The graphical function method in $2n$-dimensions

Michael Borinsky, Nikhef Amsterdam

September 18, Erlangen 2019

joint work with Oliver Schnetz

Motivation

Quantum Field theory

- Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

Quantum Field theory

- Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

- Quantifies correlation between points in space.

Quantum Field theory

- Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

- Quantifies correlation between points in space.
- $G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow$ probability of three 'scalar' events.

Quantum Field theory

- Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

- Quantifies correlation between points in space.
- $G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow$ probability of three 'scalar' events.
- $G(x_1, x_2, x_3) \in V \Rightarrow$ substructure at each point (e.g. spin).

Quantum Field theory

- Objects of interest: Correlation functions

$$G(x_1, x_2, x_3)$$

- Quantifies correlation between points in space.
- $G(x_1, x_2, x_3) \in \mathbb{R} \Rightarrow$ probability of three 'scalar' events.
- $G(x_1, x_2, x_3) \in V \Rightarrow$ substructure at each point (e.g. spin).
- Arbitary number of points can be correlated $G(x_1, x_2, x_3, \ldots)$.

Perturbation theory

- No exact formula for correlation functions!

Perturbation theory

- No exact formula for correlation functions!
- We need perturbation theory:

$$G(x_1, x_2, x_3) = G_0(x_1, x_2, x_3) + \underbrace{\hbar G_1(x_1, x_2, x_3) + \hbar^2 G_2(x_1, x_2, x_3)}_{\text{Quantum corrections!}} + \dots$$

Perturbation theory

- No exact formula for correlation functions!
- We need perturbation theory:

$$G(x_1, x_2, x_3) = G_0(x_1, x_2, x_3) + \hbar G_1(x_1, x_2, x_3) + \hbar^2 G_2(x_1, x_2, x_3) + \ldots$$

- Each $G_n(x_1, x_2, x_3)$ can be written as a sum over graphs:

$$G_n(x_1, x_2, x_3) = \sum_{\substack{\Gamma \\ \chi(\Gamma)=1-n}} \varphi(\Gamma)$$

The function φ associates an integral to each graph.

Perturbation theory

- No exact formula for correlation functions!
- We need perturbation theory:

$$G(x_1, x_2, x_3) = G_0(x_1, x_2, x_3) + \hbar G_1(x_1, x_2, x_3) + \hbar^2 G_2(x_1, x_2, x_3) + \ldots$$

- Each $G_n(x_1, x_2, x_3)$ can be written as a sum over graphs:

$$G_n(x_1, x_2, x_3) = \sum_{\substack{\Gamma \\ \chi(\Gamma)=1-n}} \varphi(\Gamma)$$

  The function φ associates an integral to each graph.
- The graphs are called Feynman graphs. The integrals are called Feynman integrals, the function φ is called Feynman rule.

Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input algebraic integrals:

$$\varphi(\Gamma) = \int \frac{d\Omega}{\mathcal{U}^{D/2}} \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^{\omega}$$

Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input algebraic integrals:

$$\varphi(\Gamma) = \int \frac{d\Omega}{\mathcal{U}^{D/2}} \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^{\omega}$$

- The renormalization group independent part is purely algebraic: The ‘period’

$$\int \frac{d\Omega}{\mathcal{U}^{D/2}}$$

is an interesting number.

Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input algebraic integrals:

$$\varphi(\Gamma) = \int \frac{d\Omega}{\mathcal{U}^{D/2}} \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^{\omega}$$

- The renormalization group independent part is purely algebraic: The 'period'

$$\int \frac{d\Omega}{\mathcal{U}^{D/2}}$$

  is an interesting number.
- For small graphs this number is mostly a linear combination of multiple zeta values.

Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input algebraic integrals:

$$\varphi(\Gamma) = \int \frac{d\Omega}{\mathcal{U}^{D/2}} \left(\frac{\mathcal{U}}{\mathcal{F}} \right)^{\omega}$$

- The renormalization group independent part is purely algebraic: The 'period'

$$\int \frac{d\Omega}{\mathcal{U}^{D/2}}$$

  is an interesting number.
- For small graphs this number is mostly a linear combination of multiple zeta values.
- There exists various number theoretic conjectures on the period: Coaction conjecture, Cosmic galois group, Motives etc.

