D-BRANE COUPLINGS & B-FIELD

work in progress with Alex Kahle

(using some calculations from [arXiv:1010.6074] + work in progress with Jim Liu)

D-brane couplings (disk amplitudes - 1 RR VO, 2 - NS VO)

copied from [K. Becker, G. Guo, D. Robbins], ... [M. Garousi], [M. Garousi, M. Mir]

$$
\begin{aligned}
S_{WZ} \supset \quad & T_p \frac{\pi^2(\alpha')^2}{24} \int_{Dp} dx^{a_1} \wedge \cdots \wedge dx^{a_{p+1}} \\
& \Big\{ \frac{1}{2} \frac{1}{(p-3)!} C^{(p-3)}_{a_1 \cdots a_{p-3}} \Big(-2\partial_{a_{p-2}}{}^{[b} h_{a_{p-1}}{}^{c]} \partial_{a_p b} h_{a_{p+1} c} + 2\partial_{a_{p-2}}{}^{[j} h_{a_{p-1}}{}^{k]} \partial_{a_p j} h_{a_{p+1} k} \\
& -\partial_{a_{p-2}}{}^{b} B_{a_{p-1}}{}^{j} \partial_{a_p b} B_{a_{p+1} j} + \partial_{a_{p-2}}{}^{j} B_{a_{p-1}}{}^{b} \partial_{a_p j} B_{a_{p+1} b} \Big) \\
& + \frac{1}{(p-2)!} C^{(p-1)}_{a_1 \cdots a_{p-2} i} \Big(2\partial_{a_{p-1}}{}^{[b} h_{a_p}{}^{c]} \partial_{a_{p+1} b} B^{i}{}_{c} - 2\partial_{a_{p-1}}{}^{[j} h_{a_p}{}^{k]} \partial_{a_{p+1} j} B^{i}{}_{k} \\
& + \partial_{a_{p-1}}{}^{b} h^{ij} \partial_{a_p b} B_{a_{p+1} j} - \partial_{a_{p-1}}{}^{j} h^{ib} \partial_{a_p j} B_{a_{p+1} b} \Big) \\
& + \frac{1}{2} \frac{1}{(p-1)!} C^{(p+1)}_{a_1 \cdots a_{p-1} i_1 i_2} \Big(-\partial_{a_p}{}^{b} h^{i_1 j} \partial_{a_{p+1} b} h^{i_2}{}_{j} + \partial_{a_p}{}^{j} h^{i_1 b} \partial_{a_{p+1} j} h^{i_2}{}_{b} \\
& -2\partial_{a_p}{}^{b} B^{i_1 c} \partial_{a_{p+1} [b} B^{i_2}{}_{c]} + 2\partial_{a_p}{}^{j} B^{i_1 k} \partial_{a_{p+1} [j} B^{i_2}{}_{k]} \Big) \Big\}
\end{aligned}
$$

- ◇ Appearance of the B-field
- ◇ RR fields - B-*twisted* differential K-theory co-cycles; D-brane couplings must take this into account

Structure of the couplings:

- "Standard" couplings $C_{p-3} \wedge X_4(W, \nu, B)$
  - ◇ Inclusion of the B-field
  - ◇ Not invariant under T-duality! (easy to check by taking $S^1 \longrightarrow W_{p-1}$)
- New structures: $C_{p-3+n,n}$ and $X_{4-n,n}$ contracted on n indices
- $X_{4-n,n}$ are **even/odd** in B for n - even/odd

Plan of the talk:

- Briefly review old ($B = 0$) couplings
- Try to (heuristically) explain new ones

D-brane couplings ($B=0$):

- co-dimension r submanifold $i: W \to X$
- W is spin^c, with curvature of the spin^c connection $-2\pi i\eta \in \Omega^2(W)$
- a vector bundle and connection $(V, \nabla_w) \to W$

$$
\begin{aligned}
I_{\text{D-brane}} &= \exp\left[-2\pi i \int_{W/T} \sqrt{\frac{\hat{A}(W/T)}{\hat{A}(\nu)}} e^{\eta/2}\, \mathrm{Ch}\, \nabla_V \wedge i^* C\right] && \text{splitting pr.} \\
&\cong \exp\left[-2\pi i \int_{W/T} e^{\eta/2} \wedge \hat{A}(W/T)\, \mathrm{Ch}\, \nabla_V \wedge i^*\left(\left(\sqrt{\hat{A}(X/T)}\right)^{-1} \wedge C\right)\right] \\
&= \exp\left[-2\pi i \int_{X/T} \sqrt{\hat{A}(X/T)}\, \mathrm{Ch}\, i_* \check{q}_V \wedge C\right] && \text{RRG trm.}
\end{aligned}
$$

