Constrained graph counting

Michael Borinsky, Nikhef - Amsterdam

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on a part of a joint work with Karen Vogtmann

Constrained graph counting

- Count graphs with restrictions on edge-induced subgraphs.

$$\sum_{\substack{\text{graphs } G \\ \text{such that } g \not\subseteq G \\ \text{for all } g \in \mathcal{P}}} \frac{\lambda^{|V_G|} w^{|E_G|}}{|\operatorname{Aut} G|}$$

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← labelled graphs

Example: $\mathcal{P} = \{ \triangle \}$ → count triangle free graphs

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- More general: Evaluate 'statistics' on graphs:

$$\sum_{\text{graphs } G} \frac{\lambda^{|V_G|} w^{|E_G|}}{|\operatorname{Aut} G|} \sum_{g \subset G} \phi(g),$$

where ϕ is a function from graphs to $\mathbb{Q}$ or a power series ring.

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Example: weight graphs by # of triangles.

Graphs and chord diagrams

What is a graph?

- Set of *half-edges* H

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In other words,

$$\text{Graph} = (\text{chord diagram}) \times (\text{Set partition})$$

Example

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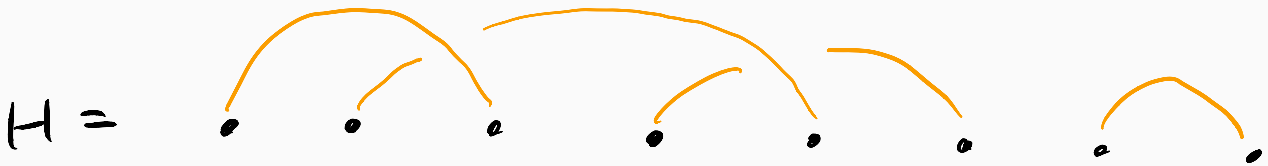
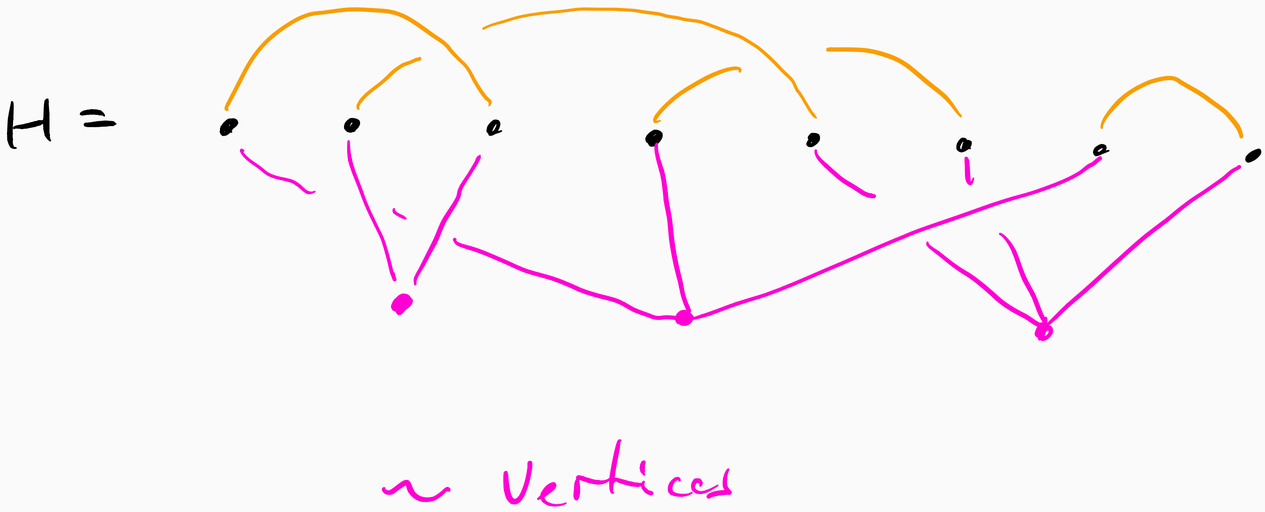


~ edges

Example

Graph = (chord diagram) × (Set partition)