Two viewpoints

Correlation functions are parametrized by the momentum of particles

Correlation functions are parametrized by the position of particles

Why position space?

Why position space?

Advantages

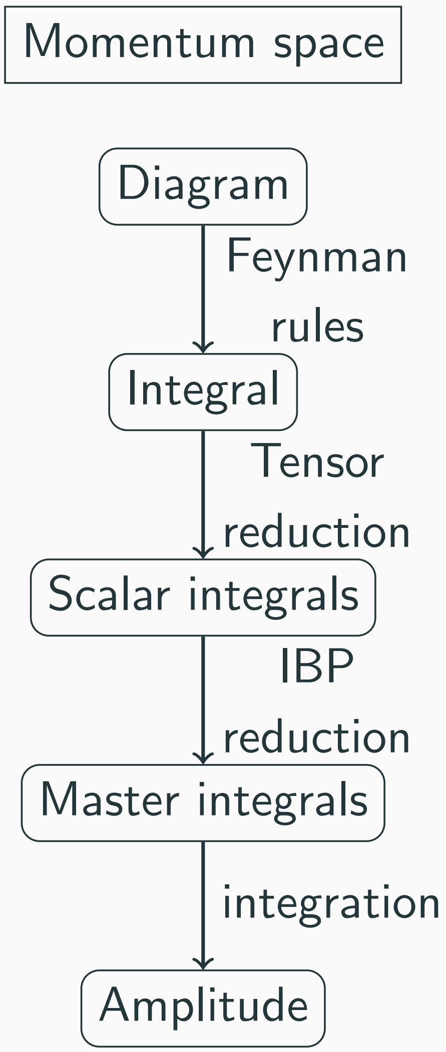
- Simpler Feynman rules
- No IBP reduction necessary
- Conceptually interesting viewpoint

Caveats

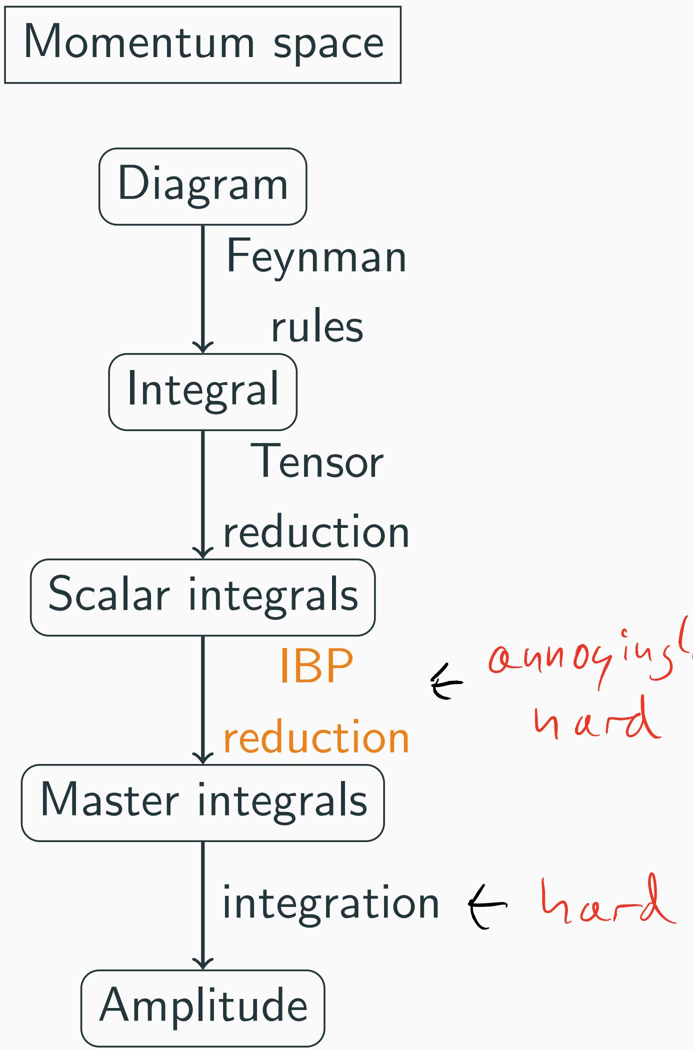
- Limited applications: only renormalization quantities so far
- New technology needed

Proof of concept:
7-loop β-function in ϕ^4 calculated in 2016 by Oliver Schnetz using graphical functions.

