Generalised geometry and flux background moduli: hypermultiplet structures

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work to appear

The problem

Type II on Calabi–Yau

Moduli spaces well understood

$$\mathrm{d}J = 0 \qquad h^{1,1}$$
$$\mathrm{d}\Omega = 0 \qquad h^{2,1}$$

with harmonic B, $C^{\pm}$

		type IIA	type IIB
hypers	quaternionic Kähler	$\Omega + C^-$	$J + B + C^+$
vectors	special Kähler	$J + B$	Ω

What about generic $\mathcal{N} = 2$ flux backgrounds?

- flux compactifications $\longrightarrow$ massless modes
- AdS-cft $\longrightarrow$ deformations
- analogues of J and Ω?
- differential conditions?
- finite dimensional? cohomology?

One approach: G structures

$$\mathrm{d}J \simeq \text{flux} \qquad\qquad \mathrm{d}\Omega \simeq \text{flux}$$

classification, but lack of integrability means moduli hard

Generalised geometry for NS backgrounds

$$J \longrightarrow \Phi^+ = e^{-\phi} e^{B - iJ}$$

$$\Omega \longrightarrow \Phi^- = e^{-\phi} e^{-B} (\Omega_1 + \Omega_3 + \Omega_5)$$

integrable pure spinors of $O(6,6)$ with $\Phi^\pm \in \Lambda^\pm T^* M$

$$d\Phi^\pm = 0$$

[*Hitchin; Gualtieri; GMPT*]

New result: generic generalised structure for hyper moduli

- full quaternionic-Kähler (hyper-Kähler cone) moduli space
- $E_{7(7)} \times \mathbb{R}^+$ generalised geometry
- integrability as infinite-dimensional hyper-Kähler quotient

describes arbitrary $N = 2$ flux background

Hyper-Kähler quotients

Moment map

- symplectic manifold (X, ω)
- action of Lie group G on X preserving ω

Infinitesimally acts by Lie derivative $\rho : \mathfrak{g} \to TM$ so for all $\xi \in \mathfrak{g}$

$$\mathcal{L}_{\rho(\xi)}\omega = \mathrm{d}(i_{\rho(\xi)}\omega) = 0$$

Moment map is $\mu : X \to \mathfrak{g}^*$ such that

$$\mathrm{d}\mu(\xi) = i_{\rho(\xi)}\omega$$

Symplectic quotient

Define $Z = \mu^{-1}(0) \subset X$, then if G acts freely on Z

$$X' = Z/G \qquad \text{is symplectic manifold}$$

Can be infinite dimensional... (eg Atiyah–Bott)

- $X =$ space of gauge connections on Riemann surface
- $G =$ group of gauge transformations
- $\mu = F$, field strength
- $X' =$ moduli space of flat connections

Hyper-Kähler quotients

- hyper-Kähler manifold $(X, \omega_1, \omega_2, \omega_3)$
- action of Lie group G on X preserving ω_a

with triplet of moment maps

$$\mathrm{d}\mu_a(\xi) = i_{\rho(\xi)}\omega_a$$

and quotient

$$X' = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/G$$

(Hitchin's equations are infinite dimensional example)

$E_{7(7)} \times \mathbb{R}^+$ generalised geometry

Type II warped dimensional reduction

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{3,1}) + \mathrm{d}s^2_d(M),$$

all fields $\{g, \phi, B, \tilde{B}, C^{\pm}, \Delta\}$ on M

Goal [*cf de Wit and Nicolai; . . .*]

- unify bosonic fields and symmetries into single geom. objects
- similar to variants of "double field theory" (strictly DFT not applicable to generic supergravity but can allow non-geom.)

[*Hull; Pacheco & DW; Berman & Perry; CSW*]

Generalised tangent space

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \Lambda^{\pm} T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)$$
$$V = v + \Lambda + \tilde{\Lambda} + \Lambda^{\pm} + \tau$$

parametrises infinitesimal symmetries – diffeos and gauge transf

$E_{7(7)} \times \mathbb{R}^+$ structure

Unique $E_{7(7)} \times \mathbb{R}^+ \supset GL(7, \mathbb{R})$ action

$$E \sim \mathbf{56_1}$$

compatible with gauge patching ($\mathbb{R}^+$ weight is $\mathbf{1_p} \sim (\det T^*M)^{p/2}$)

Generalised tensors

Vector bundles for given $E_{7(7)} \times \mathbb{R}^+$ representations, for example

$$\text{weighted adjoint} \sim \mathbf{133}_p$$

Dorfman derivative

Given $V \in E$, there is action of generalised tensors

$$L_V = \text{diffeo.} + \text{gauge transformation}$$

acts in $E_{7(7)} \times \mathbb{R}^+$

Generalised geometry and supergravity [*CSW*]

Generalised metric

$$G(V, V) \qquad \text{invariant under } SU(8) \subset E_{7(7)} \times \mathbb{R}^+$$

equivalent to giving bosonic fields $\{g, \phi, B, \tilde{B}, C^{\pm}, \Delta\}$

Generalised connections

Analogue of Levi–Civita, derivative of gen. tensor along $V \in E$

exists gen. torsion-free connection D with $DG = 0$

but not unique

Bosonic action

Analogue of Ricci tensor is unique

$$S_B = \int \sqrt{G}\, R \qquad\qquad \text{eom} = \text{gen. Ricci flat}$$

where $\sqrt{G} = \sqrt{g}\,\mathrm{e}^{2\Delta}$

Leading-order fermions and supersymmetry etc

$$D \cdot \psi = 0 \qquad\qquad \delta\psi = D \cdot \epsilon$$

$$D \cdot \lambda = 0 \qquad\qquad \delta\lambda = D \cdot \epsilon$$

full theory has local $SU(8)$ invariance

Hypermultiplet structures

Conventional geometrical structures

Tensors define G structures, at each point $p \in M$

$$g \in GL(d, \mathbb{R})/O(d)$$
$$J \in GL(2n, \mathbb{R})/Sp(n)$$
$$\Omega \in GL(2n, \mathbb{R})/SL(n, \mathbb{C})$$

$E_{7(7)} \times \mathbb{R}^+$ generalised structures

$$G \in \mathbb{R}^+ \times E_{7(7)} / SU(8)$$

what are analogous "flux-y extension" of J and Ω?

Hypermultiplet structures [*GLSW*]

$SU(2)_R \times Spin^*(12) \subset E_{7(7)}$ defines hypermultiplet structure

$$\{J_a\} \in \mathbb{R}^+ \times E_{7(7)} \,/\, Spin^*(12)$$

where $\{J_a\} \sim \mathbf{133_1}$ are $SU(2)_R$ triplet in adjoint

$$[J_a, J_b] = 2\kappa\epsilon_{abc}J_c$$
$$\mathrm{Tr}\,(J_aJ_b) = -\kappa^2\delta_{ab} \in \det T^*M$$

for Calabi–Yau gives Ω in type IIA and J in type IIB

Wolf spaces

Working at a point $p \in M$

$$W = \mathbb{R}^+ \times E_{7(7)} \,/\, Spin^*(12)$$

is (a hyper-Kähler cone over) a (pseudo-Riemmanian) Wolf space (symmetric space with quaternionic-Kähler structure)

$$TW \ni v_a = [\alpha, J_a] \qquad\qquad \alpha \in \mathfrak{e}_{7(7)} + \mathbb{R}$$

hyper-Kähler structure

$$\omega_a(v, w) = \epsilon_{abc}\,\mathrm{Tr}\,(v_b w_c)$$

Space of hypermultiplet structures

Given the bundle $\mathcal{W}$ of Wolf spaces W over M, then

$$X = \text{space of sections of } \mathcal{W}$$

and $\{J_a\} \in X$ is choice of hypermultiplet structure.

X inherits hyper-Kähler structure from fibre, if $\{v_a\}, \{w_a\} \in TX$

$$\Omega_a(v, w) = \epsilon_{abc} \int_M \mathrm{Tr}\left(v_b w_c\right)$$

given choice of $\mathbb{R}^+$ weight of $\{J_a\}$

Action of bosonic symmetries on X

The supergravity symmetries

$$G = \text{group of diffeos and gauge transformations}$$

give infinite dimensional group that preserves Ω_a and relates gauge equivalent structures $\{J_a\}$

Infinitesimally, the symmetries are generated by the Dorfman derivative, so $V \in E$ parameterise $\mathfrak{g}$ and

$$\rho(V) = \{L_V J_a\} \in TX$$

Integrability conditions as moment maps

What are the analogues of $\mathrm{d}J = 0$ and $\mathrm{d}\Omega = 0$? We have the moment maps for G

$$\mu_a(V) = -\tfrac{1}{2}\epsilon_{abc} \int \mathrm{Tr}\,(J_b\, L_V\, J_c)$$

and the integrability conditions are just

$$\mu_a(V) = 0 \qquad \text{for all } V \in E$$

(From 4d rewriting just susy condition in gauged $\mathcal{N} = 2$ [*GLSW*])

Moduli space as hyper-Kähler quotient

Since structures related by a diffeo or gauge transformation are equivalent

$$\mathcal{M} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/G$$

gives the moduli space of integrable structures, and is automatically hyper-Kähler (actually cone over quaternionic Kähler space)

Examples

$O(6,6)$ decomposition

It is useful to use the $SL(2,\mathbb{R}) \times O(6,6) \subset E_{7(7)}$ decomposition

$$\mathbf{133} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}^{\pm})$$
$$\alpha = (\alpha^i{}_j, \alpha^A{}_B, \alpha^{i\pm})$$

where

$$\mathbf{22} \sim (\det T^*M)^{-1/2} \oplus (\det T^*M)^{1/2}$$
$$\mathbf{12} \sim TM \oplus T^*M$$
$$\mathbf{32}^{\pm} \sim \Lambda^{\pm} T^*M$$

Pure spinor

$$J_+ = J_1 + \mathrm{i}J_2 = (0, 0, u^i \Phi^\pm) \qquad u^i = \begin{pmatrix} \mathrm{i}\kappa \\ -\kappa^{-1} \end{pmatrix}$$

defines $\{J_a\}$ where $\kappa^2 = \mathrm{i}\langle \Phi^\pm, \bar{\Phi}^\pm \rangle$ (Mukai pairing)

$$\begin{aligned} \mu_+(V) &= \int \langle \Lambda^\mp, \mathrm{d}\Phi^\pm \rangle \\ \mu_3(V) &= \int \langle \mathrm{d}\bar{\Phi}^\pm, i_v \Phi^\pm \rangle - \langle \bar{\Phi}^\pm, i_v \mathrm{d}\Phi^\pm \rangle \\ &\qquad - \langle \mathrm{d}\bar{\Phi}^\pm, \Lambda \wedge \Phi^\pm \rangle - \langle \bar{\Phi}^\pm, \Lambda \wedge \mathrm{d}\Phi^\pm \rangle \end{aligned}$$

vanishes iff $\mathrm{d}\Phi^\pm = 0$

D3 brane on $HK_4 \times \mathbb{R}^2$

$$\mathrm{d}s^2(M) = \mathrm{e}^{2A}\left(\mathrm{d}s^2_{\mathrm{HK}} + \mathrm{d}x^2 + \mathrm{d}y^2\right) \qquad \Delta \neq 0 \qquad C_4 \neq 0$$

then $\{J_a\} = \{\mathrm{e}^{C_4}\widehat{J}_a\}$ where ${\widehat{J}_a}{}^i{}_j = 0$ and

$$\widehat{J}_a{}^A{}_B = \tfrac{1}{2}\kappa \begin{pmatrix} j_a & 0 \\ 0 & -j_a^T \end{pmatrix} \qquad \widehat{J}_a{}^{i+} = \tfrac{1}{2}\kappa \begin{pmatrix} \mathrm{e}^{4A}\omega_a \wedge \mathrm{d}x \wedge \mathrm{d}y \\ -\mathrm{e}^{-4A}\kappa^{-2}\omega_a \end{pmatrix}$$

and

$$\mu(V)_a = 0 \qquad \text{iff} \qquad A = \Delta \qquad F_5 = \tfrac{1}{4} * \mathrm{d}(\mathrm{e}^{-4\Delta})$$

Conclusions

- same construction works for $\mathcal{N} = 2$ reductions of 11d supergravity, just different decomposition

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M),$$

- series of Wolf spaces

$$\mathbb{R}^+ \times E_{7(7)} / Spin^*(12)$$
$$\mathbb{R}^+ \times E_{6(6)} / SU^*(6)$$
$$\mathbb{R}^+ \times Spin(5,5) / SU(2) \times Spin(1,5)$$

admitting same construction for hypermultiplet structures for reductions to 4, 5 and 6 dimensions

- for infinitesimal deformations, part of complex

$$\dots \longrightarrow E \xrightarrow{L_{\cdot} J_a} TX \xrightarrow{\mathrm{d}\mu_a} E^* \otimes \mathfrak{su}(2) \longrightarrow \dots .$$

elliptic? then $\mathcal{M}$ is finite dimensional, understand cohomology to identify moduli

- structure is candidate U-duality extension of target space theory of A and B topological string, moment map construction plays role of Kähler or Kodaira-Spencer gravity