Graphical functions applied to ϕ^3 in $D = 6$

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joint work with Oliver Schnetz

Motivation

Quantum Field theory

- Objects of interest: Correlation functions

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- $G(x_1, x_2, x_3) \in V \Rightarrow$ substructure at each point (e.g. spin).
- Arbitary number of points can be correlated $G(x_1, x_2, x_3, \ldots)$.

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- Each $G_n(x_1, x_2, x_3)$ can be written as a sum over graphs:

$$G_n(x_1, x_2, x_3) = \sum_{\substack{\Gamma \\ \chi(\Gamma)=1-n}} \varphi(\Gamma)$$

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- The graphs are called Feynman graphs. The integrals are called Feynman integrals, the function φ is called Feynman rule.

Algebraic integrals: Periods

- The Feynman integrals are except for the dependence on the physical input algebraic integrals:

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  is an interesting number.
- For small graphs this number is mostly a linear combination of multiple zeta values.
- There exists various number theoretic conjectures on the period: Coaction conjecture, Cosmic galois group, Motives etc.

Two viewpoints

Correlation functions are parametrized by the momentum of particles

Correlation functions are parametrized by the position of particles

Why position space?

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Advantages

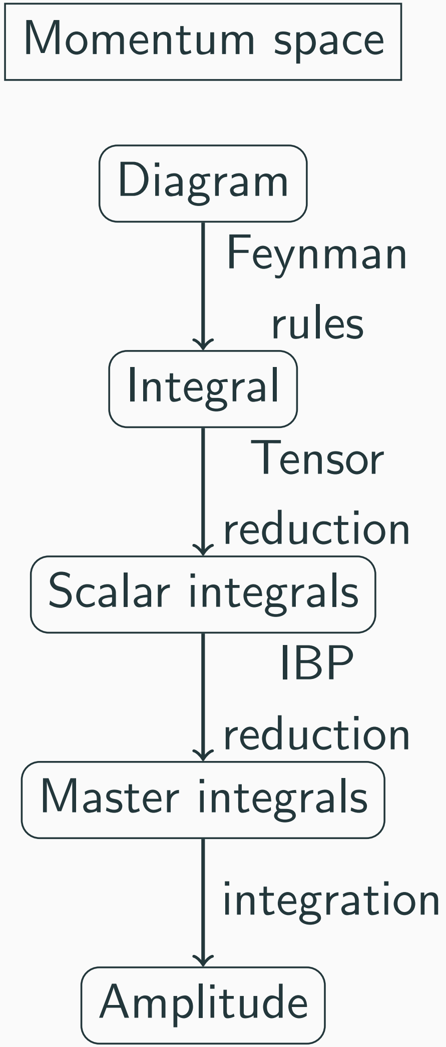
- Simpler Feynman rules
- No IBP reduction necessary
- Conceptually interesting viewpoint