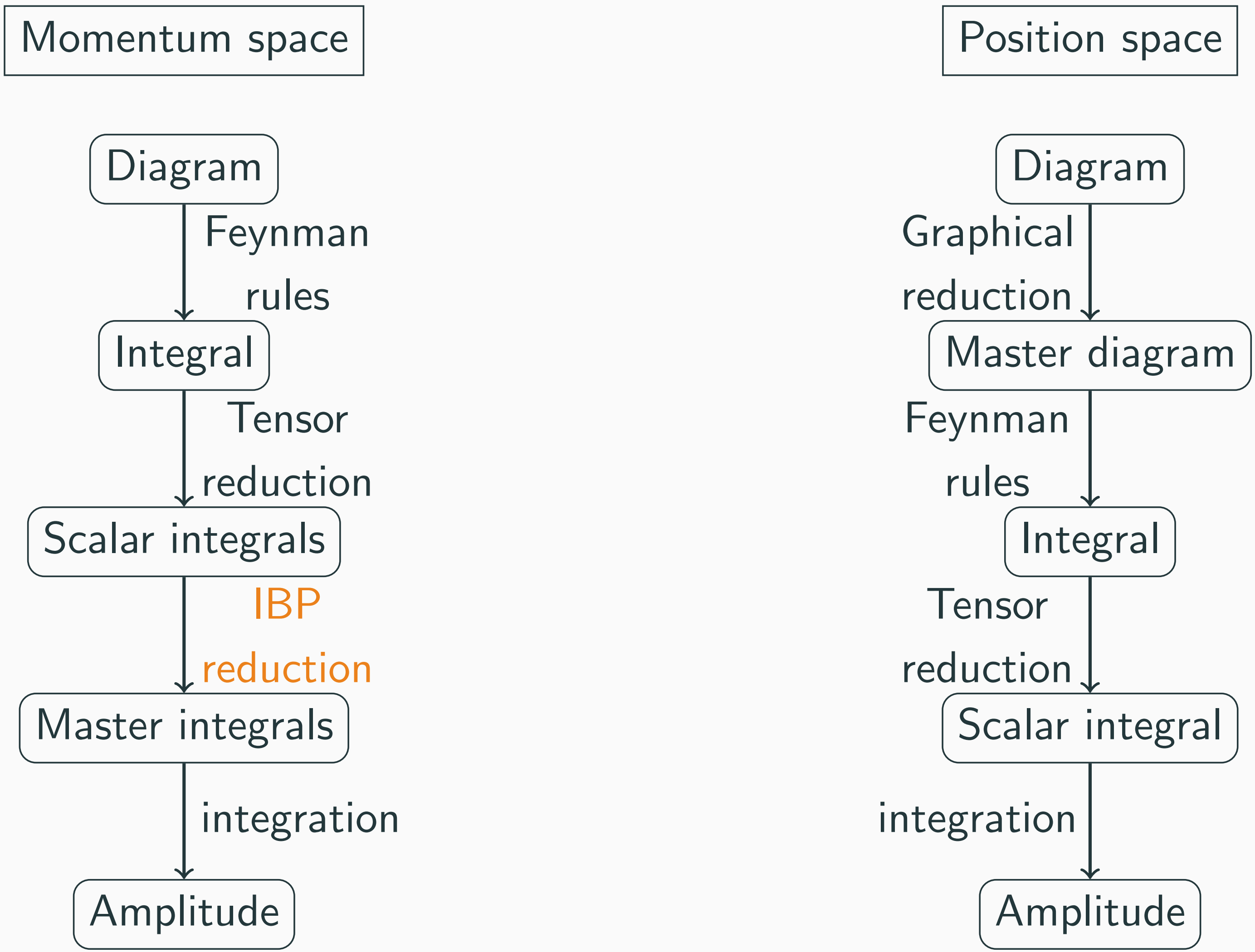
Loop integral workflow



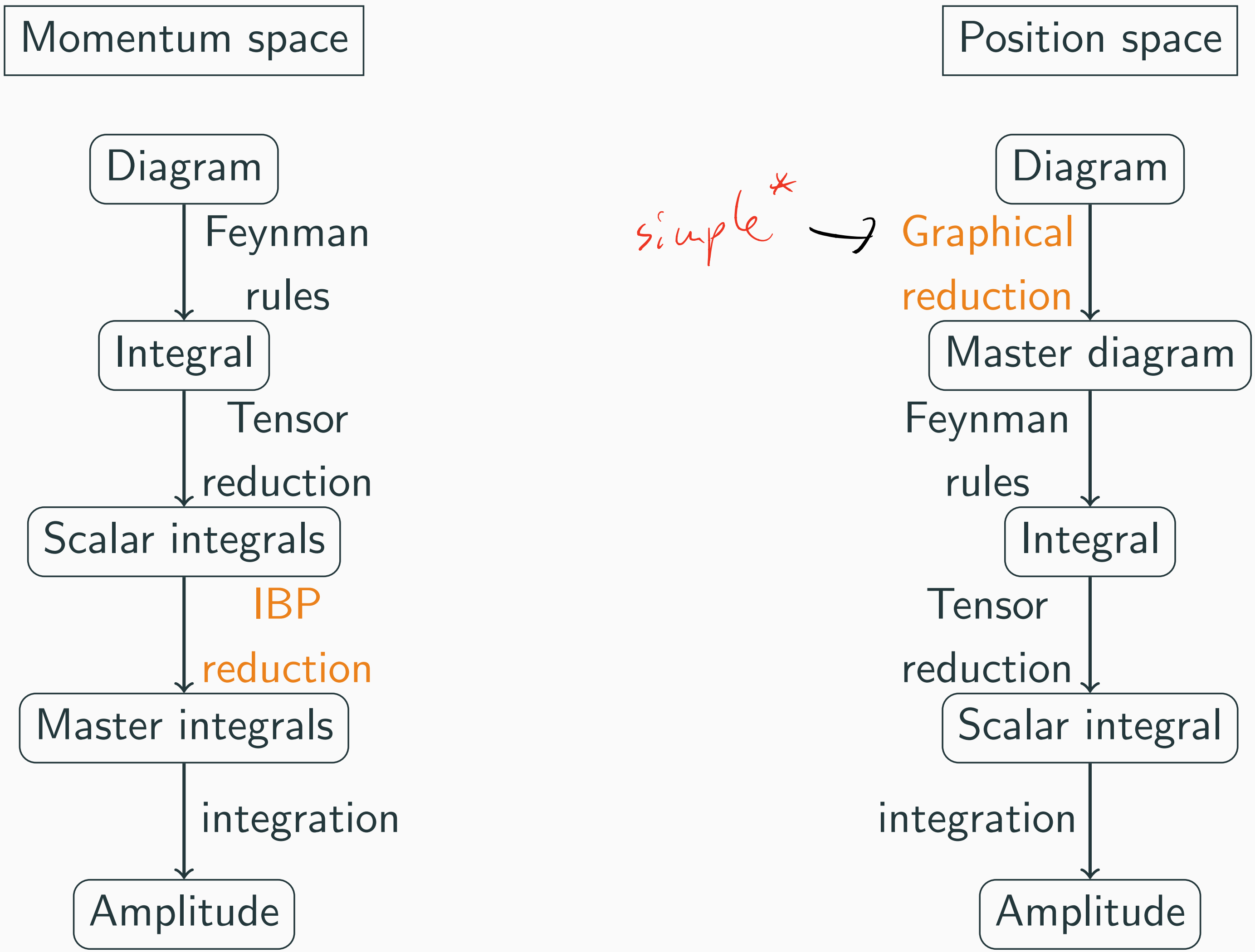
Loop integral workflow



Loop integral workflow



Loop integral workflow



Feynman integral in momentum space

$$\widetilde{G}(p_1,\ldots,p_n) = \left(\prod_{e\in E}\int d^D k_e \widetilde{\Delta}(k_e)\right)\underbrace{\left(\prod_{v\in V_{\text{int}}}\delta^{(D)}\left(\sum_{e\ni v}k_e\right)\right)}$$