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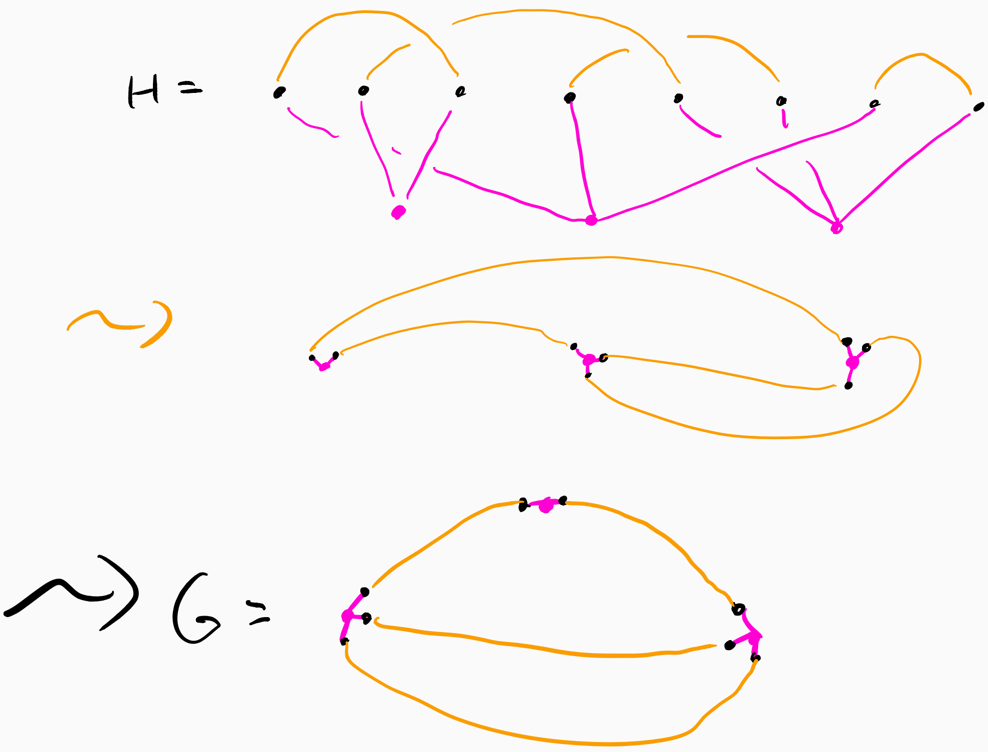

Translation to generating functions

$$\text{Graph} = (\text{chord diagram}) \times (\text{Set partition})$$

$$\sum_{\text{graphs } G} \frac{w^{|E_G|}\lambda^{|V_G|}}{|\operatorname{Aut} G|} = \sum_{m\geq 0} w^m(2m-1)!![x^{2m}]\exp\left(\lambda(e^x-1)\right)$$

# of chord diagrams with m chords

gen. fun. of set partitions

Straightforward generalization

Keep information on degree distribution

$$\Leftrightarrow$$

keep information on signature of the (vertex) partition

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keep information on signature of the (vertex) partition

'Configuration model' of graphs **Bender, Canfield 1978**:

$$\sum_{\text{graphs } G} \frac{w^{|E_G|} \prod_{v \in V_G} \lambda_{|v|}}{|\operatorname{Aut} G|} = \sum_{m \geq 0} w^m (2m-1)!! [x^{2m}] \exp\left(\sum_{k \geq 0} \lambda_k \frac{x^k}{k!} \right)$$

gen. fun of set partitions with specified part sizes.

Applications

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- Critical phenomena
- Topological invariants
  e.g. $\mathcal{M}_{g,n}$ **Kontsevich 1994**

Another generalization

Statistics on subgraphs:

Theorem (MB, Vogtmann 2019)

$$\sum_{\text{graphs } G} \sum_{g \subset G} \phi(g) \frac{w^{|E_{G/g}|}}{|\operatorname{Aut} G|}$$

$$= \sum_{m \geq 0} w^m (2m-1)!![x^{2m}] \exp\left(\sum_{\text{cntd graphs with legs } g} \phi(g) x^{|L_g|} \right)$$

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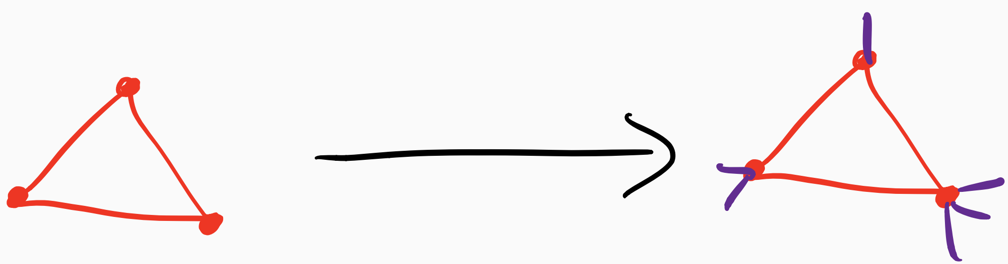
\# of legs



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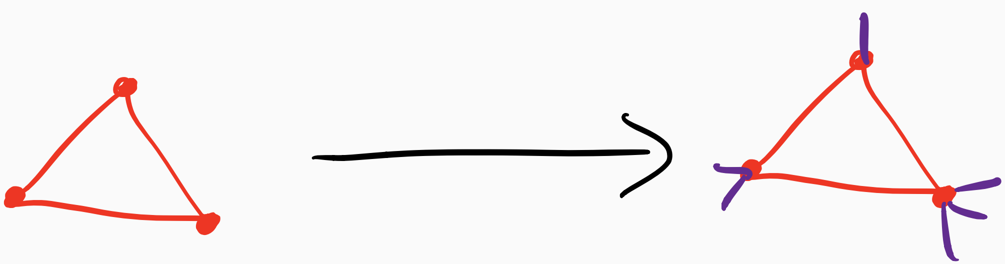
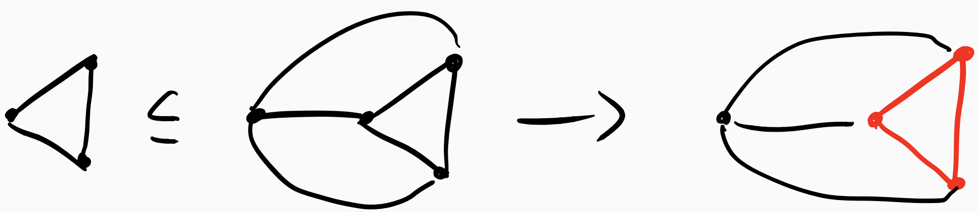


legs ≙ unmatched half-edges

Proof idea

(Pair of graph and subgraph) = (Bi-edge-colored graph)



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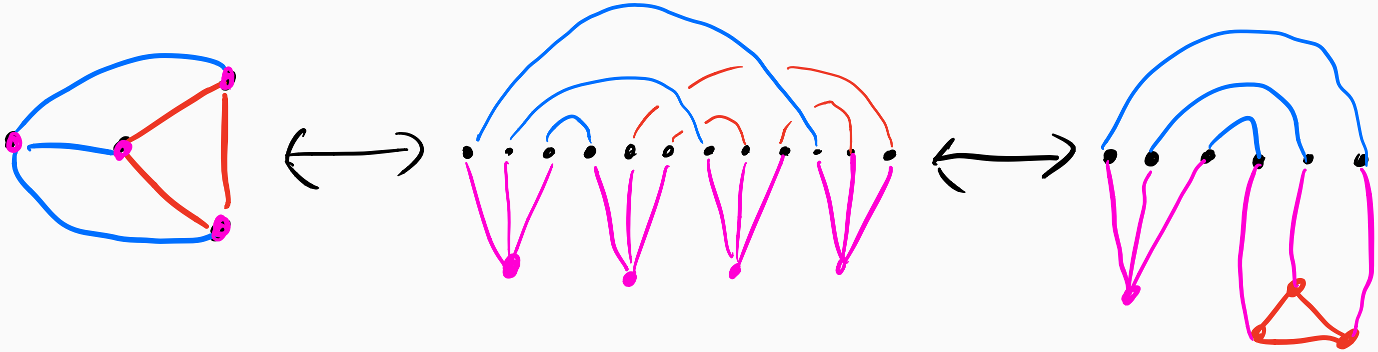

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Natural to define a convolution product on functions ϕ

$$= \phi \star \psi \left(\sum_{\text{graphs } G} \frac{G}{|\operatorname{Aut} G|} \right),$$

where $\psi(G) = w^{|E_G|}$.

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where $\psi(G) = w^{|E_G|}$.

Under mild conditions on the functions, they form a group under this product.

(Possible) applications

$$\sum_{\text{graphs } G} \sum_{g \subset G} \phi(g) \frac{w^{|E_{G/g}|}}{|\operatorname{Aut} G|}$$

- Evaluation of topological invariants e.g. $\operatorname{Out}(F_n)$
  MB, Vogtmann 2019

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- Evaluation of topological invariants e.g. $\mathrm{Out}(F_n)$
  MB, Vogtmann 2019
- Constrained graph counting
- Estimate the number of isomorphism classes of graphs

Example: $\chi(\mathbf{Out}(F_n))$ MB, Vogtmann 2019

$$T(z,x) = \sum_{\text{cntd graphs with legs } g} \tau(g) x^{|L_g|} z^{\chi(G)},$$

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for all non-trivial graphs G and $\tau(\emptyset) = 1$.

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- Can be 'solved' for $T(z,x)$.