The large Δ-recursion for conformal blocks

$$\begin{aligned} F_\Delta(c,\Delta_i|x) &= (16y)^{\Delta-\frac{c-1}{24}} x^{\frac{c-1}{24}-\Delta_1-\Delta_2}(1-x)^{\frac{c-1}{24}-\Delta_1-\Delta_3} \\ &\quad \theta_3^{\frac{c-1}{2}-4\sum_{i=1}^4 \Delta_i}(y) H_\Delta(c,\Delta_i|y) \end{aligned}$$

with

$$H_\Delta(c,\Delta_i|y) = 1 + \sum_{m,n} (16y)^{mn} \frac{R_{mn}(c,\Delta_i)}{\Delta-\Delta_{mn}(c)} H_{\Delta_{mn}+mn}(c,\Delta_i|y)$$

where

$$x = \frac{\theta_2^4(\nu)}{\theta_3^4(\nu)}\,, \qquad y = \exp \pi i \nu\,.$$

Comparison in massless limit

Simplification in massless limit: only massive parameter aside from $\epsilon_{1,2}$ is a.

$\Rightarrow$ a keeps track of joint order in g_s and s: $\frac{\sum_{n+g=k} \# g_s^{2g-2} s^n}{a^{2k-2}}$.

$$F_\triangle(c,\Delta_i|x) = (16y)^{\frac{a^2}{g_s^2}} x^{-\frac{s^2}{4g_s^2}} (1-x)^{-\frac{s^2}{4g_s^2}} \theta_3(y)^{3\frac{s^2}{g_s^2}} H_\triangle(c,\Delta_i|y)$$

with

$$H_\triangle(c,\Delta_i|y) = 1 + \sum_{m,n} (16y)^{mn} \frac{R_{mn}(c,\Delta_i)}{\Delta - \Delta_{mn}(c)} H_{\Delta_{mn}+mn}(c,\Delta_i|y)$$

$\Rightarrow H_\triangle = \exp \sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n$.

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with

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From topological string to CFT

We can obtain all order results in q for H_Δ.

E.g. $g+n=2$: isolate order a^2 contribution:

$$H_\Delta(c,\Delta_i|y) = 1 + \sum_{m,n} \frac{(16y)^{mn} R_{mn}(c,\Delta_i)}{-\frac{a^2}{g_s^2} + \alpha_{mn}} H_{\Delta_{mn}+mn}(c,\Delta_i|y)$$

$$\Rightarrow -\frac{g_s^2}{a^2} \sum_{m,n} (16y)^{mn} R_{mn}(c,\Delta_i) H_{\Delta_{mn}+mn}(c,\Delta_i|y)$$

$$= \left[\frac{s^2}{g_s^2} F^{(2,0)} + sF^{(1,1)} + g_s^2 F^{(0,2)}\right]'$$

$$= \left(\frac{s^2}{g_s^2} - 4s + 3g_s^2\right) \frac{[E_2(\nu)]'}{96a^2}$$

From CFT to topological string I

Likewise, we can obtain all-order results in g_s and s for topological string.

E.g., calculate first non-vanishing order in y:

$$\begin{aligned} H_\Delta(c,\Delta_i) &= 1 + (16y)^2 \left(\frac{R_{12}}{\Delta - \Delta_{1,2}} + \frac{R_{21}}{\Delta - \Delta_{2,1}} \right) + \dots \\ &= 1 + \left[\sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n \right]_1 e^{2\pi i \tau} + \dots \end{aligned}$$

From CFT to topological string II

$$\frac{R_{12}}{\Delta - \Delta_{1,2}} + \frac{R_{21}}{\Delta - \Delta_{2,1}} = \frac{\epsilon_2^5}{2^9 \epsilon_1 (\epsilon_1^2 - \epsilon_2^2) a^2} \frac{1}{(1 - \frac{(\epsilon_1 + 2\epsilon_2)^2}{4a^2})} + (\epsilon_1 \leftrightarrow \epsilon_2) \, .$$

$\Rightarrow$ at each order in y, the powers $\frac{1}{a^{2(g+n)-2}}$ in $F^{(n,g)}$ can be resummed via a geometric series.

Since the $F^{(n,g)}$ are polynomials in the Eisenstein series E_2, E_4, E_6 of fixed weight $2(n+g)-2$, the *exact* result to all orders in y can be determined from the knowledge of a finite number of expansion coefficients.

Physical significance of poles in a? They are cancelled by the perturbative contribution to Z_{top} as given by the DOZZ formula.

From CFT to topological string II

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DOZZ and the perturbative contribution to Z_{top} I

AGT

Perturbative contribution to Z_{top} is obtained from 3-point function of Liouville theory, as determined by Dorn-Otto-Zamolodchikov-Zamolodchikov (DOZZ).

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \frac{\prod_i \Upsilon(2\alpha_i)}{\Upsilon(\sum \alpha_i + Q) \prod_i \Upsilon(\alpha_i + \alpha_{i+1} - \alpha_{i+2})}$$

in terms of $\alpha : \Delta = \alpha(Q - \alpha)$, where

$$\Upsilon(a) = \frac{1}{\Gamma_b(a)\Gamma_b(Q-a)} , \quad \Gamma_b(a) \sim \prod_{m,n\geq 0} \frac{1}{mb + \frac{n}{b} + a}$$

DOZZ and the perturbative contribution to Z_{top} II

$$G_{43}^{21} = \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} \left| F_{\Delta}(c, \Delta_i | x) \right|^2$$

3-point function contributions to 4-point function

$$C(Q - \alpha_4, \alpha_3, \alpha) C(Q - \alpha, \alpha_2, \alpha_1)$$

distributed among F_{top} and $\bar{F}_{top}$ such that

$$G_{43}^{21} = \int \frac{d\alpha}{2\pi} |Z_{top}|^2 \, .$$

DOZZ and the perturbative contribution to Z_{top} III

$$\Gamma_b(a) \sim \prod_{m,n\geq 0} \frac{1}{mb + \frac{n}{b} + a}$$

Poles at $a = -mb - \frac{n}{b} \Rightarrow C(\alpha_1, \alpha_2, \alpha_3)$ vanishes at degenerate weights, canceling poles in conformal block.

g_s, s expansion of DOZZ via integral representation of $\log \Gamma_b$. Closed formula for $\epsilon_1 = -\epsilon_2$, i.e. $s = 0$:

$$[\log \Gamma_b(a)]_g = \frac{B_{2g}}{2g(2g-2)} \frac{1}{a^{2g-2}} \quad \text{for} \quad g > 1\,,$$

but analytic properties obscured.

DOZZ and the perturbative contribution to Z_{top} III

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Lessons so far, and where to go from here

- Lessons from topological string: modularity in conformal blocks
- Lessons from CFT: analytic structure of all genus result

Next:

- Modularity and crossing symmetry, topological string suggests symmetry *before* the sum over exchanged momenta is performed.
- Holomorphic anomaly in CFT?