Caveats

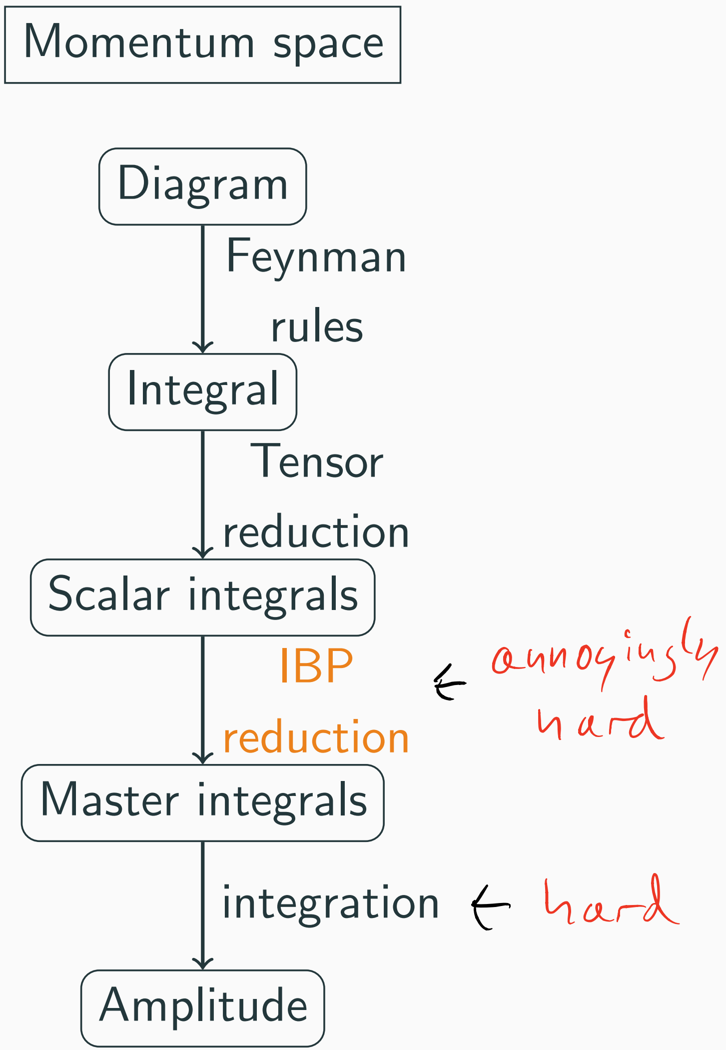
- New technology needed
- Only position space quantities accessible

Proof of concept:
7-loop β-function in ϕ^4 calculated in 2016 by Oliver Schnetz using graphical functions.

Loop integral workflow



Loop integral workflow



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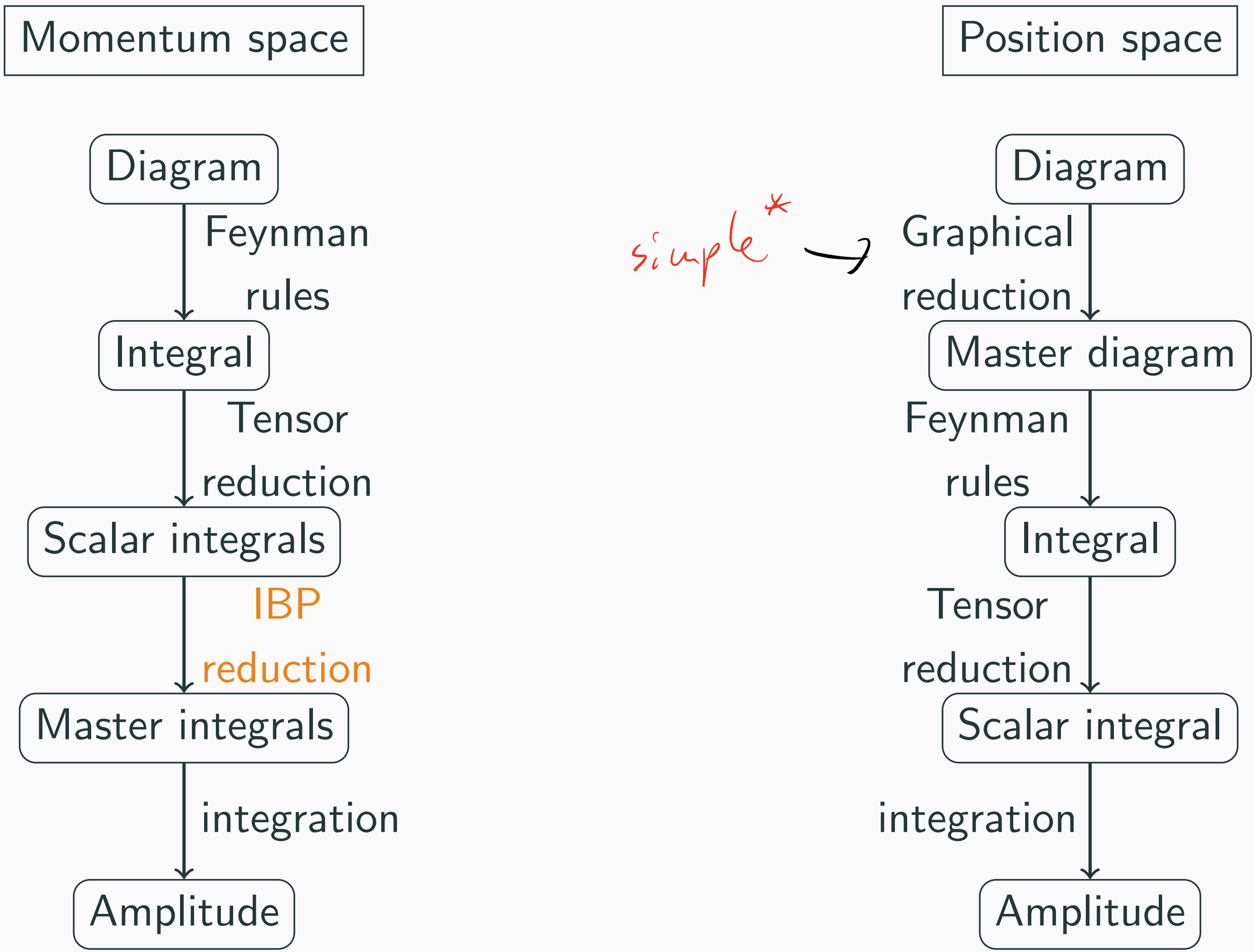

