The Euler characteristic of $\mathrm{Out}(F_n)$

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joint work with Karen Vogtmann

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Introduction I: Groups

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  for each $h \in G$.
- Outer automorphisms: $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$

Automorphisms of the free group

- Consider the free group with n generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

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- The group $\mathrm{Out}(F_n)$ is our main object of interest.

Some properties of $\mathrm{Out}(F_n)$

- Generated by

$$\begin{array}{lllll} & a_1 \mapsto a_1 a_2 & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \dots \\ \text{and} & a_1 \mapsto a_1^{-1} & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \dots \end{array}$$

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  and permutations of the letters.
- The fundamental group of a graph is always a free group,

$$\mathrm{Out}(F_n) = \mathrm{Out}(\pi_1(\Gamma))$$

  for a connected graph Γ with n independent cycles.

Mapping class group

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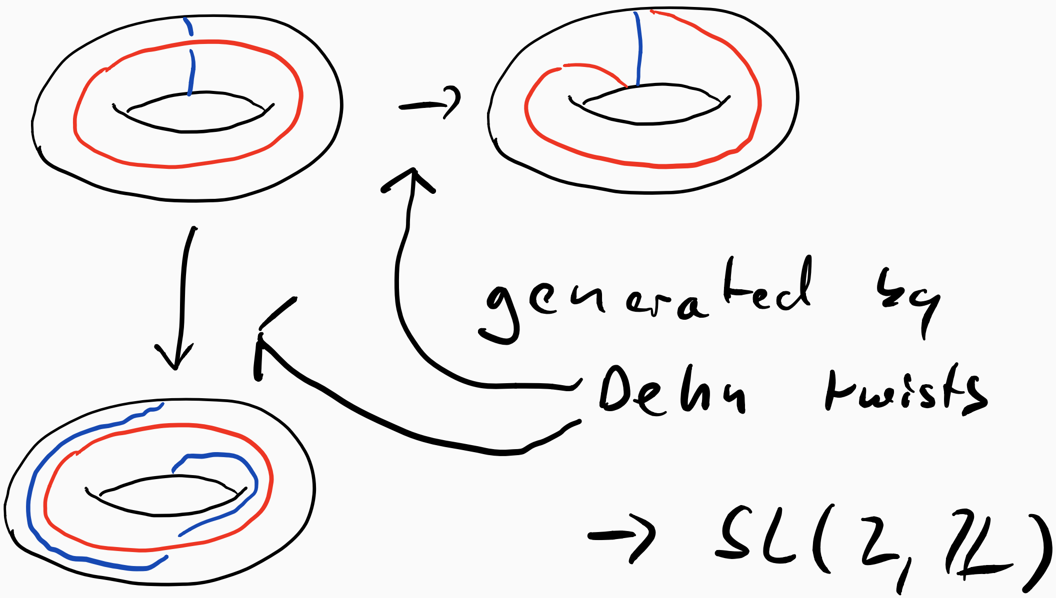
- Another example of an outer automorphism group: the mapping class group
- The group of homeomorphisms of a closed, connected and orientable surface S_g of genus g up to isotopies

$$\mathrm{MCG}(S_g) := \mathrm{Out}(\pi_1(S_g))$$

Example: Mapping class group of the torus

$$\mathsf{MCG}(\mathbb{T}^2) = \mathsf{Out}(\pi_1(\mathbb{T}^2))$$

The group of homeomorphisms $\mathbb{T}^2 \to \mathbb{T}^2$ up to an isotopy:



Introduction II: Spaces

How to study such groups?

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How to study groups such as $\mathrm{MCG}(S)$ or $\mathrm{Out}(F_n)$?

Main idea

Realize G as symmetries of some geometric object.

