Lifshitz as a deformation of AdS

Kostas Skenderis

University of Amsterdam *and*
University of Southampton

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Reference

- Based on
  Yegor Korovin, KS, Marika Taylor,
  Lifshitz as a deformation of AdS
  to appear

Introduction

- Gauge/gravity dualities have been an important new tool in extracting strong coupling physics.
- In particular, there has been a lot of activity recently aiming at using holography in order to model strongly interacting condensed matter systems.
- More precisely, several interesting condensed matter systems exhibit strongly interacting non-relativistic scale invariant fixed points and one may hope to use gauge/gravity duality to study them.
- To this end supergravity solutions with Schrödinger and Lifshitz isometries were constructed and studied.

Introduction

- The idea here is that the holographic models may allows to uncover new universality classes, not easily accessible with conventional perturbative methods.
- One should emphasize however that there is very little *a priori* evidence that the holographic models actually describe physics appropriate for the condensed matter systems.

Introduction

- The predominant approach has been to proceed phenomenologically and probe the relevance of these models by computing observables holographically and comparing to experimental results.
- Our goal is to understand better the dual theory from first principles.

Introduction: main idea

- The main problems with getting a handle on the dual theory are:
  - These models have been constructed in a bottom-up approach so we can't directly use string theory to get clues about the dual theory.
  - The geometries are not asymptotically AdS so the usual holographic dictionary and intuition does not directly apply.
- Our main idea is try to tune the parameters of the gravitational solutions so that they become deformations of asymptotically AdS solutions.
  - Then, at least for such parameters, one can use standard AdS/CFT to understand the dual theory.

Holography for Schrödinger

This approached was exploited in [Guica, KS, Taylor, van Rees (2010)] for the Schrödinger spacetimes

$$ds^2 = -\frac{b^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- For $z = 2$, this metric realizes geometrically the Schrödinger group in $(d-1)$ dimensions [Son (2008)], [K. Balasubramanian, McGreevy (2008)].
- For $d = 2$ this is the "null warped AdS_3" solution of topologically massive gravity with $\mu = 3$ [Anninos et al (2008)].

Schrödinger as a deformation of AdS

$$ds^2 = -\frac{b^2 du^2}{r^4} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- This metric is not asymptotically AdS.
- The metric becomes the AdS metric when b=0: Schrödinger is a deformation of AdS

➠ Considering the small b limit the geometry is a small perturbation of AdS and standard AdS/CFT applies.

The QFT dual to Schrödinger

- The dual QFT is a deformation of a relativistic CFT:

$$S_{CFT} \to S_{CFT} + \int d^d x \, bX_i$$

→ X_i is an irrelevant operator from the perspective of the original CFT.

→ X_i is exactly marginal operator from the perspective of the Schrödinger symmetry [Guica, KS, Taylor, van Rees (2010)] [Kraus, Perlmutter (2011)] .

Lifshitz spacetimes

- In this work we aim to reach a similar understanding for Lifshitz spacetimes [Kachru, Liu Mulligan (2008)].

$$ds^2 = dr^2 - e^{2zr/l}dt^2 + e^{2r/l}dx^a dx_a$$

- The Lifshitz symmetry is realized by the following isometry,

$$t \to e^{z\lambda}t, \qquad x^a \to e^{\lambda}x^a, \qquad r \to r - \lambda l.$$

- This solution in not asymptotically AdS.
- The metric becomes the AdS metric when $z = 1$: when $z \approx 1$ the solution is a deformation of AdS.

Condensed matter systems with $z \approx 1$

A number of theoretical models with dynamical exponent close to one have appeared in the condensed matter literature.

- Quantum spin systems with quenched disorder.
- Quantum Hall systems.
- Graphene.
- Spin liquids in the presence of non-magnetic disorder.
- Quantum transitions to and from the superconducting state in high T_c superconductors.

$z \approx 1$ systems are also relevant for the study of the IR limit of Hořava-Lifshitz gravity.

Experimental evidence for $z \approx 1$ systems

There is also experimental evidence for quantum critical behavior with $z \approx 1$ in:

- the transition from the insulator to superconductor in the underdoped region of certain high T_c superconductors [Zuev et al PRL(2005)], [Matthey etal PRL(2007)] [Broun etal PRL (2007)].
- the transition from the superconductor to metal in the overdoped region of certain high T_c superconductors [Lemberger etal PRB (2011)].

The holographic model

- We will use the formulation in terms gravity coupled to a massive vector [Taylor (2008)]

$$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{G}\left[R + d(d-1) - \frac{1}{4}F^2 - \frac{1}{2}M^2A^2\right]$$

  Relative to prior literature we rescaled the fields such that the AdS critical point has AdS radius $l^2 = 1$.
- This model admits a Lifshitz solution

$$\begin{aligned} ds^2 &= dr^2 - e^{2zr/l}dt^2 + e^{2r/l}dx^a dx_a; \\ A &= \mathcal{A}e^{zr/l}dt, \qquad \mathcal{A}^2 = \frac{2(z-1)}{z}, \end{aligned}$$

  provided the mass is given by

$$M^2 = \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}$$

The holographic model

- Standard AdS/CFT correspondence implies that this Lagrangian, expanded around the AdS critical point, describes a CFT with a vector primary operator J_i of dimension

$$\Delta = \frac{d}{2} + \sqrt{(\frac{d}{2} - 1)^2 + \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}}$$

- The same theory admits a Lifshitz critical point with dynamic critical exponent z, if

$$M^2 = \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}$$

viewed an equation for z has real solutions with $z > 1$.