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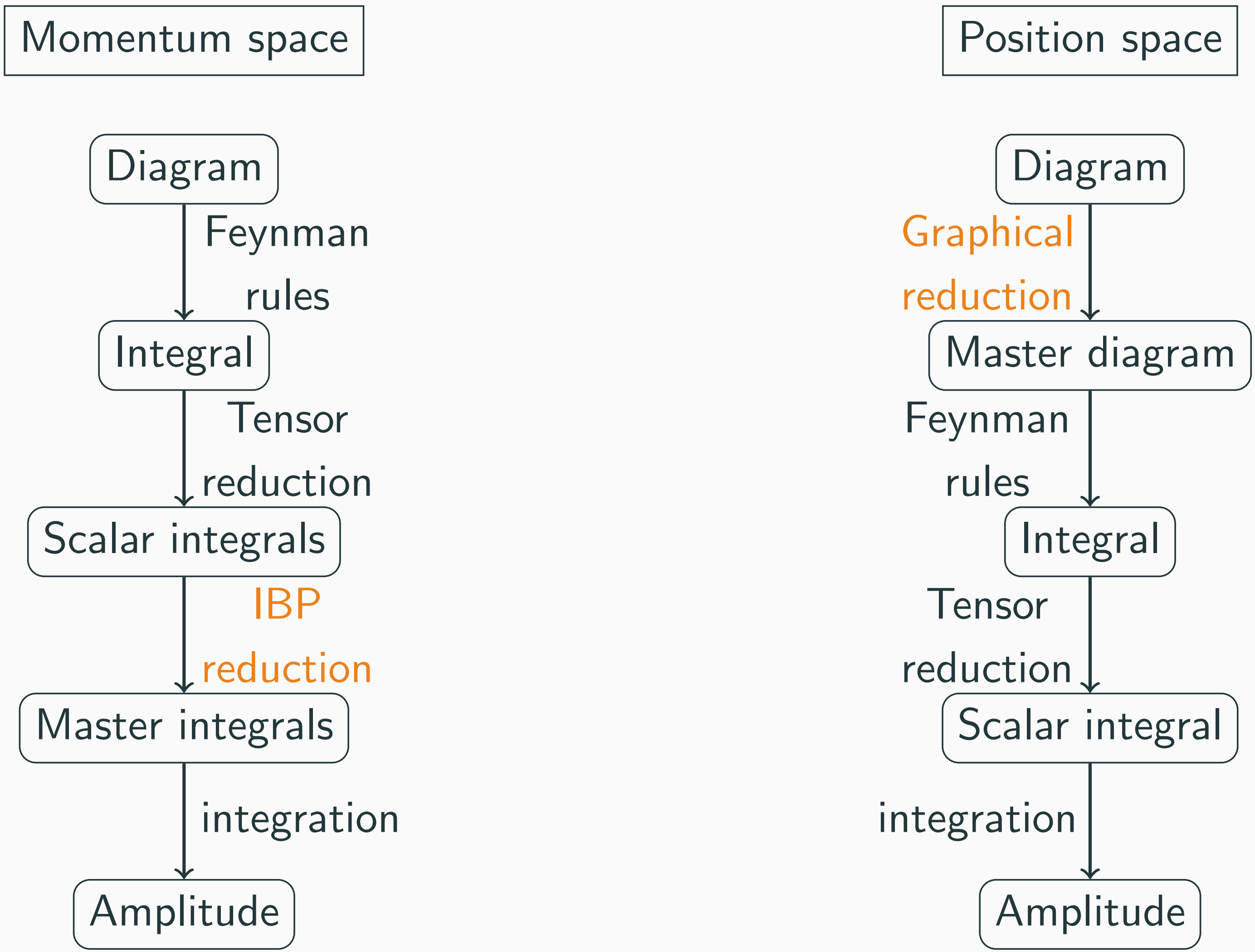