Lower dimensional integral

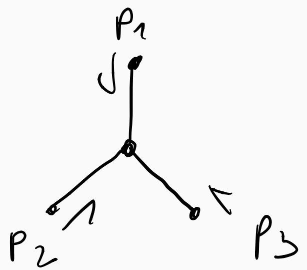
Feynman integral in position space

$$G(x_1,\ldots,x_n) = \left(\prod_{v\in V_{\text{int}}}\int d^D x_v\right)\underbrace{\left(\prod_{\{a,b\}\in E}\Delta(x_a - x_b)\right)}$$

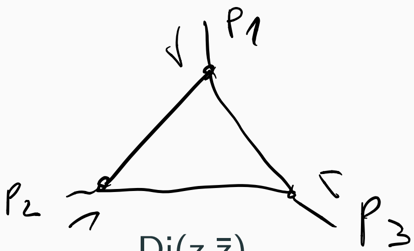
Better factorization properties

Examples

Momentum space



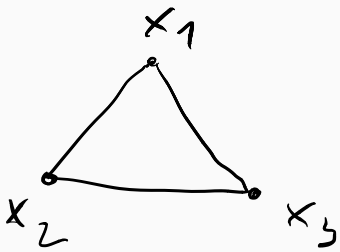
$\widetilde{\Delta}(p_{12})\widetilde{\Delta}(p_{23})\widetilde{\Delta}(p_{31})$



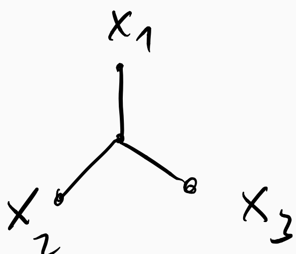
$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(p_{12}^2,p_{23}^2,p_{31}^2)}}$$

$$\widetilde{\Delta}(p) = \frac{1}{\|p\|^2}$$

Position space



$\Delta(x_{12})\Delta(x_{23})\Delta(x_{31})$



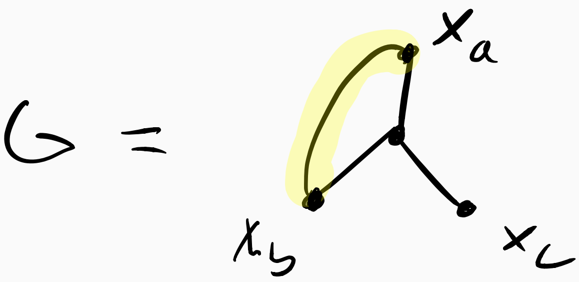
$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(x_{12}^2,x_{23}^2,x_{31}^2)}}$$

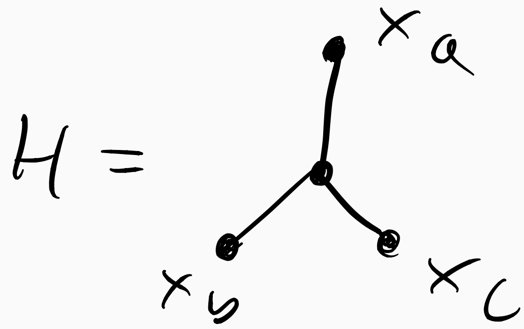
$$\Delta(x) = \frac{1}{\|x\|^2}$$

Graphical reductions

1. rule: propogators between external vertices

$$G(x_a, x_b, x_c) = \int d^D y \Delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)\Delta(x_a - x_b)$$
$$= \Delta(x_a - x_b) H(x_a, x_b, x_c)$$

