

Holography, Unfolding and Higher-Spin Theories

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HS theory

Higher derivatives in interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots, \quad S^3 = \sum_{p,q,r} (D^p \varphi)(D^q \varphi)(D^r \varphi) \rho^{p+q+r+\frac{1}{2}d-3}$$

HS Gauge Theories ($m = 0$):

Fradkin, M.V. (1987)

$$AdS_d : \quad [D_n, D_m] \sim \rho^{-2} = \lambda^2$$

AdS/CFT:

$$(3d, m = 0) \otimes (3d, m = 0) = \sum_{s=0}^{\infty} (4d, m = 0) \quad \text{Flato, Fronsdal (1978);}$$

Sundborg (2001), Sezgin, Sundell (2002,2003), Klebanov, Polyakov (2002),

Giombi, Yin (2009)... , Maldacena, Zhiboedov (2011,2012)

Results

CFT_3 dual of AdS_4 HS theory: 3d conformal HS theory

Holography: Unfolding

Plan

- I Unfolded dynamics and holographic duality
- II Free massless HS fields in AdS_4
- III Conserved currents and massless equations
- IV AdS_4 HS theory as 3d conformal HS theory
- V Conclusion

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \quad \text{initial values: } q^i(t_0)$$

Unfolded dynamics: multidimensional covariant generalization

$$\frac{\partial}{\partial t} \rightarrow d, \quad q^i(t) \rightarrow W^\Omega(x) = dx^{n_1} \wedge \dots \wedge dx^{n_p}$$

$$dW^\Omega(x) = G^\Omega(W(x)), \quad d = dx^n \partial_n$$

$G^\Omega(W)$: function of “supercoordinates” W^Φ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega_{\Phi_1 \dots \Phi_n} W^{\Phi_1} \wedge \dots \wedge W^{\Phi_n}$$

$d > 1$: Nontrivial compatibility conditions

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

Any solution: FDA Sullivan (1968); D’Auria and Fre (1982)

The unfolded equation is invariant under the gauge transformation

$$\delta W^\Omega(x) = d\varepsilon^\Omega(x) + \varepsilon^\Phi(x) \frac{\partial G^\Omega(W(x))}{\partial W^\Phi(x)},$$

Vacuum geometry

$\omega = \omega^\alpha T_\alpha$: h valued 1-form.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

the unfolded equation with $W = \omega$ has the zero-curvature form

$$d\omega + \omega \wedge \omega = 0.$$

Compatibility condition: Jacobi identity for h .

FDA: usual gauge transformation of the connection ω .

Zero-curvature equations: background geometry in a coordinate independent way.

If h is Poincare or anti-de Sitter algebra it describes Minkowski or AdS_d space-time

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms
- Exterior algebra formalism
- Interactions: nonlinear deformation of $G^\Omega(W)$
- Local degrees of freedom are in 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$)
infinite-dimensional module dual to the space of single-particle states
- Independence of ambient space-time
Geometry is encoded by $G^\Omega(W)$

Unfolding and holographic duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time:

ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$d = dX^n \frac{\partial}{\partial X^n} \rightarrow \tilde{d} = dX^n \frac{\partial}{\partial X^n} + d\hat{X}^{\hat{n}} \frac{\partial}{\partial \hat{X}^{\hat{n}}}, \quad dX^n W_n \rightarrow dX^n W_n + d\hat{X}^{\hat{n}} \hat{W}_{\hat{n}},$$

$\hat{X}^{\hat{n}}$ are additional coordinates

$$\tilde{d}W^\Omega(X, \hat{X}) = G^\Omega(W(X, \hat{X}))$$

Particular space-time interpretation of a universal unfolded system, e.g., whether a system is on-shell or off-shell, depends not only on $G^{\Omega}(W)$ but, in the first place, on space-time M^d and chosen vacuum solution $W_0(X)$.

Two unfolded systems in different space-times are equivalent (dual) if they have the same unfolded form.

Direct way to establish holographic duality between two theories: unfold both to see whether their unfolded formulations coincide.

Given unfolded system generates a class of holographically dual theories in different dimensions.