Feynman integral in momentum space

$$\widetilde{G}(p_1,\ldots,p_n)=\left(\prod_{e\in E}\int d^D k_e \widetilde{\Delta}(k_e)\right)\underbrace{\left(\prod_{v\in V_{\text{int}}}\delta^{(D)}\left(\sum_{e\ni v}k_e\right)\right)}$$

Lower dimensional integral

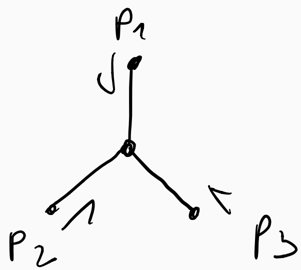
Feynman integral in position space

$$G(x_1,\ldots,x_n)=\left(\prod_{v\in V_{\text{int}}}\int d^D x_v\right)\underbrace{\left(\prod_{\{a,b\}\in E}\Delta(x_a-x_b)\right)}$$

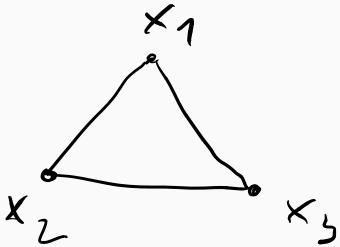
Better factorization properties

Examples

Momentum space	Position space



$\widetilde{\Delta}(p_{12})\widetilde{\Delta}(p_{23})\widetilde{\Delta}(p_{31})$



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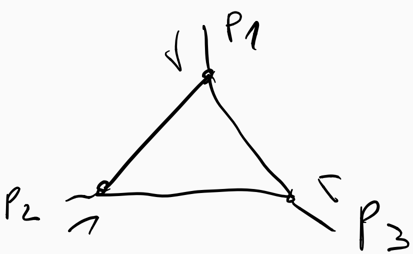
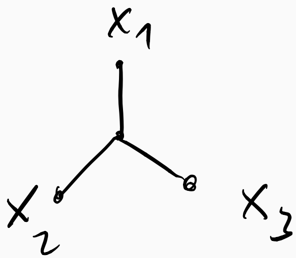


$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(p_{12}^2,p_{23}^2,p_{31}^2)}}$$