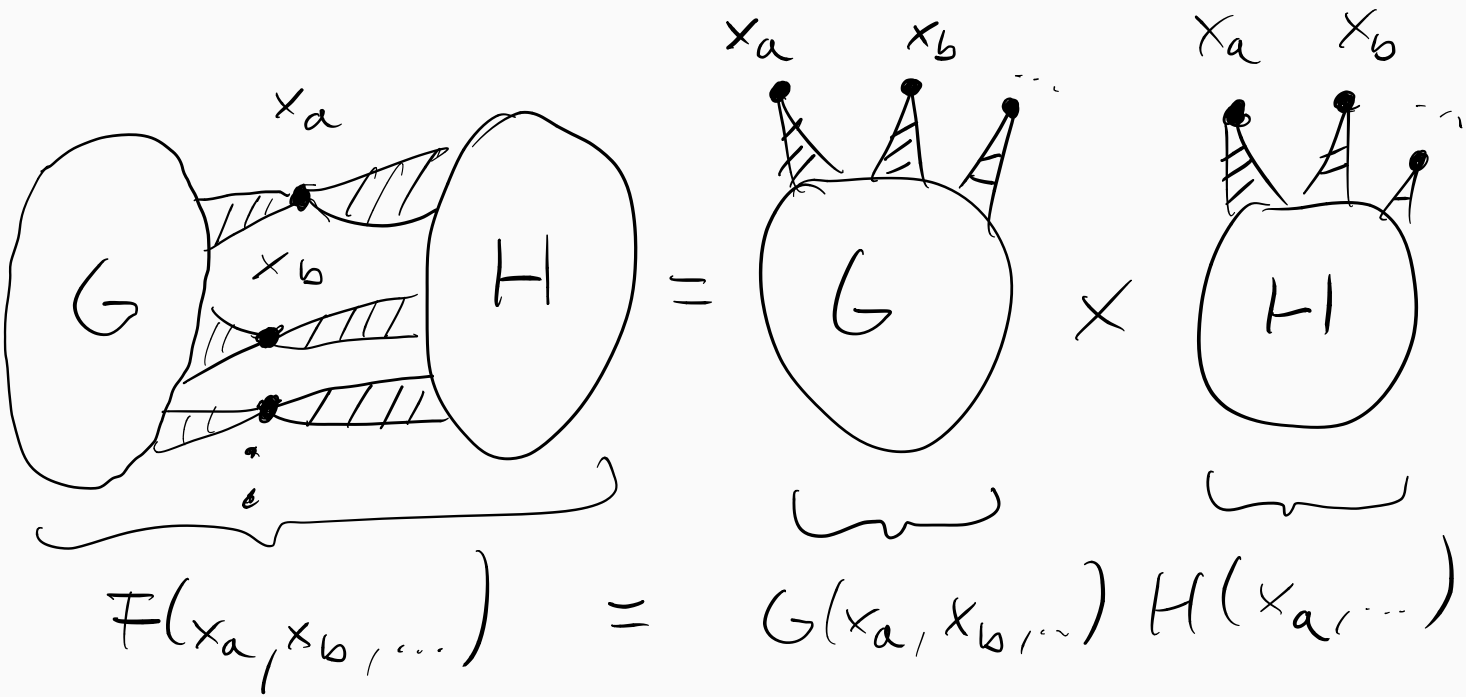




⇒ edges between external vertices factorize.

Graphical reduction rules

2. rule: split graph



⇒ factorizes if split along external vertices.

Graphical reduction rules

Intermezzo: amputating a propagator

Recall the definition of the propagator, Δ, as *Green's function for the free field equation*

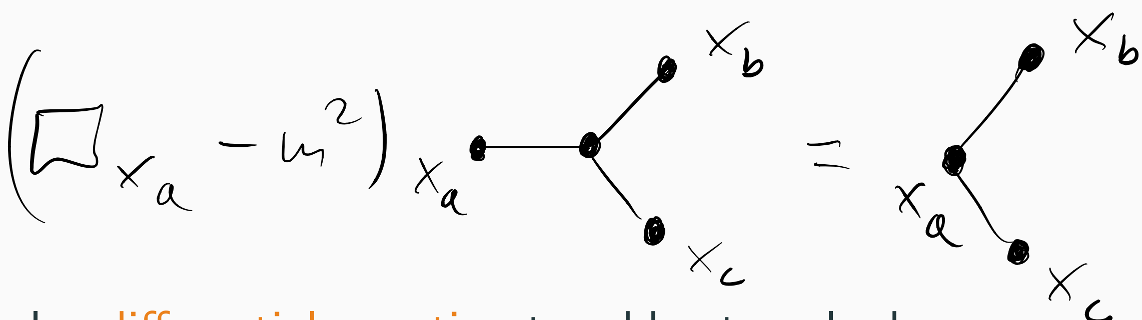
$$(\Box_x - m^2)\Delta(x-y) = \delta^{(D)}(x-y)$$

We can use this equation to *amputate* free external edges.