Free massless fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2 \dots$

Fermions require doubling of fields

$$\omega^{ii}(y, \bar{y} | x), \quad C^{i1-i}(y, \bar{y} | x), \quad i = 0, 1,$$

$$\bar{\omega}^{ii}(y, \bar{y} | x) = \omega^{ii}(\bar{y}, y | x), \quad \bar{C}^{i1-i}(y, \bar{y} | x) = C^{1-i i}(\bar{y}, y | x)$$

$$A(y, \bar{y} | x) = i \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \dots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \dots \bar{y}_{\dot{\beta}_m} A^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x)$$

The unfolded system for free massless fields is **MV (1989)**

$$\star \quad R_1^{ii}(y, \bar{y} | x) = \eta \bar{H}^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} C^{1-i i}(0, \bar{y} | x) + \bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^{i1-i}(y, 0 | x)$$

$$\star \quad \tilde{D}_0 C^{i1-i}(y, \bar{y} | x) = 0$$

$$R_1(y, \bar{y} | x) = D_0^{ad} \omega(y, \bar{y} | x) \quad H^{\alpha\beta} = e^\alpha_{\dot{\alpha}} \wedge e^{\beta\dot{\alpha}}, \quad \bar{H}^{\dot{\alpha}\dot{\beta}} = e_\alpha^{\dot{\alpha}} \wedge e^{\alpha\dot{\beta}}$$

$$D_0^{ad} \omega = D^L - \lambda e^{\alpha\dot{\beta}} \left(y_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} + \frac{\partial}{\partial y^\alpha} \bar{y}_{\dot{\beta}} \right), \quad \tilde{D}_0 = D^L + \lambda e^{\alpha\dot{\beta}} \left(y_\alpha \bar{y}_{\dot{\beta}} + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} \right)$$

$$D^L = d_x - \left(\omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \right)$$

Non-Abelian HS algebra

Star product

$$(f * g)(Y) = \int dS dT f(Y + S) g(Y + T) \exp -i S_A T^A$$

$$[Y_A, Y_B]_* = 2i C_{AB}, \quad C_{\alpha\beta} = \epsilon_{\alpha\beta}, \quad C_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}$$

Non-Abelian HS curvature

$$R_1(y, \bar{y}|x) \rightarrow R(y, \bar{y}|x) = d\omega(y, \bar{y}|x) + \omega(y, \bar{y}|x) * \omega(y, \bar{y}|x)$$

$$\tilde{D}_0 C(y, \bar{y}|x) \rightarrow \tilde{D}C(y, \bar{y}|x) = dC(y, \bar{y}|x) + \omega(y, \bar{y}|x) * C(y, \bar{y}|x) - C(y, \bar{y}|x) * \omega(y, -\bar{y}|x)$$

3d conformal equations

Conformal invariant massless equations in $d = 3$

$$dx^{\alpha\beta} \left(\frac{\partial}{\partial x^{\alpha\beta}} \pm \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) C(y|x) = 0, \quad \alpha, \beta = 1, 2 \quad \text{Shaynkman, MV (2001)}$$

Rank r unfolded equations: tensoring of Fock modules Gelfond, MV (2003)

$$dx^{\alpha\beta} \left(\frac{\partial}{\partial x^{\alpha\beta}} + \eta_{ij} \frac{\partial^2}{\partial y_i^\alpha \partial y_j^\beta} \right) C(y|x) = 0, \quad i, j = 1, \dots, r.$$

For diagonal η^{ij} higher-rank equations are satisfied by

$$C(y_i|x) = C_1(y_1|x) C_2(y_2|x) \dots C_r(y_r|x).$$

Rank-two equations: conserved currents

$$\left\{ \frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha} \partial u^{\beta)}} \right\} T(u, y|x) = 0$$

$T(u, y|x)$: generalized stress tensor. Rank-two equation is obeyed by

$$T(u, y|x) = \sum_{i=1}^N C_{+i}(y-u|x) C_{-i}(u+y|x)$$

Rank-two fields: bilocal fields in the twistor space.

