Simulations of Dyon Configurations in SU(2) Yang-Mills Theory

Benjamin Maier

in collaboration with

Falk Bruckmann, Simon Dinter, Ernst-Michael Ilgenfritz,
Michael Müller-Preußker, Marc Wagner

Physics Department
Humboldt-University of Berlin

March 28, 2011

HUMBOLDT-UNIVERSITÄT ZU BERLIN

SU(2) Yang-Mills Theory

- describes gluons (and infinitely heavy quarks)
- rough approximation of QCD
- defined by action

$$S[A] = \frac{1}{4g^2} \int d^4x \; F^a_{\mu\nu} \; F^a_{\mu\nu},$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$$

- calculation of observable O with path integral

$$\langle O \rangle = \frac{1}{Z} \int DA \; O[A] \exp\left(-S[A]\right)$$
$$Z = \int DA \; \exp\left(-S[A]\right)$$

$\Rightarrow$ obtain qualitative unterstanding of YM theory and confinement

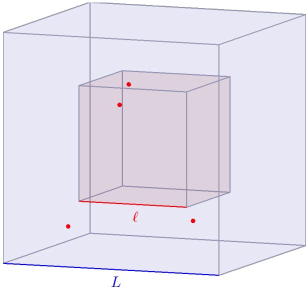
Dyons

- dyons: approximative classical solutions with small action
- gauge field is superposition of Coulombic gauge field
- Diakonov, et al.: "Confining ensemble of dyons" (Phys. Rev. D 76, 056001) - attempt to treat dyon ensembles analytically
- first numerical attempt: "Cautionary remarks on the moduli space metric for multi-dyon simulations" by other members of collaboration (arXiv:0903.3075v1)

Problems with Long-range Dyon Fields

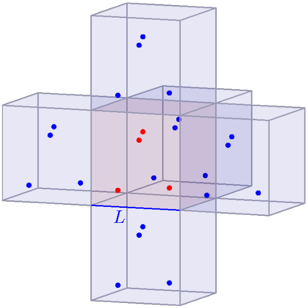
problem long-range potential $q/r \Rightarrow$ rather large volume is needed
$\Rightarrow$ two possible solutions

1. simulating a cubic spatial volume of length L, but evaluate observables within a spatial volume of length $\ell < L$



$\Rightarrow$ extrapolation to infinite volume in ℓ and L

2. copy the cubic volume of length L infinitely often in all directions



$\Rightarrow$ extrapolation in L

$\Rightarrow$ Ewald's method

Ewald's Method

- pedagogical introduction: "Ewald Summation for Coulombic Interactions in a Periodic Supercell" by H. Lee & W. Cai
- split potential into short-range part and long-range part

$$A_0(\mathbf{r}) = A_0^{\text{Short}}(\mathbf{r}) + A_0^{\text{Long}}(\mathbf{r})$$

- A_0^{Short} converges exponentially

$$A_0^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \sum_{j=1}^{n_D} \frac{q_j}{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|} \,\text{erfc}\left(\frac{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

- A_0^{Long} converges exponentially in Fourier space (with momenta $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$)

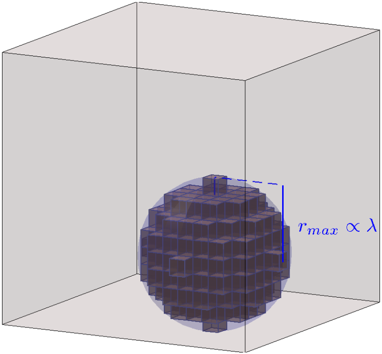
$$A_0^{\text{Long}}(\mathbf{r}) = \frac{4\pi}{L^3} \sum_{\mathbf{k}\neq 0} \sum_{j=1}^{n_D} q_j e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)} \frac{e^{-\lambda^2\mathbf{k}^2/2}}{\mathbf{k}^2}$$

- divergencies cancel in case of neutral box

Ewald's Method more in Detail

Short-range

$$A_0^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \sum_j \frac{q_j}{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|} \operatorname{erfc}\left(\frac{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

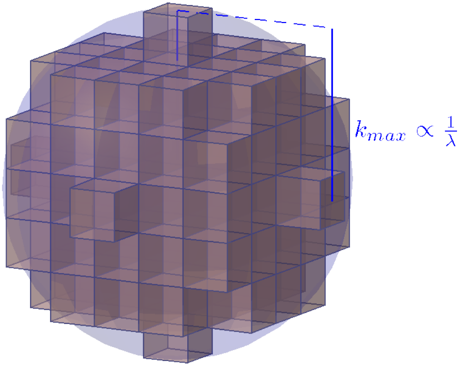


- λ: arbitrary parameter which controls the tradeoff between A_0^{Short} and A_0^{Long}
- due to exponential convergence of A_0^{Short}, evaluation can be restricted to dyons within a sphere of radius $r_{\text{max}} \propto \lambda$

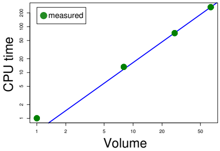
Ewald's Method more in Detail

Long-range

$$A_0^{\text{Long}}(\mathbf{r}) = \frac{4\pi}{L^3} \sum_{\mathbf{k}\neq 0} e^{+i\mathbf{k}\mathbf{r}} \frac{e^{-\lambda^2\mathbf{k}^2/2}}{\mathbf{k}^2} \left(\sum_{j=1}^{n_D} q_j e^{-i\mathbf{k}\mathbf{r}_j} \right), \qquad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}$$



- perform a sum over momenta pointing on volume copies (within a sphere of radius $k_{\text{max}} \propto \frac{1}{\lambda}$)
- calculate the structure functions once
- performance for evaluating A_0 on a grid of $M \propto V$ points **r**
  - Shortrange: $\mathcal{O}(V\lambda^3)$
  - Longrange: $\mathcal{O}(V^2/\lambda^3)$



$\Rightarrow$ choose $\lambda^3 \propto \sqrt{V}$ for scaling of $\mathcal{O}(V^{3/2})$

Numerical Results

Setup and method of computation

- choose density
- put dyons on random positions in a cubic spatial volume for 30 to 800 configurations (non-interacting dyon model)
- vary dyon number (between 1 000 and 125 000) at fixed density to extrapolate to infinite volume
- evaluate $A_0(\mathbf{r})$ at various points $\mathbf{r}$ using Ewald's method
- the Polyakov-loop correlator $\left\langle P(\mathbf{r})P^\dagger(\mathbf{r}')\right\rangle$ can directly be obtained from $A_0(\mathbf{r})$ using

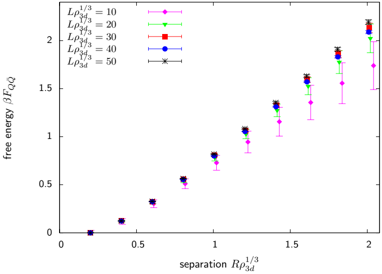
$$P(\mathbf{r}) = \sin\left(\frac{1}{2T}A_0(\mathbf{r})\right)$$

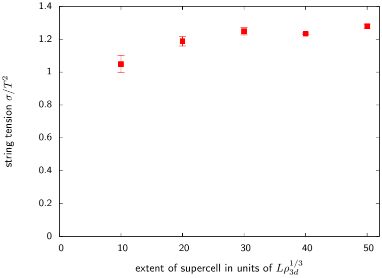
- calculate free energy between a static quark antiquark pair at separation $R = |\mathbf{r} - \mathbf{r}'|$ from

$$F_{Q\bar{Q}}(R) = -T\ \ln\left\langle P(\mathbf{r})P^\dagger(\mathbf{r}')\right\rangle$$

Free Energy and String Tension

Free energy $F_{Q\bar{Q}}$ for non-interacting dyons from Polyakov-loop correlators obtained with Ewald's method





Results for non-interacting dyons

- $F_{Q\bar{Q}}$ linear in quark antiquark separation ⇒ confinement
- converging string tension with increasing volume ⇒ controlled extrapolation to infinite volume

Summary & Outlook

Summary

- Ewald's method: efficient algorithm for superposition of long-range objects in field theories
- controlled extrapolation of observables to infinite volume (e.g. string tension)
- dyon model (even without interactions) generates confinement

Ongoing Projects / Future Plans

- simulate an interacting dyon model by expanding an "effective action" $S_{\text{eff}} = \frac{1}{2} \sum_j \sum_i \ln \left(1 - \frac{2 q_i q_j}{\pi \left| \mathbf{r}_i - \mathbf{r}_j \right|} \right)$ (based on the moduli space metric of calorons) in inverse powers of r using Ewald's method
- understand effects of interacting/non-interacting dyon model on string tension

Backup Slides

Ewald's Method for $1/r^p$

- using the gamma function $\Gamma(x)$ one is able to find Ewald sums for all potentials $\Phi(\mathbf{r}) = \frac{1}{r^p}, p \in \mathbb{R} | p \geq 1$

$$\Phi^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}} \sum_{j=1}^{n_D} \frac{q_j}{|\mathbf{r} - \mathbf{r}_j + \mathbf{n}L|^p} \, g_p\left(\frac{|\mathbf{r} - \mathbf{r}_j + \mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

$$\Phi^{\text{Long}}(\mathbf{r}) = \frac{\pi^{3/2}}{V\left(\sqrt{2}\lambda\right)^{p-3}} \sum_{j} \sum_{\mathbf{k}} q_j \, \exp\left(i\,\mathbf{k}(\mathbf{r} - \mathbf{r}_j)\right) f_p\left(\frac{k\lambda}{\sqrt{2}}\right)$$

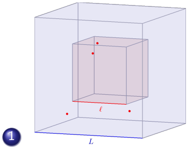
- with the decay functions

$$g_p(x) = \frac{2}{\Gamma(p/2)} \int_x^{\infty} s^{p-1} \exp\left(-s^2\right) \, ds,$$

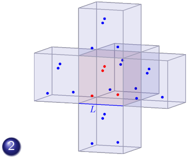
$$f_p(x) = \frac{2x^{p-3}}{\Gamma(p/2)} \int_x^{\infty} s^{2-p} \exp\left(-s^2\right) \, ds.$$

$\Rightarrow$ long-range potentials can be evaluated in powers of $1/r$ for an efficient algorithm

Finite Volume Effects



- small $\ell \Rightarrow$ small finite volume effects
- reduces dyon number which can be treated numerically
- increased statistical errors
- extrapolation to infinite volume difficult (controlling two parameters ℓ and L)
- when considering interacting dyons they tend to accumulate at boundaries



- easier extrapolation to infinite volume (only one parameter L)
- homogenous configurations considering interacting dyons
- divergencies in case of non-neutral box
- performing the infinite sum yields to dielectric effects in case of naive $1/r$-summation

$\Rightarrow$ Ewald's method

Dyon Gauge Field

- gauge field of single dyon (for our preliminary computations relevant: a_0)

$$a_0^3(\mathbf{r};q) = \frac{q}{r}; \qquad a_1^3(\mathbf{r};q) = -\frac{qy}{r(r-z)};$$

$$a_2^3(\mathbf{r};q) = +\frac{qx}{r(r-z)}; \qquad a_3^3(\mathbf{r};q) = 0$$

$\Rightarrow$ electric and magnetic charges with $q = \pm 1$ and $\mathbf{E} = \pm\mathbf{B}$

- gauge field of a superposition of dyons

$$A_\mu(\mathbf{r}) = \sum_j a_\mu(\mathbf{r} - \mathbf{r}_j; q_j)$$

Effective Action S_{eff}

- coordinate transformation

$$\int DA \to \prod_{j=1}^{n_D} \int d^3\mathbf{r}_j$$

- from determinant of moduli space metric

$$S_{\text{eff}} = \frac{1}{2} \sum_j \sum_i \ln \left(1 - \frac{2 q_i q_j}{\pi \left| \mathbf{r}_i - \mathbf{r}_j \right|} \right)$$

- series expansion at $r_0 = \infty$

$$S_{\text{eff}} = -\frac{2 q_i q_j}{\pi r} - \frac{2}{\pi^2 r^2} - \frac{8 q_i q_j}{3 \pi^3 r^3}$$

$\Rightarrow$ **Treatable with Ewald's method!**