Top-down models?

Thus a necessary condition for obtaining such a Lifshitz theory with critical exponent z_+ or z_- is that the spectrum of the *AdS critical point* contains vector modes with mass within the following range

	z_+	z_-
$d = 2$	$0 < M^2 \leq 1$	$M^2 = 1$
$d = 3$	$0 < M^2 \leq 2.4$	$2.33 < M^2 \leq 2.4$
$d = 4$	$0 < M^2 \leq 9/2$	$4.29 < M^2 \leq 9/2$

This is only a necessary condition because one needs to additionally check that it is consistent to retain only the massive vector mode.

Top-down models?

The simplest AdS compactifications do contain modes in the allowed range:

- The $AdS_3 \times S^3$ spectrum contains vector modes with $M^2 = 1$, leading to $z = 1$.
- The $AdS_4 \times S^7$ spectrum contains two such massive vectors:
  - $M^2 = 3/4$, leading to $z \approx 14.72$.
  - $M^2 = 2$, leading to either $z = 4$ or $z = 1$.
- The $AdS_5 \times S^5$ spectrum contains one massive vector in the allowed range, $M^2 = 3$, leading to either $z = 9$ or $z = 1$.

Consistent truncation to only these modes is a non-trivial condition that still needs to be checked (in progress).

The $z \approx 1$ case

Leaving aside the top-down models, we now focus on the case $z = 1 + \epsilon^2$, with $\epsilon \ll 1$.

- This can be achieved by taking

$$M^2 = (d-1) + (d-2)\epsilon^2 + \frac{1 + d - d^2}{d(d-1)}\epsilon^4 + \cdots$$

- The Lifshitz solution now becomes Asymptotically AdS and its dual interpretation can be obtained using the standard AdS/CFT dictionary.

The QFT dual to Lifshitz with $z = 1 + \epsilon^2$

The dual theory is a deformation of a relativistic CFT by the time component of a vector primary J^μ of dimension d,

$$S_{CFT} \to S_{CFT} + \sqrt{2} \int d^d x \epsilon J^t .$$

Holographic dictionary

We are now going set up the holographic dictionary working perturbatively in ϵ.

- Holographic renormalization for general z was studied by a number of authors [Ross (2011)], [Baggio et al (2011)] [Griffin et al (2011)].
- When $\epsilon \sim 0$, the solution is Asymptotically AdS, so well established techniques can be used [de Haro, KS, Solodukhin (2000)].
- The results obtained in this fashion are more easily comparable with those of the dual QFT.

Holographic renormalization for $z = 1 + \epsilon^2$

- We need to obtain the most general asymptotic solution with Dirichlet data $\mathbf{g}_{[0](0)ij}(\mathbf{x}), \mathbf{A}_{(0)i}(\mathbf{x})$,

$$ds^2 = dr^2 + e^{2r}(\mathbf{g}_{[0](0)ij} + \ldots)dx^i dx^j, \quad A_i = (\epsilon e^r \mathbf{A}_{(0)i} + \ldots)$$

- Using the asymptotic solution one can compute the most general divergences that can appear in the on-shell action and then work out the counterterm action needed to remove them. For $d = 2$:

$$S_{\text{ct}} = -\frac{1}{8\pi G_3}\int d^2x\sqrt{\gamma}(1 + \frac{1}{2}R\mathbf{r_0})$$
$$+\frac{1}{32\pi G_3}\int d^2x\sqrt{\gamma}(\gamma^{ij}A_iA_j - \mathbf{r_0}\Big((\nabla_i A^i)^2 - \frac{1}{2}F_{ij}F^{ij}\Big).$$

- The renormalized action is

$$S_{\text{ren}} = \lim_{r_0\to\infty}(S_{\text{reg}} + S_{\text{ct}})$$

One-point functions

1-point functions in the presence of sources are derived by functionally differentiating $S_{\rm ren}$ wrt source.

- Stress energy tensor:

$$\langle \mathcal{T}_{ij} \rangle = \langle T_{ij} \rangle_{[0]} + \epsilon^2 \langle T_{ij} \rangle_{[2]}$$

  where the correction at order ϵ^2 contains in particular,

$$\langle T_{ij} \rangle_{[2]} = -\frac{1}{8\pi G_3} g_{[2](2)ij} - \frac{1}{2} A^k_{(0)} \langle T_{kl} \rangle_{[0]} A^l_{(0)} g_{[0](0)ij} + \cdots$$

- Vector operator:

$$\langle \mathcal{J}^i \rangle = \frac{2\epsilon}{16\pi G_3} A^i_{(2)} - \frac{1}{2} \langle T_{ij} \rangle_{[0]} (\epsilon A_{(0)j}).$$

Ward identities

These 1-point functions satisfy the correct Ward identities as expected.