`Due to Stallings, Thurston, Gromov, ... (1970-)`

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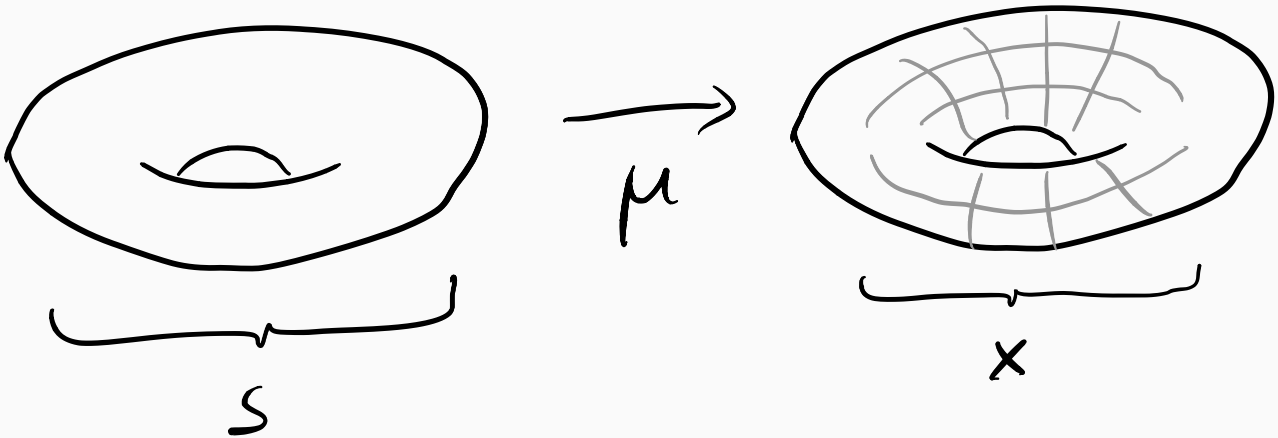
- A Riemann surface X.
- A *marking*: a homeomorphism $\mu : S \to X$.



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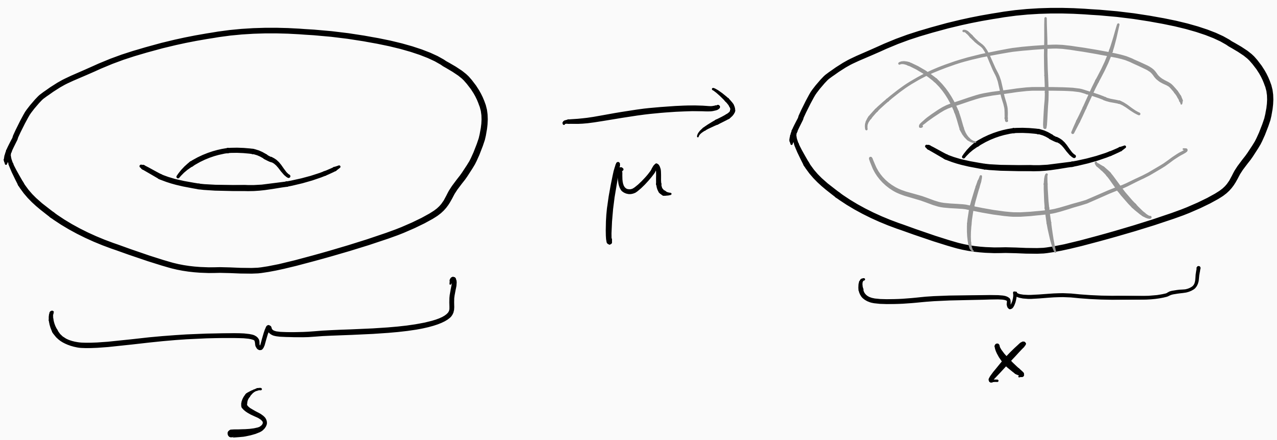

MCG(S) acts on $T(S)$ by composing to the marking:

$$(X, \mu) \mapsto (X, \mu \circ g^{-1}) \text{ for some } g \in \mathrm{MCG}(S).$$

For Out(F_n): Outer space

Idea: Mimic previous construction for Out(F_n).

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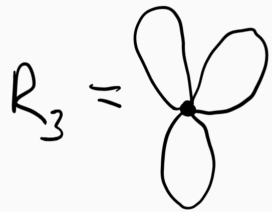

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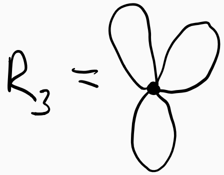

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- A connected graph G with a length assigned to each edge.
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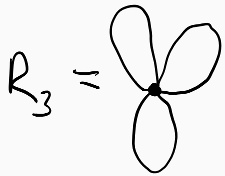