Dynamical currents (primaries)

$$J(u|x) = T(u, 0|x), \quad \tilde{J}(y|x) = T(0, y|x) \quad \text{Gelfond, MV (2003)}$$

$$J^{asym}(u, y|x) = u_\alpha y^\alpha \left(\frac{\partial^2}{\partial u^\beta \partial y_\beta} T(u, y|x) \Big|_{u=y=0} \right)$$

$J(u|x)$ generates $3d$ currents of all integer and half-integer spins

$$J(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1} \dots u^{\alpha_{2s}} J_{\alpha_1 \dots \alpha_{2s}}(x), \quad \tilde{J}(u|x) = \sum_{2s=0}^{\infty} u^{\alpha_1} \dots u^{\alpha_{2s}} \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x).$$

$$J^{asym}(u, y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1 \dots \alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(x) = s + 1 \quad \Delta J^{asym}(x) = 2$$

Differential equations: conventional conservation condition

$$\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_\alpha \partial u_\beta} J(u|x) = 0, \quad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} \tilde{J}(y|x) = 0$$

3d conformal setup in AdS_4 HS theory

For manifest conformal invariance introduce

$$y_\alpha^+ = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha), \quad y_\alpha^- = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha), \quad [y_\alpha^-, y^{+\beta}]_* = \delta_\alpha^\beta$$

3d conformal realization of the algebra $sp(4; \mathbb{R}) \sim o(3, 2)$

$$L^\alpha{}_\beta = y^{+\alpha}y_\beta^- - \frac{1}{2}\delta_\beta^\alpha y^{+\gamma}y_\gamma^-, \quad D = \frac{1}{2}y^{+\alpha}y_\alpha^-$$

$$P_{\alpha\beta} = iy_\alpha^-y_\beta^-, \quad K^{\alpha\beta} = -iy^{+\alpha}y^{+\beta}$$

Conformal weight of HS gauge fields

$$[D, \omega(y^\pm|X)] = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} - y_\alpha^- \frac{\partial}{\partial y_\alpha^-} \right) \omega(y^\pm|X).$$

Pullback $\hat{\omega}(y^\pm|x)$ of $\omega(y^\pm|x)$ to Σ : 3d conformal HS gauge fields

Holography at infinity

AdS_4 **foliation:** $x^n = (\mathbf{x}^a, z)$: \mathbf{x}^a are coordinates of leafs ($a = 0, 1, 2,$) z is a foliation parameter

Poincaré coordinates

$$W = \frac{i}{z} d\mathbf{x}^{\alpha\beta} y_{\alpha}^{-} y_{\beta}^{-} - \frac{dz}{2z} y_{\alpha}^{-} y^{+\alpha}$$

$$e^{\alpha\dot{\alpha}} = \frac{1}{2z} dx^{\alpha\dot{\alpha}}, \quad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta}, \quad \bar{\omega}^{\dot{\alpha}\dot{\beta}} = \frac{i}{4z} d\mathbf{x}^{\dot{\alpha}\dot{\beta}}$$

$$\left[d\mathbf{x} + \frac{i}{z} d\mathbf{x}^{\alpha\beta} \left(y_{\alpha} \frac{\partial}{\partial y^{\beta}} - \bar{y}_{\alpha} \frac{\partial}{\partial \bar{y}^{\beta}} + y_{\alpha} \bar{y}_{\beta} - \frac{\partial^2}{\partial y^{\alpha} \partial \bar{y}^{\beta}} \right) \right] C(y, \bar{y} | \mathbf{x}, z) = 0$$

Rescaling y^{α} and $\bar{y}^{\dot{\alpha}}$ **via**

$$C(y, \bar{y} | \mathbf{x}, z) = z \exp(y_{\alpha} \bar{y}^{\alpha}) T(w, \bar{w} | \mathbf{x}, z),$$

$$w^{\alpha} = z^{1/2} y^{\alpha}, \quad \bar{w}^{\alpha} = z^{1/2} \bar{y}^{\alpha}$$

$T(w, \bar{w} | \mathbf{x}, z)$ satisfies the 3d conformal invariant current equation

$$\left[d\mathbf{x} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^{\alpha} \partial \bar{w}^{\beta}} \right] T(w, \bar{w} | \mathbf{x}, z) = 0$$