- Diffeomorphism Ward identity

$$\nabla^j \langle T_{ij} \rangle = A_{(0)i} \nabla_j \langle J^j \rangle - \langle J^j \rangle F_{(0)ij}.$$

  This is indeed the correct Ward identity.

- Trace Ward identity

$$\langle T^i_i \rangle = A_{(0)i} \langle J^i \rangle + \mathcal{A},$$

  This is the expected trace Ward identity with a trace anomaly $\mathcal{A}$,

$$\mathcal{A} = -\frac{1}{2} A^i_{(0)} \langle T_{ij} \rangle_{[0]} A^j_{(0)} + \cdots$$

Recovering Lifshitz invariance

Let us now evaluate the trace Ward identity at the Lifshitz critical point.

- In this case,

$$A_{(0)t} = \sqrt{2}, \qquad g_{(0)ij} = \eta_{ij}$$

- Then the trace Ward identity leads to

$$\langle T^i_i \rangle = -\epsilon^2 \langle T^t_t \rangle + O(\epsilon^4) \quad \Rightarrow \quad \langle z T^t_t \rangle + \langle T^a_a \rangle = O(\epsilon^4)$$

- This is precisely the condition for Lifshitz invariance!

The QFT side: remarks

The Lifshitz theory we obtained is markedly different than any other Lifshitz invariant theory that appeared before.

- For example, the scalar theory with action

$$S = \int dt d^3x (\dot{\phi}^2 + \phi(-\partial^2)^z \phi) \qquad (1)$$

  which is often used in the literature as an illustrative example (especially when $z = 2$) does not become of the form we find when $z \sim 1 + \epsilon^2$.
- This suggests that this field theory model is unlikely to share key features of the holographic modle.
- None of the condensed matter Lifshitz models are of this form, which leads to the question:

➠ Is this type of Lifshitz model special to strongly coupled models or not?

The QFT story

- The answer to this question is:

These models are generic. If one deforms any CFT by any dimension d vector primary, the theory becomes Lifshitz invariant with $z = 1 + \epsilon^2 + O(\epsilon^4)$.

- Whether this fixed point is stable or not depend on the particular CFT/vector primary.

Sketch of proof

Our theory is defined by

$$S = S_{CFT} + \int d^d x \epsilon J^t.$$

where J^i is a conformal vector primary of dimension d.

- ➢ This theory still has a conserved stress-energy tensor, but it is not symmetric anymore.
- ➢ Classically, the stress-energy tensor is traceless (because the dimension of J^μ is d).
- ➠ The classical theory is a $z = 1$ non-relativistic CFT.

The quantum theory

We can analyze the quantum theory using conformal perturbation theory.

- The first non-trivial effects are at order ϵ^2, where we find

$$Z[\epsilon] = \left\langle \epsilon^2 \int d^d x d^d y J^t(x) J^t(y) \right\rangle_{CFT} + O(\epsilon^3)$$

- To compute this we may use the OPE of $J^i J^j$.

OPE of $J_i J_j$

- This OPE contains the following universal terms

$$J_i(x)J_j(0) \sim \frac{I_{ij}}{x^{2d}} + \cdots + \mathcal{A}_{ij}{}^{kl}\frac{T_{kl}}{x^d} + \cdots ,$$

  where $I_{ij} = \delta_{ij} - 2x_i x_j / x^2$.

➢ In $d = 2$, the OPE coefficient $\mathcal{A}_{ij}{}^{kl}$ is completely fixed by conformal invariance.

➢ In $d > 2$, the OPE coefficient $\mathcal{A}_{ij}{}^{kl}$ is fixed up to two constants by conformal invariance.

Lifshitz invariance

- Inserting the OPE in $Z[\epsilon]$ one finds a logarithmic divergence proportional to T_{ij}.
- To remove the infinity one adds a counterterm in the action.
- This results in a beta function β_g^{ij} for the source of T_{ij}, i.e. the background metric, leading to

$$\langle T_i^i \rangle = -\beta_g^{ij} T_{ij}, \qquad \beta_g^{ij} \sim A_{(0)}^i A_{(0)}^j$$

- Using the specific coefficient of the OPE we find that this is precisely the Ward identity we found at strong coupling using holographic renormalization!
- The same argument we gave earlier establishes Lifshitz invariance to this order.

Conclusions

Gravity: We showed that the Lifshitz solution with $z \approx 1$ is dual to a deformation of a relativistic CFT by the time component of a vector conformal primary of dimension d.

QFT: Such a deformation of a relativistic CFT always leads to Lifshitz invariant theory with $z \approx 1$.

Outlook

- Correlation functions: compute holographically 2-point functions.
- Black holes: obtain black hole solutions.