$$\frac{\mathrm{Di}(z,\bar{z})}{\sqrt{-\lambda(x_{12}^2,x_{23}^2,x_{31}^2)}}$$

$$\tilde{\Delta}(p) = \frac{1}{\|p\|^2}$$

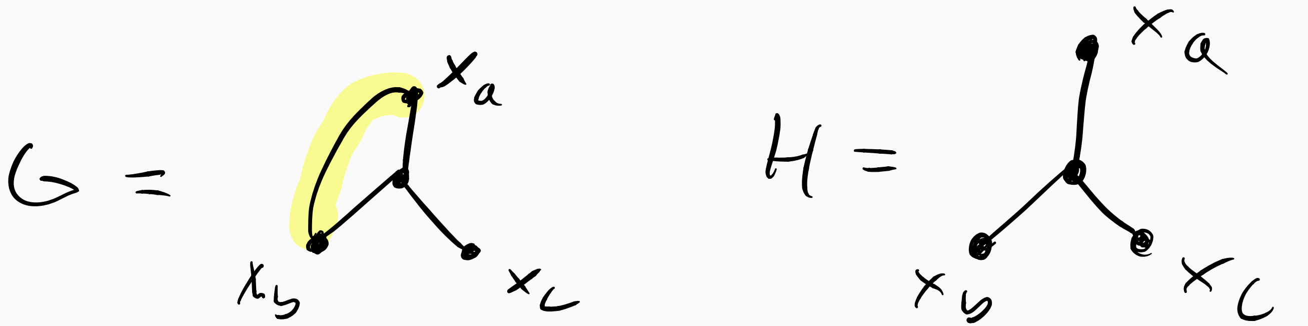
$$\Delta(x) = \frac{1}{\|x\|^2}$$

Graphical reductions

Graphical reduction rules

1. rule: propogators between external vertices

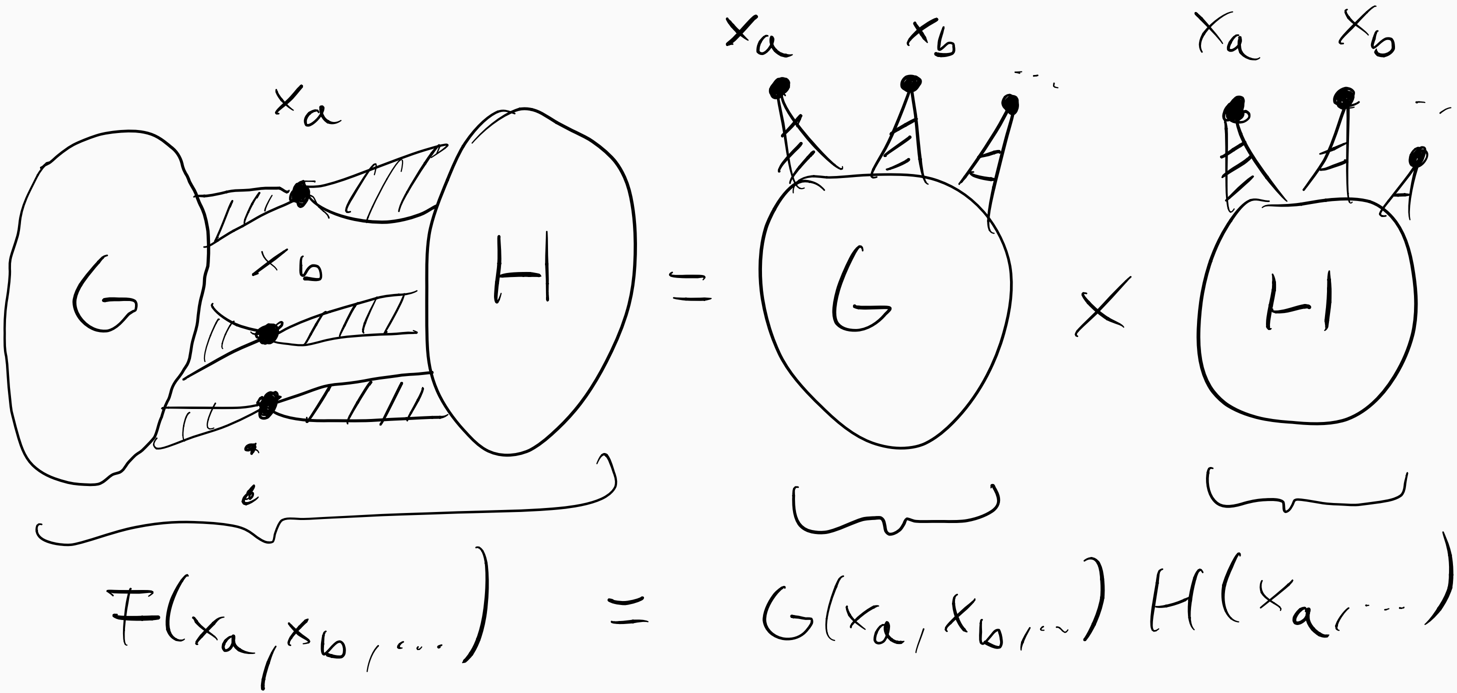
$$G(x_a, x_b, x_c) = \int d^D y \Delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)\Delta(x_a - x_b)$$
$$= \Delta(x_a - x_b) H(x_a, x_b, x_c)$$