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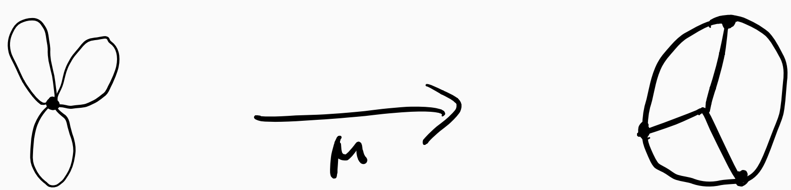
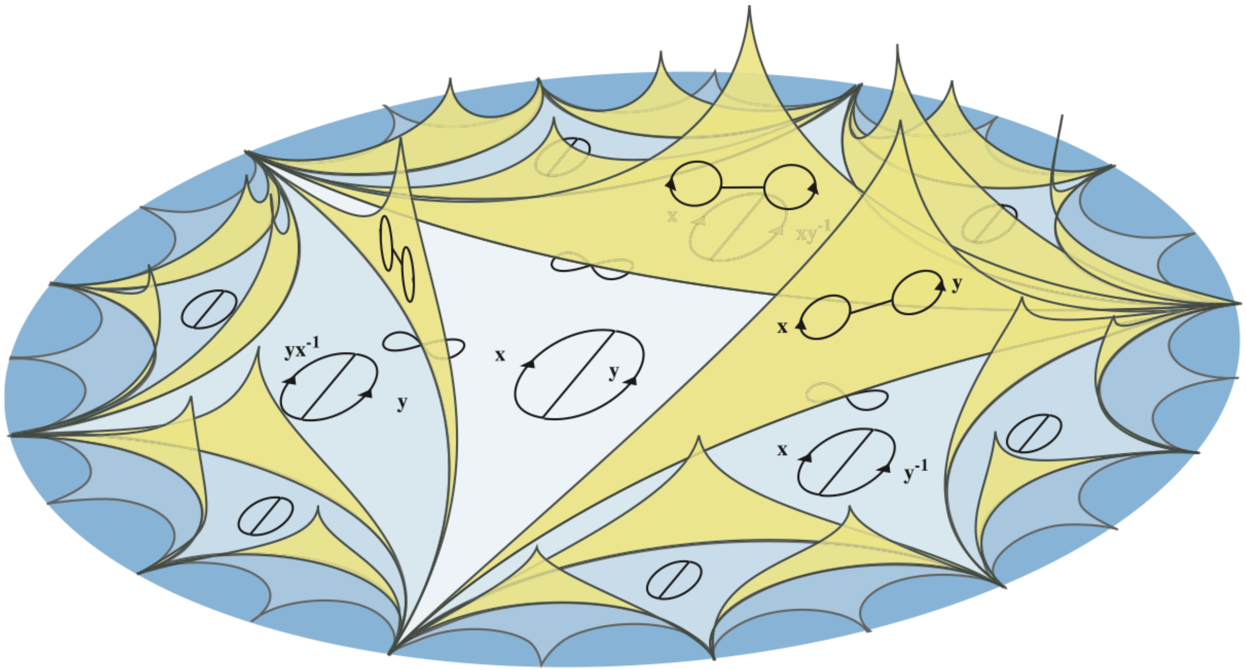


Out(F_n) acts on $\mathcal{O}_n$ by composing to the marking:

$$(\Gamma, \mu) \mapsto (\Gamma, \mu \circ g^{-1}) \text{ for some } g \in \mathrm{Out}(F_n) = \mathrm{Out}(\pi_1(R_n)).$$

$\mathcal{O}_2$



Vogtmann 2008

Examples of applications of Outer space

- The group $\mathrm{Out}(F_n)$
- Moduli spaces of punctured surfaces
- Tropical curves
- Invariants of symplectic manifolds
- Classical modular forms
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$$\text{Scalar QFT} \sim \text{Integrals over}\, \mathcal{O}_n \,/\, \mathrm{Out}(F_n)$$

analogous to

$$\text{2D Quantum gravity} \sim \text{Integral over}\, T(S)/\,\mathrm{MCG}(S)$$

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- Both can be used to study the respective groups.

Summary of the respective groups and spaces

	$\mathrm{MCG}(S_g)$	$\mathrm{Out}(F_n)$
acts freely and properly on	Teichmüller space $\mathcal{T}(S_g)$	Outer space $\mathcal{O}_n$
Quotient X/G	Moduli space of curves $\mathcal{M}_g$	Moduli space of graphs $\mathcal{G}_n$

Invariants

Algebraic invariants

- $H_\bullet(\mathrm{Out}(F_n); \mathbb{Q}) \simeq H_\bullet(\mathcal{O}_n \,/\, \mathrm{Out}(F_n); \mathbb{Q}) = H_\bullet(\mathcal{G}_n; \mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).

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- One simple invariant: Euler characteristic

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$\Rightarrow$ $\mathcal{T}_n$ does not have finitely-generated homology if $\chi(\mathrm{Out}(F_n)) \neq 0$.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

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Theorem Bestvina, Bux, Margalit (2007)

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Results: $\chi(\mathrm{Out}(F_n)) \neq 0$

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- Where does all this homology come from?

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- In this talk: Focus on proof of Theorem B

Analogy to the mapping class group

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

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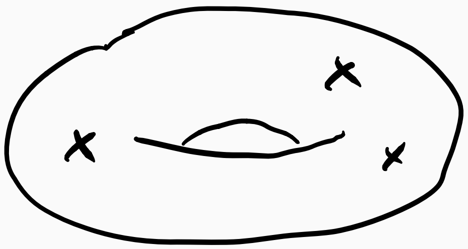
- Simplified proof by Kontsevich (1992) based on TFT's.

$\Rightarrow$ Kontsevich's proof served as a blueprint for $\chi(\mathrm{Out}(F_n))$.

Sketch of Kontsevich's TFT proof of the Harer-Zagier formula

Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.



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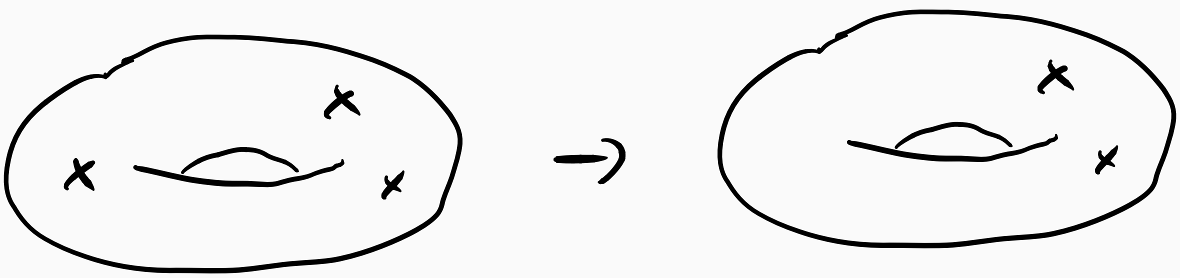

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Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.

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- Γ has n boundary components: $h_0(\partial\Gamma) = n$
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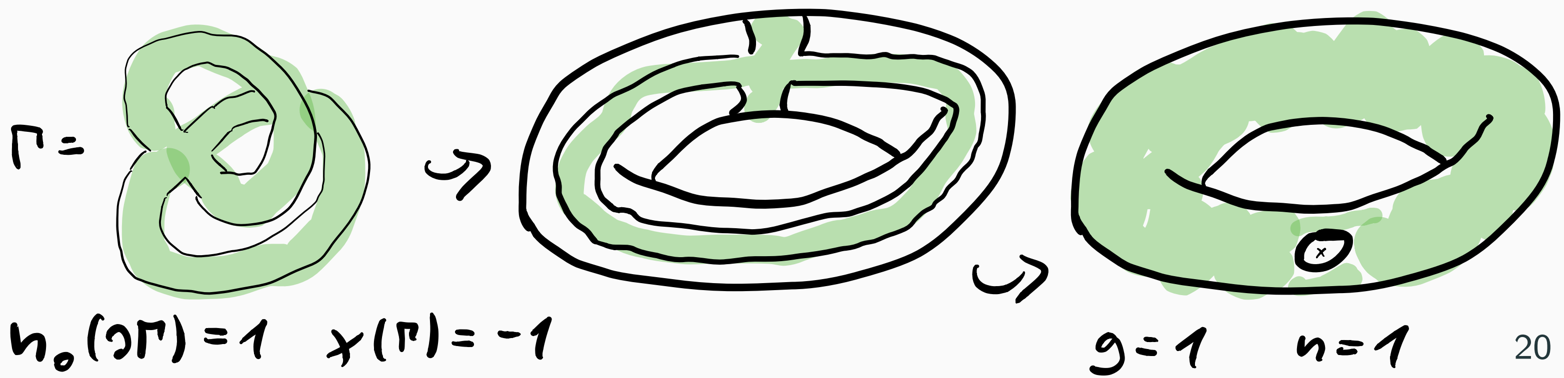