Graphical reduction rules

3. rule: amputating an external edge

$$
\begin{aligned}
(\Box_{x_a} - m^2)G(x_a, x_b, x_c) &= \int d^D y (\Box_{x_a} - m^2)\Delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y) \\
&= \int d^D y \delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y) \\
&= \Delta(x_b - x_a)\Delta(x_c - x_a) = H(x_a, x_b, x_c)
\end{aligned}
$$



$\Rightarrow$ solve differential equation to add external edge.

Differential equations

For rule 3, a differential equation needs to be solved:

$(\Box_{x_a} - m^2)G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

Can be solved systematically if (Schnetz 2013)

- particles are massless, $m = 0$,
- only 3-point functions are considered
- in $D = 4 - \epsilon$ Euklidean space.

3-point configuration space is 2-dimensional $\Rightarrow$

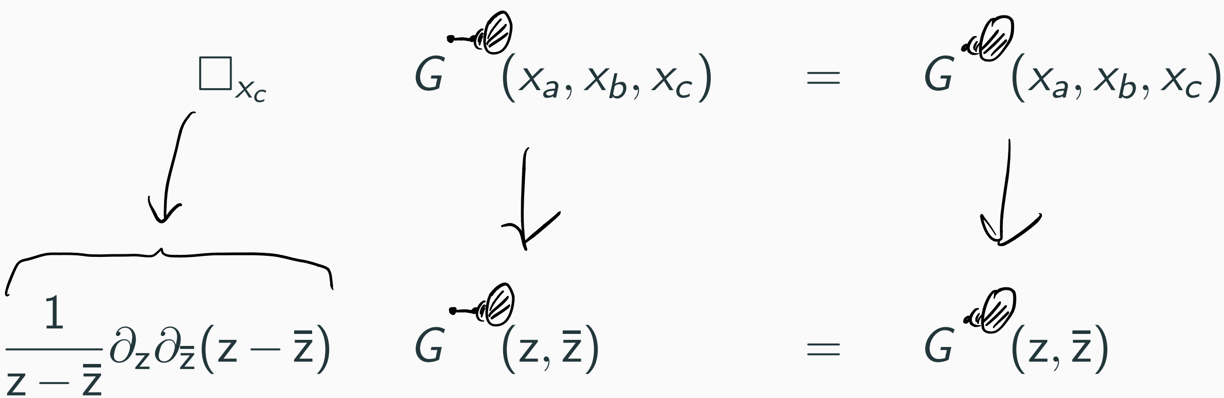
Use complex paramater z such that

$$\mathrm{z}\,\bar{\mathrm{z}} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-\mathrm{z})(1-\bar{\mathrm{z}}) = \frac{x_{bc}^2}{x_{ab}^2}$$

3-point configuration space is 2-dimensional $\Rightarrow$

Use complex paramater z such that

$$z\,\bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}$$



The ∂_z and $\partial_{\bar{z}}$ operators can be inverted in the function space of generalized single-valued hyperlogarithms (Chavez, Duhr 2012, Schnetz 2014, Schnetz 2017).

Graphical functions

- Rules 1,2,3 are part of a larger framework: graphical functions (Schnetz 2013).
- Graphical functions can also be applied in a broader context, e.g. to conformal amplitudes (Basso, Dixon 2017).
- Calculation within this framework are extremely efficient, due to the rapid reductions and small numbers of irreducible *master diagrams*.

Graphical functions for gauge theory

Beyond scalar

Only change: adding an edge

For instance, for abelian gauge theory:

$$\Box_x \to \not{\partial} \text{ and } \eta^{\mu\nu}\Box_x$$

Only change: adding an edge

For instance, for abelian gauge theory:

$$\Box_x \to \not{\partial} \text{ and } \eta^{\mu\nu}\Box_x$$

The differential equation for appending an edge,

$\Box_{x_a} G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

becomes a system of differential equations

$\not{\partial}_{x_a} G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

Paramatrizing non-scalar graphical functions

$$\not{\partial}_{x_c} \qquad G(x_a, x_b, x_c) = G(x_a, x_b, x_c)$$

Paramatrizing non-scalar graphical functions

$$\not{\partial}_{x_c} \qquad G(x_a, x_b, x_c) = G(x_a, x_b, x_c)$$

$$\left(\not{\lambda}\,\partial_z + \not{\bar{\lambda}}\,\partial_{\bar{z}} - \frac{P^{\mu\nu}}{z-\bar{z}} \left(\partial^{\nu}_{\lambda} - \partial^{\nu}_{\bar{\lambda}}\right) \right) \qquad G(z, \bar{z}, \lambda, \bar{\lambda}) = G(z, \bar{z}, \lambda, \bar{\lambda})$$

Using light-cone-like parametrization $z, \bar{z}, \lambda^\mu, \bar{\lambda}^\mu$ such that

$$z\,\bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}$$

$$x_{ab}^\mu = \lambda^\mu + \bar{\lambda}^\mu \qquad x_{ac}^\mu = z\,\lambda^\mu + \bar{z}\,\bar{\lambda}^\mu \qquad x_{bc}^\mu = (1-z)\,\lambda^\mu + (1-\bar{z})\,\bar{\lambda}^\mu$$

$$\lambda^\mu\,\lambda_\mu = \bar{\lambda}^\mu\,\bar{\lambda}_\mu = 0$$

Actual inversion becomes more complicated: $D \neq 4$ dimensional Laplacian has to be inverted.

Extension to $D \neq 4$

- For general dimension D we need to solve,

$$\left(\frac{1}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{D-4}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \right) \quad G$$

$$(z, \bar{z}) \quad = \quad G$$

$$(z, \bar{z}).$$

- For general dimension D we need to solve,

$$\left(\frac{1}{z-\bar{z}}\partial_z\partial_{\bar{z}}(z-\bar{z}) - \frac{D-4}{z-\bar{z}}(\partial_z - \partial_{\bar{z}})\right) G$$

$$(z,\bar{z}) \quad = \quad G$$

$$(z,\bar{z}).$$

- This is also possible for arbitrary even D using a non-trivial linear combination of integration operators.

Extension to $D \neq 4$

- For general dimension D we need to solve,

$$\left(\frac{1}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) - \frac{D-4}{z - \bar{z}} (\partial_z - \partial_{\bar{z}}) \right) G$$

$$(z, \bar{z}) = G$$

$$(z, \bar{z}).$$

- This is also possible for arbitrary even D using a non-trivial linear combination of integration operators.

$\Rightarrow$ Opens the door to calculations in quantum electro dynamics.

Extension to $D \neq 4$

- For general dimension D we need to solve,

$$\left(\frac{1}{z-\bar{z}} \partial_z \partial_{\bar{z}} (z-\bar{z}) - \frac{D-4}{z-\bar{z}} (\partial_z - \partial_{\bar{z}}) \right) \quad G \quad (z, \bar{z}) \quad = \quad G \quad (z, \bar{z}).$$

- This is also possible for arbitrary even D using a non-trivial linear combination of integration operators.

$\Rightarrow$ Opens the door to calculations in quantum electro dynamics.

$\Rightarrow$ Immediately possible with Oliver's tools: ϕ^3-theory. With applications to percolation theory.

Summary

- Efficient graphical reduction replaces IBP reduction in x-space.

Summary

- Efficient graphical reduction replaces IBP reduction in x-space.
- Work in progress: extension to gauge theory.

Summary

- Efficient graphical reduction replaces IBP reduction in x-space.
- Work in progress: extension to gauge theory.
- Intermediate step finished: extension to arbitrary even D.

Summary

- Efficient graphical reduction replaces IBP reduction in x-space.
- Work in progress: extension to gauge theory.
- Intermediate step finished: extension to arbitrary even D.
- Application of ϕ^3-theory: Critical exponents in percolation theory.

Example of a master diagram, which is irreducible w.r.t. rules 1–3: