Simulations of Dyon Configurations in SU(2) Yang-Mills Theory

Benjamin Maier

Physics Department
Humboldt-University of Berlin

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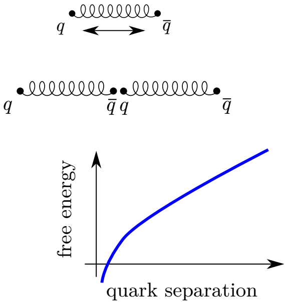
HUMBOLDT-UNIVERSITÄT ZU BERLIN

SU(2) Yang-Mills Theory

- rough approximation of QCD
- describes gluons (and infinitely heavy quarks)
- defined via specific action $S_{\mathrm{YM}}[A]$
- evaluation of observable O with path integral

$$\langle O \rangle = \frac{1}{Z} \int DA \; O[A] \exp\left(-S[A]\right)$$

$$Z = \int DA \; \exp\left(-S[A]\right)$$



$\Rightarrow$ obtain qualitative understanding of YM theory and confinement

Dyons

- dyons: approximative classical solutions with small action
- carry electric charge as well as magnetic charge, abstract charge $q = \pm 1$

$\Rightarrow$ Transformation in the path integral $\int DA \rightarrow \prod_{j=1}^{n_D} \int d^3\mathbf{r}_j \;\det G$

- Diakonov, et al.: "Confining ensemble of dyons" (Phys. Rev. D 76, 056001)
- first numerical attempt: "Cautionary remarks on the moduli space metric for multi-dyon simulations" (F. Bruckmann, S. Dinter, E.-M. Ilgenfritz, M. Müller-Preußker, M. Wagner) (arXiv:0903.3075v1)

Procedure of Observable Evaluation

- choose dyon density ρ, temperature T
- consider dyons to be located in a volume at positions $\{\mathbf{r}_k\}$
- compute gauge field (superposition of relevant component of the dyon gauge field)

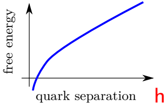
$$\Phi(\mathbf{r}) = \sum_{i=1}^{n_D} \frac{q_i}{|\mathbf{r}_i - \mathbf{r}|}$$

- evaluate Polyakov-loop correlator $\langle P(\mathbf{r})P(\mathbf{r}')\rangle$, which can directly be obtained from $\Phi(\mathbf{r})$ using

$$P(\mathbf{r}) = \sin\left(\frac{1}{2T}\Phi(\mathbf{r})\right)$$

- calculate free energy between a static quark antiquark pair at separation $d = |\mathbf{r} - \mathbf{r}'|$ from

$$F_{Q\bar{Q}}(d) = -T\ \ln\left\langle P(\mathbf{r})P(\mathbf{r}')\right\rangle$$

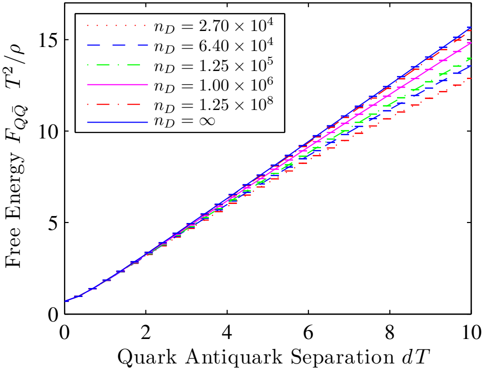


hopefully!

$\Rightarrow$ investigate behavior of free energy for growing quark separation

Analytical Results for Non-Interacting Dyons

- moduli space metric: $\det G = 1 \Rightarrow$ analytical Polyakov loop averaging over dyon positions possible



Result

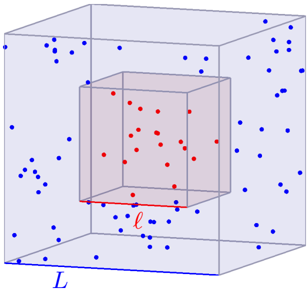
$F_{Q\bar{Q}}$ linear in quark antiquark separation $\Rightarrow$ confinement

Problems with Long-range Dyon Fields

problem long-range potential $q/r \Rightarrow$ rather large volume is needed

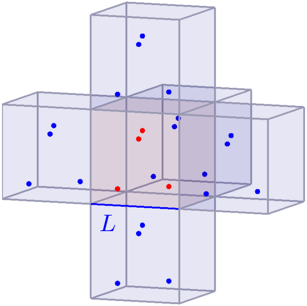
$\Rightarrow$ two possible solutions

1. simulating a cubic spatial volume of length L, but evaluate observables within a spatial volume of length $\ell < L$



$\Rightarrow$ extrapolation to infinite volume in ℓ and L

2. copy the cubic volume of length L infinitely often in all directions



$\Rightarrow$ extrapolation in L

$\Rightarrow$ Ewald's method

Ewald's Method

- pedagogical introduction: "Ewald Summation for Coulombic Interactions in a Periodic Supercell" by H. Lee & W. Cai
- split superposition of dyon potentials into short range part and long range part

$$\Phi(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \sum_{j=1}^{n_D} \frac{q_j}{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|} \quad = \quad \Phi^{\text{Short}}(\mathbf{r}) + \Phi^{\text{Long}}(\mathbf{r})$$

- Φ^{Short} converges exponentially

$$\Phi^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \sum_{j=1}^{n_D} \frac{q_j}{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|} \operatorname{erfc}\left(\frac{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

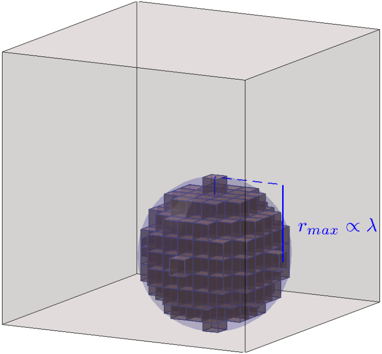
- Φ^{Long} converges exponentially in Fourier space (with momenta $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$)

$$\Phi^{\text{Long}}(\mathbf{r}) = \frac{4\pi}{L^3} \sum_{\mathbf{k}\neq 0} \sum_{j=1}^{n_D} q_j e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)} \frac{e^{-\lambda^2\mathbf{k}^2/2}}{\mathbf{k}^2}$$

Ewald's Method more in Detail

Short-range

$$\Phi^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \sum_j \frac{q_j}{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|} \, \text{erfc}\left(\frac{|\mathbf{r}-\mathbf{r}_j-\mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

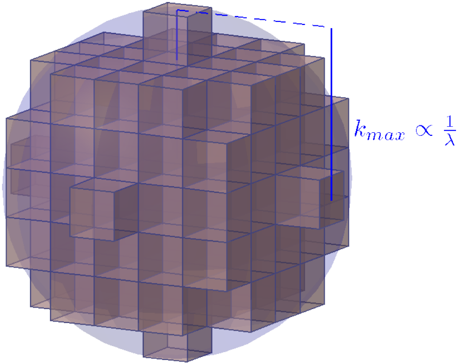


- λ: arbitrary parameter which controls the tradeoff between Φ^{Short} and Φ^{Long}
- due to exponential convergence of Φ^{Short}, evaluation can be restricted to dyons within a sphere of radius $r_{\text{max}} \propto \lambda$

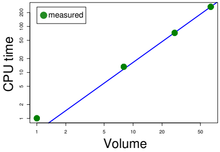
Ewald's Method more in Detail

Long-range

$$\Phi^{\text{Long}}(\mathbf{r}) = \frac{4\pi}{L^3} \sum_{\mathbf{k}\neq 0} e^{+i\mathbf{kr}} \frac{e^{-\lambda^2\mathbf{k}^2/2}}{\mathbf{k}^2} \left(\sum_{j=1}^{n_D} q_j e^{-i\mathbf{kr}_j} \right), \qquad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}$$



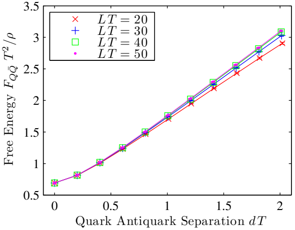
- perform a sum over momenta pointing on volume copies (within a sphere of radius $k_{\text{max}} \propto \frac{1}{\lambda}$)

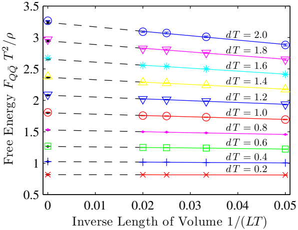


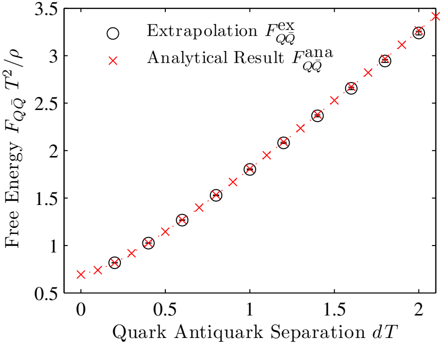
$\Rightarrow$ scaling: $\mathcal{O}(V^{3/2})$

Free Energy by Means of Ewald Summation

Free energy $F_{Q\bar{Q}}$ for non-interacting dyons from Polyakov-loop correlators obtained with Ewald's method







Result

Ewald summation: efficient numerical method to treat long-range objects

Interacting Dyons

- coordinate transformation $\int DA \rightarrow \int \prod_{k=1}^{n_D} d^3\mathbf{r}_k \;\; \det G$
- moduli space metric for a pair (i, j) of dyons proposed by Diakonov, et. al. (already known for two opposite kind dyons: caloron)
- approximate interaction of n_D dyons by superposition of two-body interactions

$$Z = \int d^3\mathbf{r}_k \exp\left(S_{\text{eff}}\right)$$

$$S_{\text{eff}} = \frac{1}{2}\sum_{j=1}^{n_D}\sum_{i=1}^{n_D} \ln\left(1 - \frac{2q_i q_j}{\pi\left|\mathbf{r}_i - \mathbf{r}_j\right|}\right)$$

- $\ln\left(1 + \frac{\#}{r}\right) \propto \frac{1}{r}$ for $r \gg 1$

$\Rightarrow$ same problems as for gauge field computation - Ewald's method!

Interaction by Means of Ewald Summation

- action

$$S = \frac{1}{2} \sum_{j=1}^{n_D} \sum_{i=1}^{n_D} \ln \left(1 - \frac{2 q_i q_j}{\pi \left| \mathbf{r}_i - \mathbf{r}_j \right|} \right)$$

- series expansion

$$S^{\text{Exp}} = -\frac{2 q_i q_j}{\pi r} - \frac{2}{\pi^2 r^2} - \frac{8 q_i q_j}{3 \pi^3 r^3}$$

- Ewald splitting

$$S^{\text{Ewald}} = S^{\text{Long}} + S^{\text{Short}}$$

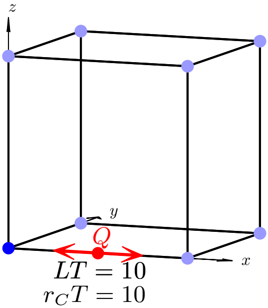
- problem: logarithmic core for small r not represented sufficiently

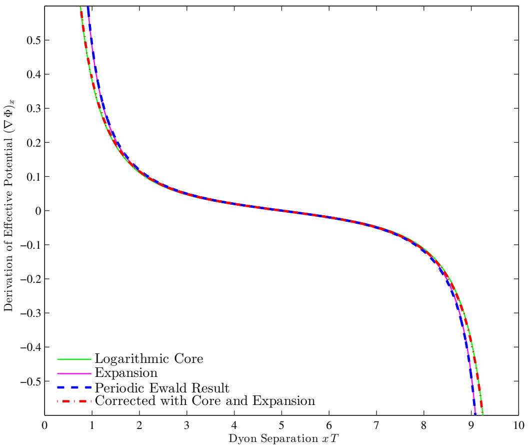
- solution: correct periodic Ewald result for small r (within a sphere of radius r_C)

$$S^{\text{Corr}} = S^{\text{Ewald}} - S^{\text{Exp}} + S^{\text{Log}}$$

Explanation of Correction

Setup:





$$\Phi(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}} \ln\left(1 - \frac{2qQ}{\pi T|\mathbf{n}L - \mathbf{r}|}\right)$$

$$\Phi^{\text{Log}}(\mathbf{r}) = \ln\left(1 + \frac{\#}{|x|}\right) + \ln\left(1 + \frac{\#}{|L-x|}\right)$$

$$\Phi^{\text{Exp}}(\mathbf{r}) = \frac{\#}{|x|} - \frac{\#}{|x|^2} + \frac{\#}{|x|^3} + \frac{\#}{|L-x|} - \frac{\#}{|L-x|^2} + \frac{\#}{|L-x|^3}$$

$$\Phi^{\text{Ewald}}(\mathbf{r}) = \Phi_3^{\text{Short}}(\mathbf{r}) + \Phi_3^{\text{Long}}(\mathbf{r})$$

$$\Phi^{\text{Corr}}(\mathbf{r}) = \Phi^{\text{Ewald}}(\mathbf{r}) - \Phi^{\text{Exp}}(\mathbf{r}) + \Phi^{\text{Log}}(\mathbf{r})$$

Summary & Outlook

Summary

- non-interacting dyon model generates confinement
- Ewald's method: efficient algorithm for superposition of long-range objects in field theories
- controlled extrapolation of observables to infinite volume (e.g. free energy)
- first Metropolis algorithm to approximate dyon interactions is known

Ongoing Projects / Future Plans

- implement Metropolis algorithm and run simulations
- understand effects of interacting/non-interacting dyon model on the free energy

Backup Slides

Ewald's Method for $1/r^p$

- using the gamma function $\Gamma(x)$ one is able to find Ewald sums for all potentials $\Phi(\mathbf{r}) = \frac{1}{r^p}, p \in \mathbb{R} | p \geq 1$

$$\Phi^{\text{Short}}(\mathbf{r}) = \sum_{\mathbf{n}} \sum_{j=1}^{n_D} \frac{q_j}{|\mathbf{r} - \mathbf{r}_j + \mathbf{n}L|^p} \, g_p\left(\frac{|\mathbf{r} - \mathbf{r}_j + \mathbf{n}L|}{\sqrt{2}\lambda}\right)$$

$$\Phi^{\text{Long}}(\mathbf{r}) = \frac{\pi^{3/2}}{V\left(\sqrt{2}\lambda\right)^{p-3}} \sum_{j} \sum_{\mathbf{k}} q_j \, \exp\left(i\,\mathbf{k}(\mathbf{r} - \mathbf{r}_j)\right) f_p\left(\frac{k\lambda}{\sqrt{2}}\right)$$

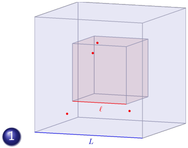
- with the decay functions

$$g_p(x) = \frac{2}{\Gamma(p/2)} \int_x^\infty s^{p-1} \exp\left(-s^2\right) \, ds,$$

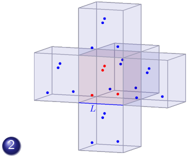
$$f_p(x) = \frac{2x^{p-3}}{\Gamma(p/2)} \int_x^\infty s^{2-p} \exp\left(-s^2\right) \, ds.$$

$\Rightarrow$ long-range potentials can be evaluated in powers of $1/r$ for an efficient algorithm

Finite Volume Effects



- small ℓ $\Rightarrow$ small finite volume effects
- reduces dyon number which can be treated numerically
- increased statistical errors
- extrapolation to infinite volume difficult (controlling two parameters ℓ and L)
- when considering interacting dyons they tend to accumulate at boundaries



- easier extrapolation to infinite volume (only one parameter L)
- homogenous configurations considering interacting dyons
- divergencies in case of non-neutral box
- performing the infinite sum yields to dielectric effects in case of naive $1/r$-summation

$\Rightarrow$ Ewald's method

Dyon Gauge Field

- gauge field of single dyon (for our preliminary computations relevant: a_0)

$$a_0^3(\mathbf{r};q) = \frac{q}{r}; \qquad a_1^3(\mathbf{r};q) = -\frac{qy}{r(r-z)};$$

$$a_2^3(\mathbf{r};q) = +\frac{qx}{r(r-z)}; \qquad a_3^3(\mathbf{r};q) = 0$$

$\Rightarrow$ electric and magnetic charges with $q = \pm 1$ and $\mathbf{E} = \pm\mathbf{B}$

- gauge field of a superposition of dyons

$$A_\mu(\mathbf{r}) = \sum_j a_\mu(\mathbf{r} - \mathbf{r}_j; q_j)$$

Dyon Gauge Field

$$S[A] = \frac{1}{4g^2} \int d^4x \; F^a_{\mu\nu} \; F^a_{\mu\nu},$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$$