Step 3 of Kontsevich's proof

$$\chi(\mathcal{M}_{g,n}) = \sum_{\sigma} \frac{(-1)^{\mathsf{dim}(\sigma)}}{|\,\mathsf{Stab}(\sigma)|}$$

Step 3 of Kontsevich's proof

dimension of resp. strata

$= |V_\Gamma| \bmod 2$

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Used by Penner (1988) to calculate $\chi(\mathcal{M}_g)$ with Matrix models.

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Kontsevich's simplification:

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$$= \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

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Evaluation is classic (Stirling/Euler-Maclaurin formulas)

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

Last step of Kontsevich's proof

$$\sum_{\substack{g,n \\ 2-2g-n=k}} \frac{\chi(\mathcal{M}_{g,n})}{n!} = \frac{B_{k+1}}{k(k+1)}$$

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⇒ recover Harer-Zagier formula using the identity

$$\chi(\mathcal{M}_{g,n+1}) = (2-2g-n)\chi(\mathcal{M}_{g,n})$$

Analogous proof strategy for $\chi(\mathrm{Out}(F_n))$ using renormalized TFTs

Step 1

Generalize from $\mathrm{Out}(F_n)$ to $A_{n,s}$ and from $\mathcal{O}_n$ to $\mathcal{O}_{n,s}$, Outer space of graphs of rank n and s legs.

Contant, Kassabov, Vogtmann (2011)

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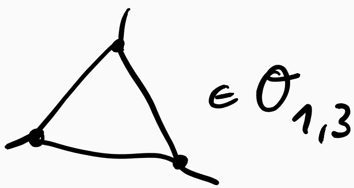
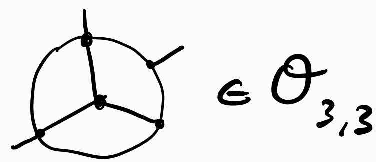
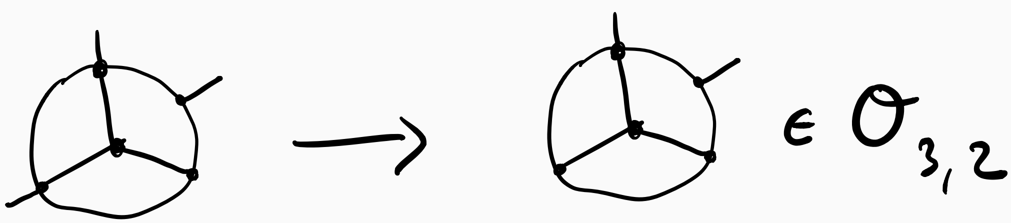




Forgetting a leg gives the short exact sequence of groups

$$1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$$



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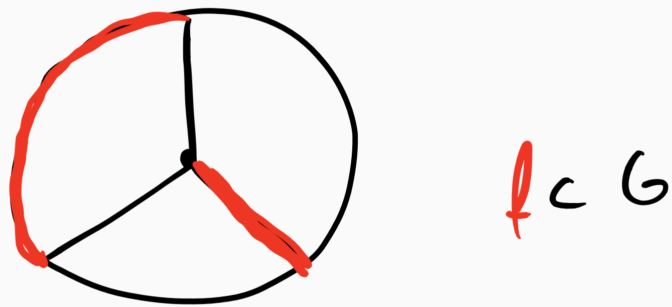

(G, f)

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Renormalized TFT interpretation MB-Vogtmann (2019):

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The group invariants $\chi(A_{n,s})$ are encoded in a renormalized TFT.

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

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This gives the implicit result in Theorem B.

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- Can the TFT analysis be explained with a duality between $\mathrm{MCG}(S_g)$ and $\mathrm{Out}(F_n)$? Obvious candidate: Koszul duality
- Can renormalized TFT arguments also be used for other groups? For instance RAAGs.