Connections

Setting

$$W^{jj}(y^\pm | \mathbf{x}, z) = \Omega^{jj}(v^-, w^+ | \mathbf{x}, z)$$

$$\mathbf{v}^\pm = z^{-1/2} \mathbf{y}^\pm, \quad \mathbf{w}^\pm = z^{1/2} \mathbf{y}^\pm$$

manifest z -dependence disappears

$$D_{\mathbf{x}} \Omega^{jj}(v^-, w^+ | \mathbf{x}, z) = \left(d_{\mathbf{x}} + 2i d_{\mathbf{x}}^{\alpha\beta} v_{\alpha}^- \frac{\partial}{\partial w^{+\beta}} \right) \Omega^{jj}(v^-, w^+ | \mathbf{x}, z)$$

Using

$$w_{\alpha} = w_{\alpha}^+ + izv_{\alpha}^-, \quad \bar{w}_{\alpha} = iw_{\alpha}^+ + zv_{\alpha}^-$$

in the limit $z \rightarrow 0$ free HS equations take the form

$$\star \quad D_{\mathbf{x}} \Omega_{\mathbf{x}}^{jj}(v^-, w^+ | \mathbf{x}, 0) = d_{\mathbf{x}\alpha}^{\gamma} d_{\mathbf{x}\beta\gamma} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} \mathcal{T}^{jj}(w^+, 0 | \mathbf{x}, 0),$$

$$\star \quad \left[d_{\mathbf{x}} - i d_{\mathbf{x}}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{-\beta}} \right] T^{j1-j}(w^+, w^- | \mathbf{x}, 0) = 0.$$

$$\mathcal{T}^{jj}(w^+, w^- | \mathbf{x}, 0) = \eta T^{j1-j}(w^+, w^- | \mathbf{x}, 0) - \bar{\eta} T^{1-jj}(-iw^-, iw^+ | \mathbf{x}, 0)$$

Towards nonlinear 3d conformal HS theory

Conformal HS theory is nonlinear since conformal HS curvatures inherited from the AdS_4 HS theory are non-Abelian Fradkin, Linetsky (1990)

$$R_{\mathbf{xx}}(v^-, w^+ | \mathbf{x}) = d_{\mathbf{x}}\Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x}) + \Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x}) \star \Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x})$$

It is important

$$[v_{\alpha}^-, w^{+\beta}]_{\star} = \delta_{\alpha}^{\beta}$$

The equation on 0-forms deforms to nonlinear twisted adjoint representation

$$dT(w^{\pm}|x) + \Omega\left(\frac{\partial}{\partial w^{+\beta}}, w_{\alpha}^{+}\right) \circ T(w^{\pm}|x) - T(w^{\pm}|x) \circ \Omega\left(-i\eta\frac{\partial}{\partial w^{-\alpha}}, -i\eta w^{-}|x\right) = O(T^2).$$

Matter fields can be added via the Fock module

$$(d + \Omega_0(v^-, w^+ | \mathbf{x})) \star C^i(w^+ | \mathbf{x}) \star F = 0$$

Reduction to free CFT_3

The unfolded equation

$$D_{\mathbf{x}}\Omega_{\mathbf{x}}^{jj}(v^-, w^+ | \mathbf{x}, 0) = \mathcal{H}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} \mathcal{T}^{jj}(w^+, 0 | \mathbf{x}, 0)$$

remains free if

$$\mathcal{T}^{jj} = 0 \quad \longrightarrow \quad J^{asym} = 0 \quad \text{or} \quad J^{sym} = 0$$

depending on whether A -model or B -model is considered. For these cases the model remains free in accordance with the Klebanov-Polyakov-Sezgin-Sundell conjecture.

Free models are equivalent to the reductions of the HS theory with respect to P -involution $y \leftrightarrow \bar{y}$ which is possible for the A and B models.

For HS theory with general phase η parameter such reduction is not possible: no realization as a free conformal theory.

Non-Abelian contribution of conformal HS connections has to be taken into account.

Conclusions

Holographic duality relates theories that have equivalent unfolded formulation: equivalent twistor space description.

AdS_4 HS theory is dual to nonlinear $3d$ conformal HS theory of $3d$ currents

Both of holographically dual theories are HS theories of gravity

Beyond $1/N$

Free boundary theories are dual to truncations of HS theories

To do

Nonlinear $3d$ conformal HS theory

Actions

Correlators

AdS_3/CFT_2 and Gaberdiel-Gopakumar conjecture