$\Rightarrow$ edges between external vertices factorize.

Graphical reduction rules

2. rule: split graph



⇒ factorizes if split along external vertices.

Intermezzo: amputating a propagator

Recall the definition of the propagator, Δ, as *Green's function for the free field equation*

$$(\Box_x - m^2)\Delta(x-y) = \delta^{(D)}(x-y)$$

We can use this equation to *amputate* free external edges.

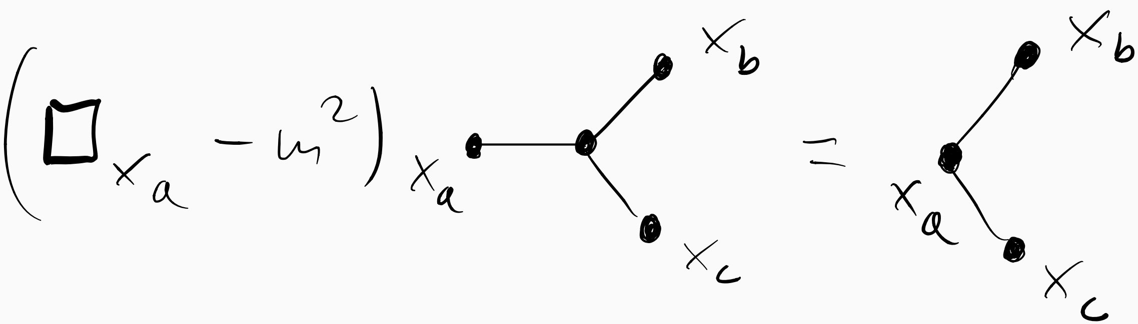
Graphical reduction rules

3. rule: amputating an external edge

$$(\Box_{x_a} - m^2)G(x_a, x_b, x_c) = \int d^D y (\Box_{x_a} - m^2)\Delta(x_a - y)\Delta(x_b - y)\Delta(x_c -$$

$$= \int d^D y \delta(x_a - y)\Delta(x_b - y)\Delta(x_c - y)$$

$$= \Delta(x_b - x_a)\Delta(x_c - x_a) = H(x_a, x_b, x_c)$$



$\Rightarrow$ solve differential equation to add external edge.

Differential equations

For rule 3, a differential equation needs to be solved:

$(\Box_{x_a} - m^2)G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

Can be solved systematically if `(Schnetz 2013)`

- particles are massless, $m = 0$,
- only 3-point functions are considered
- in $D = 4 - \epsilon$ Euklidean space.

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Related approach: (Drummond, Henn, Smirnov 2007) (Magic identities)

3-point configuration space is 2-dimensional, due to Poincare and scaling invariance:

$$G(x_a, x_b, x_c) = G(x'_a, x'_b, x'_c)$$

for

$$x'^{\,\mu}_k = \Lambda^\mu_\nu x^\nu_k$$
$$x'^{\,\mu}_k = v^\mu + x^\mu_k$$

with $\Lambda \in \mathrm{SO}(D)$ and $v \in \mathrm{R}^D$ and

$$G(\lambda x_a, \lambda x_b, \lambda x_c) = \lambda^\omega G(x_a, x_b, x_c).$$

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$\Rightarrow$ G only depends on the shape of the triangle spanned by x_a, x_b, x_c.