◇ The pushforward: $i_*: \check{K}^{0-\check{w}_2(\nu)}(W) \to \check{K}^{10-r}(X)$

◇ $\check{w}_2(\nu)$ is a (differential) charactaristic class induced by the second Stiefel-Whitney class of the normal bundle ν

Field-strength map (mod. Chern isomorphism) for $\check{x} \in \check{K}^\bullet(X)$:

$$\check{F}(\check{x}) = \sqrt{\hat{A}(X)}\,\mathrm{Ch}(\check{x})$$

◇ Commutes with bilinear pairing

Inclusion of B-field ($\check{B} \in \check{H}^3(X)$):

∗ *Generalised* geometry determined by $\check{B}$ used to define the pushforward

∗ *Generalised* connection $\nabla^{\pm H}$ used to form our Dirac operators

▷ appropriate index theorem for torsionful connections [Bismut]:

$$\lim_{t \longrightarrow 0} \mathrm{Tr} e^{-t \not{D}(\nabla^{\pm \check{B}})^2} = [\hat{A}(\Omega^{\mp H})] \qquad (\longleftarrow \text{SIGNS!})$$

∗ The fieldstrength map $F : \check{K}^{\bullet + \check{B}}(X) \to \Omega^{\bullet + \check{B}}(X)$ modified:

$$F : \quad \check{x} \mapsto \sqrt{\hat{A}(\Omega^{-H})}\, e^{-B}\, \mathrm{Ch}\, \check{x}$$

with $\Omega^{\pm H} = \Omega(\omega^{LC} \pm \frac{1}{2} H)$ (use also $\Omega^{\pm \check{B}}$ for $\Omega^{\pm H}$)

Mukai pairing: for $\omega_1, \omega_2 \in \Omega(M)$,

$$\langle \omega_1, \omega_2 \rangle = \sum_i (-1)^i \left(\omega_1^{2i} \wedge \omega_2^{n-2i} + \omega_1^{2i+1} \wedge \omega_2^{n-2i-1} \right),$$

where ω^i denotes the part of the differential form in degree i

B-transform:

- RR pure spinors: $C \longrightarrow C^{(\pm)} = e^{\pm B} C$
- Twisted Chern map: $\mathrm{Ch} \longrightarrow \mathrm{Ch}^{(\pm)} = e^{\pm B}\, \mathrm{Ch}$

B-transform preserves Mukai pairing:

$$\langle e^{\pm B} \omega_1, e^{\pm B} \omega_2 \rangle = \langle \omega_1, \omega_2 \rangle$$

D-brane couplings:

- " no B"
- need to be symmetrized: $B \leftrightarrow -B$

Conjectural Riemann-Roch:

Let $X \to T$ be a spin Riemannian family with compact fibres, and $\check{B} \in \check{H}^3(X)$ be a differential 3-co-cycle on X. Then there is a bilinear pairing $\check{K}^{\check{B}+\bullet}(X) \otimes \check{K}^{\check{B}+\bullet}(X) \to \check{K}^{\bullet - \dim X/T}(T)$ defined by

$$(\check{x}, \check{y}) = \int_{X/T} \check{x} \cdot \theta(\check{y})$$

where $\theta : \check{K}^{\check{B}+\bullet}(X) \to \check{K}^{-\check{B}+\bullet}(X)$ (smooth -1-Adams operation).

Furthermore,

$$\mathrm{Ch}(\check{x}, \check{y}) = \int_{X/T} \hat{A}(\Omega_{\check{B}}^{X/T}) \, \langle \mathrm{Ch}\, \check{x}, \, \mathrm{Ch}\, \check{y} \rangle \,,$$

where the pairing on ($\check{B}$-twisted) differential forms is the Mukai pairing, and $\Omega_{\check{B}}^{X/T}$ is the curvature of the torsionfull generalised (relative) connection determined by $\check{B}$.

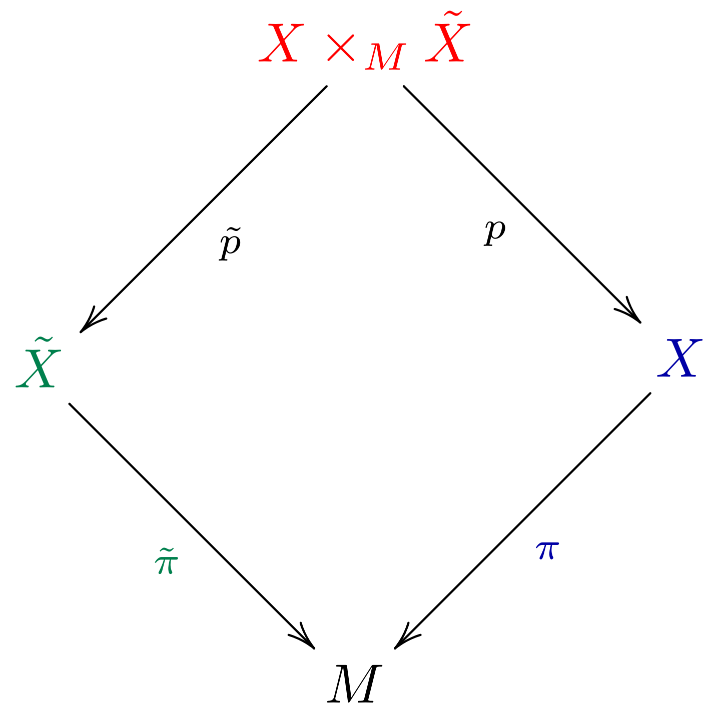
D-brane couplings:

$$
\begin{aligned}
I_{\text{D-brane}} &= (\theta(\check{j}), \check{C}) \\
&= \exp\Bigg[-2\pi i \int_{X/T} \frac{1}{2} \left\langle \sqrt{\hat{A}(\Omega^{X/T,\check{B}})}\, e^{B} i_* \operatorname{Ch} \nabla_V,\, C^{(+)} \right\rangle \\
&\qquad\qquad - 2\pi i \int_{X/T} \frac{1}{2} \left\langle \bar{C}^{(-)},\, \sqrt{\hat{A}(\Omega^{X/T,-\check{B}})}\, e^{-B} i_* \operatorname{Ch} \nabla_{\bar{V}} \right\rangle \Bigg]
\end{aligned}
$$

- The connections used in the $\hat{A}$ expressions contain the B-field, leading to new terms in the derivative of H
- These occur symmetrically because of the two terms in the expression
- Only even powers of H appear
- When pulled to W, $\hat{A}(\Omega^{X/T,\pm\check{B}})$ does NOT split into tangent and normal parts, and thus in general “mixed derivative” (contracted) terms appear

T-duality - coord-independent $O(n,n)$ transformation in a background with n isometries v^i. Acts separately on NS and RR sectors - perturbative symmetry.

Topology change:



Correspondence space $Y = X \times_M \tilde{X}$:

- a circle bundle over X with first Chern class $\pi^*(c_1(\tilde{X}))$
- a circle bundle over $\tilde{X}$ with first Chern class $\tilde{\pi}^*(c_1(\tilde{X}))$

T-duality:

$$\pi_* H = c_1(\tilde{X}) \quad \tilde{\pi}_* \tilde{H} = c_1(X) \qquad \in H^2(M)$$

D-brane couplings and T-duality:

$\mathcal{L}_v g_{10} = 0 = \mathcal{L}_v H \quad (v = \partial/\partial t)$

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_{10} \\ & & \downarrow \pi \\ & & X_9 \end{array} \qquad \mathrm{d}e = \pi^* F \qquad (\mathcal{L}_v\, e = 0)$$

On pure spinors: $\quad T_v C = \imath_v C + \mathrm{d}t \wedge C$

- Take $B = B_2 + b \wedge e \quad (H = H_3 + H_2 \wedge e) \qquad \& \qquad e = \mathrm{d}t + a$
- $T_v C^{(-)} = T_v \left(e^{-B_2} [\hat{C}_p + (\hat{C}_p \wedge b + \hat{C}_{p-1}) \wedge e] \right) = e^{-\tilde{B}} [(\mathrm{d}t + b) \wedge \hat{C}_p + \hat{C}_{p-1}]$
- Note $\tilde{B} = (B_2 + b \wedge a) + a \wedge (\mathrm{d}t + b)$ but(!) $\tilde{H}_3 = H_3$

Mukai product of two PS is T-duality invariant:

- $$\langle C^{(\pm)}, \alpha^{(\pm)} \rangle = \pm \langle T_v C^{(\pm)}, T_v \alpha^{(\pm)} \rangle$$

RR gauge invariance $(\mathrm{d}\alpha^{(\pm)} = 0)$:

$C \mapsto C + \mathrm{d}\Lambda \quad \longrightarrow \quad C_{p-1} \mapsto C_{p-1} + \mathrm{d}\Lambda_{p-2} \quad \& \quad C_p \mapsto C_p + \mathrm{d}\Lambda_{p_1} + (-)^p \Lambda_{p-2} \wedge F$

$$\langle C, \alpha \rangle \mapsto \langle C, \alpha \rangle \pm \langle \Lambda_{p-2} \wedge e, (\mathrm{d}\alpha_{10-p} - (-)^p \alpha_{9-p} \wedge F) \rangle = 0$$

Consider D5 with $B = 0$, *trivial* ν and C-P bundle and W having a U(1) isometry:

$$I_{\mathrm{D5}} = \frac{1}{4} C_2 \wedge \left(p_1(\Omega^{H}) + p_1(\Omega^{-H})\right) \wedge \eta = C_2 \wedge X \wedge \eta \sim C_2 \wedge \left(\mathrm{tr}\,\Omega^2 + \frac{1}{2}\mathrm{d}\,\mathrm{tr}\,(H \nabla H)\right) \wedge \eta$$

$$X = \pi^* X_4 + \pi^* X_3 \wedge e$$

$$\mathrm{d}X = 0 \qquad \Leftrightarrow \qquad \begin{cases} \bullet \quad \mathrm{d}X_4 - X_3 \wedge F = 0 \\ \bullet \quad \mathrm{d}X_3 = 0 \end{cases}$$

If $\mathcal{L}_v X_3^{(0)} = 0 \qquad \Rightarrow \qquad X_3$ is exact: $X_3 = \mathrm{d}Y_2$

$$\mathrm{d}[X_4 - Y_2 \wedge F] = 0$$

- $X = (X_4 - F \wedge X_2) + \mathrm{d}(Y_2 \wedge e)$
- $(X_4 - F \wedge X_2)$ is T-duality invariant!
- $Y_2(F, \imath_v H) = \tilde{Y}_2(\imath_v \tilde{H}, \tilde{F})$ and T-duality:
  $Y_2 = \mathrm{d}^{-1}(\imath_v X) = \mathrm{d}^{-1}\left(\imath_v(X^{(+)} + X^{(-)})/2\right) \quad \longleftrightarrow \quad \tilde{Y}_2 = \mathrm{d}^{-1}\left(\imath_v(X^{(+)} - X^{(-)})/2\right)$

T-duality: D5 $\longrightarrow$ D4 (with $B \neq 0$):

$$\begin{aligned} I_{\text{D5}} = T_v(I_{\text{D5}}) &= [\hat{C}_1 \wedge X_4 + \hat{C}_2 \wedge X_3] \wedge \eta \wedge \tilde{e} \\ &= (\tilde{C}_1 \wedge [X_4 - Y_2 \wedge F] - F_3 \wedge Y_2) \wedge \tilde{\eta}_5 \\ &= (\tilde{C}_1 \wedge [X_4 - Y_2 \wedge F] - \tilde{F}_{3,t} \wedge \tilde{Y}_2^t) \wedge \tilde{\eta}_5 = I_{\text{D4}} \end{aligned}$$

$$\Longleftrightarrow$$

Couplings in the bulk

$I_{\text{D-brane}} = \frac{1}{2}\left(\langle C^{(-)}, \alpha^{(-)}\rangle + \langle C^{(+)}, \alpha^{(+)}\rangle\right)$ with $\alpha^{(\pm)} = e^{\pm B}\sqrt{\hat{A}(\Omega^{X/T,\pm\check{B}})}\,\text{Ch}\,x$

- Invariant under T-duality!
- Give rise to couplings with $C_{p-3+n,n}$ and $X_{4-n,n}$ contracted on n indices
- Reproduce $X_{4-n,n}$ being **even/odd** in B for n - even/odd

$$\Longleftrightarrow$$

Tests of the general structure in backgrounds without isometries - contractions form non-split parts $\hat{A}(\Omega^{X/T,\pm\check{B}})$ pulled back to W?

Topology vs. Generalized Geometry