Exploit this symmetry by using complex paramater z such that

$$z\,\bar{z} = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar{z}) = \frac{x_{bc}^2}{x_{ab}^2}$$

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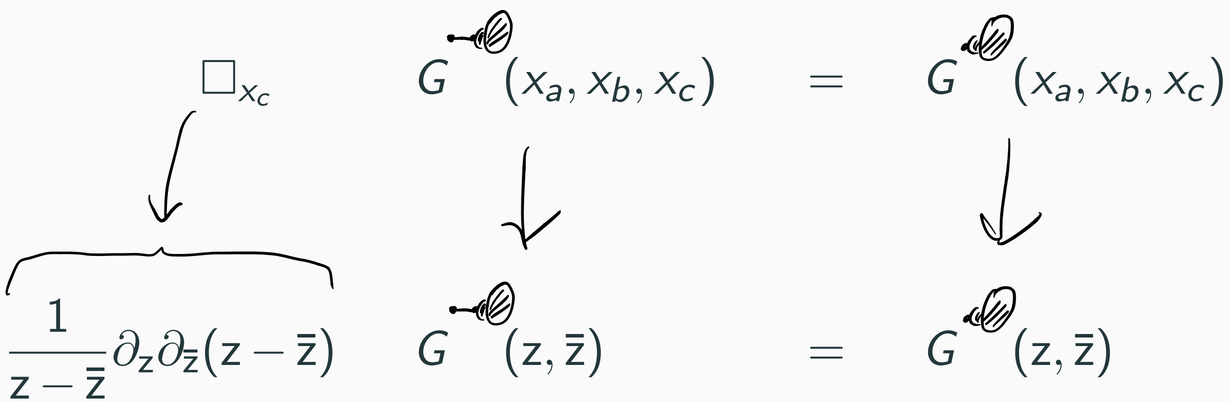

The ∂_z and $\partial_{\bar{z}}$ operators can be inverted in the function space of generalized single-valued hyperlogarithms (Chavez, Duhr 2012, Schnetz 2014, Schnetz 2017).

Graphical functions

- Rules 1,2,3 are part of a larger framework: graphical functions (Schnetz 2013).
- Graphical functions can also be applied in a broader context, e.g. to conformal amplitudes (Basso, Dixon 2017).
- Calculation within this framework are extremely efficient, due to the rapid reductions and small numbers of irreducible *master diagrams*.
- Additional identities specific to the theory (e.g. conformal transformations for scalar theories).

Graphical functions for gauge theory

Beyond scalar

Only change: adding an edge

For instance, for abelian gauge theory:

$$\Box_x \to \not{\partial} \text{ and } \eta^{\mu\nu}\Box_x$$

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For instance, for abelian gauge theory:

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The differential equation for appending an edge,

$\Box_{x_a} G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

becomes a system of differential equations

$\not\partial_{x_a} G$  $(x_a, \ldots) = G$  $(x_a, \ldots)$

Paramatrizing non-scalar graphical functions

$$\not{\partial}_{x_c} \; G$$  $$(x_a, x_b, x_c) \;=\; G$$  $$(x_a, x_b, x_c)$$

Paramatrizing non-scalar graphical functions

$$\not\partial_{x_c} \qquad G(x_a, x_b, x_c) = G(x_a, x_b, x_c)$$

$$\not\partial_{x_c} = \left(\not\lambda\, \partial_z + \not\bar\lambda\, \partial_{\bar z} - \frac{P^{\mu\nu}}{z - \bar z} \left(\partial^\nu_\lambda - \partial^\nu_{\bar\lambda} \right) \right) \qquad G(z, \bar z, \lambda, \bar\lambda) = G(z, \bar z, \lambda, \bar\lambda)$$

Using light-cone-like parametrization $z, \bar z, \lambda^\mu, \bar\lambda^\mu$ such that

$$z\,\bar z = \frac{x_{ac}^2}{x_{ab}^2} \quad \text{and} \quad (1-z)(1-\bar z) = \frac{x_{bc}^2}{x_{ab}^2}$$

$$x_{ab}^\mu = \lambda^\mu + \bar\lambda^\mu \qquad x_{ac}^\mu = z\,\lambda^\mu + \bar z\,\bar\lambda^\mu \qquad x_{bc}^\mu = (1-z)\,\lambda^\mu + (1-\bar z)\,\bar\lambda^\mu$$

$$\lambda^\mu\,\lambda_\mu = \bar\lambda^\mu\,\bar\lambda_\mu = 0$$

Actual inversion becomes more complicated: $D \neq 4$ dimensional Laplacian has to be inverted.

Diagonalization of the equation system gives,

$$\begin{pmatrix} \Delta_D & 0 & 0 \\ 0 & \Delta_{D+2} & 0 \\ 0 & 0 & \Delta_{D+4} \end{pmatrix} \widetilde{G} \quad (x_a, x_b, x_c) \quad = \quad \widetilde{G} \quad (x_a, x_b, x_c),$$


where $\Delta_D = \frac{2}{\mathsf{z}-\bar{\mathsf{z}}}\partial_\mathsf{z}\partial_{\bar{\mathsf{z}}}(\mathsf{z}-\bar{\mathsf{z}}) - \frac{D-4}{\mathsf{z}-\bar{\mathsf{z}}}(\partial_\mathsf{z} - \partial_{\bar{\mathsf{z}}})$.

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$\Rightarrow$ we would like to invert Δ_D for general even D.

Extension to $D \neq 4$

- For general dimension D we need to solve,

$\left(\frac{2}{\mathrm{z}-\bar{\mathrm{z}}}\partial_{\mathrm{z}}\partial_{\bar{\mathrm{z}}}(\mathrm{z}-\bar{\mathrm{z}}) - \frac{D-4}{\mathrm{z}-\bar{\mathrm{z}}}(\partial_{\mathrm{z}}-\partial_{\bar{\mathrm{z}}})\right) G$  $(\mathrm{z},\bar{\mathrm{z}}) = G$  $(\mathrm{z},\bar{\mathrm{z}}).$

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$$(z,\bar{z}) = G$$

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$\Rightarrow$ Opens the door to calculations in gauge theories.

$\Rightarrow$ Immediately possible tools: ϕ^3-theory. With applications to percolation theory and other variants (e.g. biadjoint ϕ^3).

An inverse to the differential operator

$$\frac{1}{2}\Delta_{2+2n} = \frac{1}{z-\bar{z}}\partial_z\partial_{\bar{z}}(z-\bar{z}) - \frac{n-1}{z-\bar{z}}(\partial_z - \partial_{\bar{z}})$$

is given by the integration operator:

$$I_n = \sum_{k,l=0}^{n} c_{n,k,l}(z-\bar{z})^{-k}\int_{SV} d\,z(z-\bar{z})^{k+l}\int_{SV} d\,\bar{z}(z-\bar{z})^{-l}$$

where $c_{n,k,l}$ are some easily determined coefficients.

Results

$$\beta_{\phi^3}(g) = \Big(\frac{5}{2016}\pi^6 - \frac{46519}{829440}\pi^4 + \frac{102052031}{6718464} + \frac{99}{16}\zeta(3)^2 + \\ + \frac{366647}{6912}\zeta(3) + \frac{151795}{3456}\zeta(5) - \frac{5495}{64}\zeta(7)\Big) g^{11} + \\ + \left(\frac{1}{192}\pi^4 - \frac{3404365}{746496} - \frac{4891}{864}\zeta(3) + \frac{5}{3}\zeta(5)\right) g^9 + \\ + \left(\frac{33085}{20736} + \frac{5}{8}\zeta(3)\right) g^7 - \frac{125}{144} g^5 + \frac{3}{4} g^3$$

4- and 3-loop results due to (Gracey 2015; de Alcantara Bonfim, Kirkham, McKane, 1980).

⇒ More accurate predictions for the critical exponents in percolation theory and for the Lee-Yang edge singularity.

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- Work in progress: extension to gauge theory.
- Intermediate step finished: extension to arbitrary even D.
- Application of ϕ^3-theory: Critical exponents in percolation theory.
- Question: Extension to odd D possible?

Example of a master diagram, which is irreducible w.r.t. rules 1–3: