5. 2-dim CFT

5.1. Notation & kinematics

Minkowski $x^{\pm} = x^1 \pm x^0$, $\partial_\pm = \frac{1}{2}(\partial_1 \mp \partial_0)$, $\partial_1 = \partial_+ + \partial_-$, $\partial_0 = \partial_+ - \partial_-$

$$ds^2 = (dx^1)^2 - (dx^0)^2 = dx^+ dx^-, \text{ i.e. } \boxed{\eta_{+-} = \tfrac{1}{2}, \ \eta_{++} = \eta_{--} = 0}$$

$$T_{++} = \tfrac{1}{2}(T_{1+} - T_{0+}) = \tfrac{1}{4}(T_{11} - T_{10} - T_{01} + T_{00})$$

$$T_{--} = \tfrac{1}{4}(T_{11} + T_{00} + 2T_{01}), \quad T_{+-} = \tfrac{1}{4}(T_{11} - T_{00})$$

conformal invariance: $T^\mu{}_\mu = 0 \iff \boxed{T_{+-} = 0}$

energy-momentum conservation:

$$\partial^\mu T_{\mu\pm} = \eta^{\mu\nu}\partial_\nu T_{\mu\pm} \Rightarrow \eta^{+-}\partial_- T_{++} = 0, \quad \eta^{-+}\partial_+ T_{--} = 0$$

i.e. T_{++} and T_{--} are chiral

$$\boxed{\begin{aligned} T_{++} &= T_{++}(x^+) \\ T_{--} &= T_{--}(x^-) \end{aligned}}$$

special conf. trafo:

$$x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2 x^2} \Rightarrow x^+ \to \frac{x^+ \cancel{(1 + c^+ x^-)}}{(1 + c^- x^+)\cancel{(1 + c^+ x^-)}}$$

i.e. $$x^+ \longrightarrow \frac{x^+}{1 + c^- x^+}, \quad x^- \longrightarrow \frac{x^-}{1 + c^+ x^-}$$

dilatation $\quad x^\pm \to \lambda x^\pm$

translation $\quad x^\pm \to x^\pm + a^\pm$

Lorentz $\quad x^+ \to e^\eta x^+ , \; x^- \to e^{-\eta} x^-$

Combination of dilat. & Lorentz:

$$\left.\begin{aligned} x^+ &\to \lambda e^\eta x^+ =: \varrho^+ x^+ \\ x^- &\to \lambda e^{-\eta} x^- =: \varrho^- x^- \end{aligned}\right\} (*)$$

See: no mixing of x^+, x^- under spec. conf. tr., transl, and $(*)$

For trafo of x^+ three real parameters: c^-, a^+, ϱ^+

$$\Rightarrow \boxed{SO(2,2) \sim \frac{SL(2,\mathbb{R})}{\mathbb{Z}_2} \times \frac{SL(2,\mathbb{R})}{\mathbb{Z}_2}}$$

action of $SL(2,\mathbb{R})$ becomes bijective after

$$\mathbb{R} \cup \infty \sim S^1$$

$\Longrightarrow$ torus comp. of $\mathbb{R}^{(1,1)}$

!!

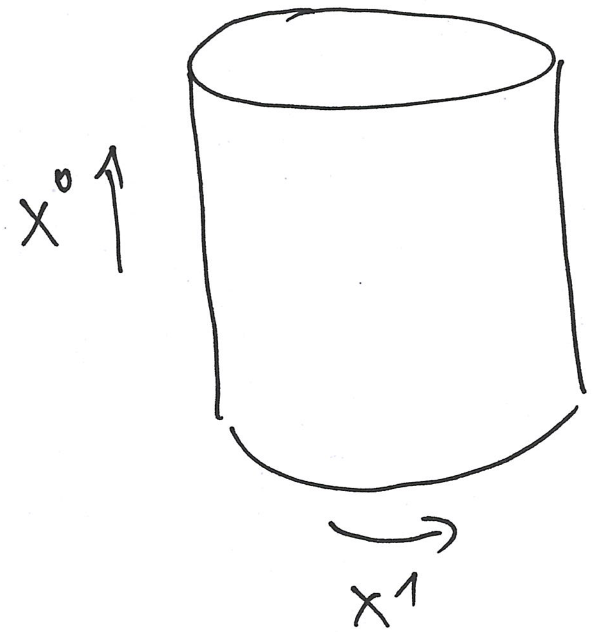
Since $N=2 \Rightarrow \exists$ further conf. trafo

$$\boxed{x^\pm \to f^\pm(x^\pm)}$$

$$\boxed{\text{Conformal group of } \mathbb{R}^{(1,1)} \sim \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)}$$

Cylinder with Lorentzian sign:

→ relevant for strings



& all fields: $\varphi(x^0, x^1+2\pi) = \varphi(x^0, x^1)$ (*)

conf. group of $\mathbb{R}\times S^1\Big|_{\text{Lorentz}} \sim \mathrm{Diff}(S^1)\times \mathrm{Diff}(S^1)$

Note: $f(x^\pm + 2\pi) = f(x^1 \pm x^0 + 2\pi) = f(x^1 \pm x^0) = f(x^\pm)$

see (*)

Euclidean $\mathbb{R}^2$:

$$z^+ = z = x^1 + i x^2 \qquad \partial = \tfrac{1}{2}(\partial_1 - i\partial_2) \, , \quad \partial_1 = \partial + \bar\partial$$

$$z^- = \bar z = x^1 - i x^2 \qquad \bar\partial = \tfrac{1}{2}(\partial_1 + i\partial_2) \, , \quad \partial_2 = i(\partial - \bar\partial)$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dz\, d\bar z \, , \quad \eta_{++} = \eta_{--} = 0 \, , \; \eta_{+-} = \tfrac{1}{2}$$

$$T_{++} = \tfrac{1}{4}(T_{11} - T_{22} - 2i\, T_{12}) \, , \quad T_{--} = \tfrac{1}{4}(T_{11} - T_{22} + 2i\, T_{12})$$

$$T_{+-} = \tfrac{1}{4}(T_{11} + T_{22})$$

conformal invariance: $\quad T_{+-} = 0$

en.-mom. conservation: $\quad T_{++} = T_{++}(z) \, , \; T_{--} = T_{--}(\bar z) = \overline{T_{++}(z)}$

i.e. T_{++} holomorph !

spec. conf. $z \to \dfrac{z}{1+cz}$, $\bar{z} \to \dfrac{\bar{z}}{1+\bar{c}\bar{z}}$

$SO(1,3) \sim \dfrac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$ Möbius group

$z \to \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$
$ad-bc=1$

counting parameters:

$SO(1,3)$: $\binom{4}{2} = 6$ real

Möbius: 3 complex = 6 real

Trafo properties of quasiprimaries:

For generic dims.

$$U(K)\varphi_i(x')U^{-1}(K) = \left|\det \frac{\partial x'}{\partial x}\right|^{-\frac{d}{N}} D_i{}^j(R)\,\varphi_j(x)$$

$$R^\alpha{}_\beta = \frac{\partial_\beta x'^\alpha}{\left|\det \frac{\partial x'}{\partial x}\right|^{1/2}}$$

Now $N=2$: $\det \frac{\partial x'}{\partial x} = \frac{\partial x'^+}{\partial x^+}\frac{\partial x'^-}{\partial x^-}$ $\Rightarrow$

scalars

$$U(K)\varphi(x')U^{-1}(K) = (\partial_+ x'^+ \partial_- x'^-)^{-\frac{d}{2}}\varphi(x) \quad \text{Lorentz}$$

$$U(K)\varphi(z',\bar z')U^{-1}(K) = (\partial z' \bar\partial \bar z')^{-d/2}\varphi(z,\bar z) \quad \text{Euclid}$$

Vectors

$$U(K)\varphi^{+}(x')U^{-1}(K) = \left(\partial_+ x'^+ \partial_- x'^-\right)^{-d/2} \frac{\partial_+ x'^+}{\left(\partial_+ x'^+ \partial_- x'^-\right)^{1/2}} \varphi^{+}(x)$$

$$= \left(\partial_+ x'^+\right)^{-\frac{d-1}{2}} \left(\partial_- x'^-\right)^{-\frac{d+1}{2}} \varphi^{+}(x)$$

similarly $U \varphi^{-}(x') U^{-1} = \left(\partial_+ x'^+\right)^{-\frac{d+1}{2}} \left(\partial_- x'^-\right)^{-\frac{d-1}{2}} \varphi^{-}(x)$

one says: φ^+ has spin -1, φ^- has spin $+1$

Generalization, including all conf. trafo in 2 Dim.

talk then about conformal primaries

$$U(K)\,\varphi(x')\,U^{-1}(K) = \left(\partial_+ x'^+\right)^{-\frac{d+s}{2}} \left(\partial_- x'^-\right)^{-\frac{d-s}{2}} \varphi(x)$$

$$h_\pm = \frac{d \pm s}{2} \quad , \quad h, \bar{h} \text{ in Euclidean case}$$

φ stands for a component of tensor or spinor field

Fields unique on $\mathbb{R}^2 \Leftrightarrow 2s$ integer

$2s$ non integer $\Leftrightarrow$ parafermions

Conformal Ward identities (with respect to Möbius)

<u>dilatations:</u>

$$\sum_{j=1}^{n} \left(x_{j+} \frac{\partial}{\partial x_{j+}} + h_{j+} \right) \langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle = 0$$

<u>special conf.:</u>

$$\sum_{j=1}^{n} \left((x_j^+)^2 \frac{\partial}{\partial x_j^+} + 2 h_{j+} x_j^+ \right) \langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle = 0$$

4.2. (General) Conformal Ward identities, OPE with energy-momentum tensor

From now: Euclidean

meaning of $\delta\varphi$

classically: $\delta\varphi = \varphi'(x) - \varphi(x) = \varphi'(x) - \varphi'(x') + \varphi'(x') - \varphi(x)$

$= \varphi'(x') - \varphi(x) - \delta x \, \partial\varphi + \mathcal{O}((\delta x)^2)$

quantum: $\delta\varphi = U(k)\varphi(x)U^{-1}(k) - \varphi(x) - \delta x^\mu \partial_\mu \varphi$

then with $z \to z + \epsilon(z)$ holomorph

$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ antiholomorph

$$\delta\varphi = -h_+ \partial\epsilon\,\varphi - h_- \bar\partial\bar\epsilon\,\varphi - \epsilon\partial\varphi - \bar\epsilon\bar\partial\varphi$$

$$\delta\langle\varphi_1(x_1)\cdots\varphi_n(x_n)\rangle = \int d^2x\,\partial_\mu\langle T^{\mu\nu}\epsilon_\nu(x)\ \varphi_1(x_1)\cdots\varphi_n(x_n)\rangle$$

used: $j^\mu = T^\mu{}_\nu\,\epsilon^\nu$; $\partial_\mu j^\mu = \partial_\mu T^{\mu\nu}\epsilon_\nu$

$$= \lim_{M \,\text{“}\!\to\!\text{”}\, \mathbb{R}^2} \int_M d^2x\,\partial_\mu\langle\cdots\rangle$$

$$= \oint_{C=\partial M} dx^\alpha\,\epsilon_{\mu\alpha}\langle T^{\mu\nu}(x)\,\epsilon_\nu(x)\,\varphi_1(x_1)\cdots\varphi_n(x_n)\rangle$$

Now $\epsilon_{\mu\alpha} dx^\alpha T^{\mu\nu} \epsilon_\nu = dx^2 T^{1\nu} \epsilon_\nu - dx^1 T^{2\nu} \epsilon_\nu$

$$\vdots$$

$$= i T_{--} \bar{\epsilon} d\bar{z} - i T_{++} \epsilon dz - i T_{+-} (\epsilon dz - \bar{\epsilon} d\bar{z})$$

$T_{+-} = 0$ on shell $\Rightarrow$ gives only contact terms,

for $x_1, \dots, x_n \notin C$ no contribution

Standard redefinition $T_{++} = -\frac{T}{2\pi}$, $T_{--} = -\frac{\bar{T}}{2\pi}$

$$\boxed{\delta_\epsilon \langle \varphi_1(z_1,\bar{z}_1) \cdots \varphi_n(z_n,\bar{z}_n) \rangle = -\frac{1}{2\pi i} \oint dz\, \epsilon(z) \langle T(z)\, \varphi_1(z_1,\bar{z}_1) \cdots \varphi_n \rangle + \frac{1}{2\pi i} \oint d\bar{z}\, \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z})\, \varphi_1 \cdots \varphi_n \rangle}$$

- integrand on r.h.s. holomorph / antiholomorph up to contact points $(z_1, \bar z_1), \ldots, (z_n, \bar z_n)$
- formula valid for generic hol./antihol $\epsilon(z), \bar\epsilon(\bar z)$
- non vanishing l.h.s. can result only via residua theorem & poles at $z_j, \bar z_j$, $j = 1, \ldots, n$

$$\boxed{T(z)\, \varphi_j(z_j, \bar z_j) = \frac{1}{z - z_j}\, \partial \varphi_j(z_j, \bar z_j) + \frac{h_j}{(z - z_j)^2}\, \varphi_j(z_j, \bar z_j) + \text{regular}}$$

& analog for $\bar T \varphi$

Apply this for $\varphi_j \to T$:

$\dim T = 2$ classically, no quantum correction

$\text{spin}\, T = 2$ $\quad \to \quad h_T = \frac{2+2}{2} = 2$

$$\overset{?}{\Longrightarrow} \quad T(z)\,T(w) = \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w) + \text{regul.}$$

- then we would have $\langle T T \rangle = 0$, since $\langle T \rangle = \langle \partial T \rangle = 0$
- but $\langle T(z)\,T(w) \rangle = \frac{c/2}{(z-w)^4}$, c is called central charge

$$\boxed{T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w) + \text{regul.}}$$

Example: massless free field

$$S = \tfrac{1}{2} \int d^2x \, \partial_\mu \varphi \, \partial^\mu \varphi$$

$$T_{\mu\nu} = \partial_\mu \varphi \, \partial_\nu \varphi - \tfrac{1}{2} \eta_{\mu\nu} \partial_\alpha \varphi \, \partial^\alpha \varphi \qquad \text{(no improvement in } N=2 \text{ case)}$$

$$T_{++} = \partial\varphi \, \partial\varphi \qquad \text{i.e. } T = -2\pi \, \partial\varphi \, \partial\varphi$$

$$T_{--} = \bar\partial\varphi \, \bar\partial\varphi$$

$$\langle \varphi(z,\bar z) \, \varphi(w,\bar w) \rangle = -\frac{1}{4\pi} \log\left((z-w)(\bar z - \bar w) \right)$$

$$\Rightarrow \langle T(z) \, T(w) \rangle = \frac{4\pi^2}{16\pi^2} \left(\partial_z \partial_w \log |z-w|^2 \right)^2 \cdot 2$$

$$= \tfrac{1}{2} \left(\frac{1}{(z-w)^2} \right)^2 \quad \Rightarrow \quad \boxed{c = 1 \text{ for free scalar}}$$

$\exists$ correspondence $\quad c \Leftrightarrow$ number of field degrees of freedom

central charge & trace anomaly

embedding of CFT in curved manifold $\Rightarrow T^\mu{}_\mu = \frac{c}{48\pi} \cdot R$

$\Rightarrow$ add. terms in page 14 bottom

see Eguchi, Ooguri Nucl. Phys. B282 (1987) 308

generate second insertion of T by diff. w.r.t. $g_{\mu\nu}$ <u>before</u> flat limit.

$\Rightarrow$ c in $\langle T T \rangle_{flat}$ and c in $T^\mu{}_\mu$ curved

are equal !!

Applying conformal Ward identity to T itself $\Rightarrow$

$$\delta_\epsilon T(z) = -\frac{c}{12}\partial^3\epsilon(z) - 2\partial_z\epsilon\cdot T(z) - \epsilon(z)\partial_z T(z)$$

without the c-term this <u>would</u> mean

$$U(k)T(w)U^{-1}(k) = \left(\frac{\partial z}{\partial w}\right)^2 T(z)\,, \quad w = k(z)$$

$\hat{=}$ T primary with dim. 2

make Ansatz $U(k)T(w)U^{-1}(k) = \left(\frac{\partial z}{\partial w}\right)^2 \Big(T(z)$ [redacted] $S(w,z)\Big)$

group property for $z \to w \to v$

$$s(v,z) = s(w,z) + \left(\frac{\partial w}{\partial z}\right)^2 s(v,w)$$

$\Rightarrow$ $s(w,z) = \frac{c}{12} S(w,z)$, S Schwarz derivative

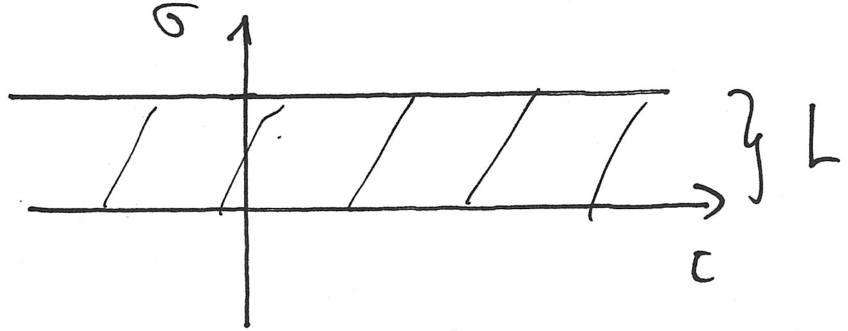
$$S(w,z) = \frac{w'''}{w'} - 3/2\left(\frac{w''}{w'}\right)^2$$

Note: $S(w,z) = 0$ for $w = \frac{az+b}{cz+d}$

i.e. $T(z)$ is quasiprimary

but $T(z)$ no true primary

Physical effects of central charge:



$$\varphi(\sigma+L,\tau)=\varphi(\sigma,\tau)$$

$w=\tau+i\sigma$ mapping to plane $z=e^{\frac{2\pi w}{L}}$, $w=\frac{L}{2\pi}\log z$

$$\Rightarrow \quad S(w,z)=\frac{1}{2}\,\frac{1}{z^2}\,,\qquad w'=\frac{L}{2\pi}\frac{1}{z},\quad w''=\dots$$

$$\Rightarrow \quad U(k)\,T(w)\,U^{-1}(k)=\left(\frac{2\pi}{L}\right)^2 z^2\left(T(z)-\frac{c}{24}\cdot\frac{1}{z^2}\right)$$

Index s for strip

p for plane

$$|0\rangle_p = U(k)\,|0\rangle_s$$

$$\Rightarrow\ {}_s\langle 0|T(w)|0\rangle_s = {}_s\langle 0|\,U^{-1}\,U\,T(w)\,U^{-1}U\,|0\rangle_s$$

$$= {}_p\langle 0|\left(\frac{2\pi}{L}\right)^2\left(z^2T(z)-\frac{c}{24}\right)|0\rangle_p$$

now ${}_p\langle 0|\,T\,|0\rangle_p = 0 \quad \Longrightarrow$

$${}_s\langle 0|T|0\rangle_s = -\frac{c\pi^2}{6L^2}\,,\quad \left\{\begin{aligned} E_0 &= \int_0^L d\sigma\langle T_{00}\rangle \\ &= -\frac{1}{2\pi}\int_0^L d\sigma\,(T_{++}+T_{--}) \\ &= -\frac{L}{2\pi}\left(-\frac{c\pi^2}{6L^2}-\frac{c\pi^2}{6L^2}\right)\end{aligned}\right.$$

$$\boxed{E_0 = \frac{\pi c}{6L}}$$ Casimir effect

5.3. Virasoro-algebra & state space

radial quantization: correlation fcts. as vev's of fields <u>ordered</u> w.r.t. $r = |z|$

$$R\left(\varphi_1(z,\bar z)\,\varphi_2(w,\bar w)\right) = \begin{cases} \varphi_1\,\varphi_2\,, & |z| > |w| \\ \varphi_2\,\varphi_1\,, & |z| < |w| \end{cases}$$

$$\mathbb{R}\times S^1\Big|_{\text{Minkowski}} \xrightarrow{\text{Wick}} \mathbb{R}\times S^1\Big|_{\text{Euclid}}$$: coord. σ, τ, $\varphi(\sigma + 2\pi) = \varphi(\sigma)$

$$\begin{aligned} \tau \to -\infty &\;\hat{=}\; z = 0 \\ \tau \to +\infty &\;\hat{=}\; z = \infty \end{aligned} \quad \longleftarrow \quad z := e^{w}, \quad w = \tau + i\sigma$$

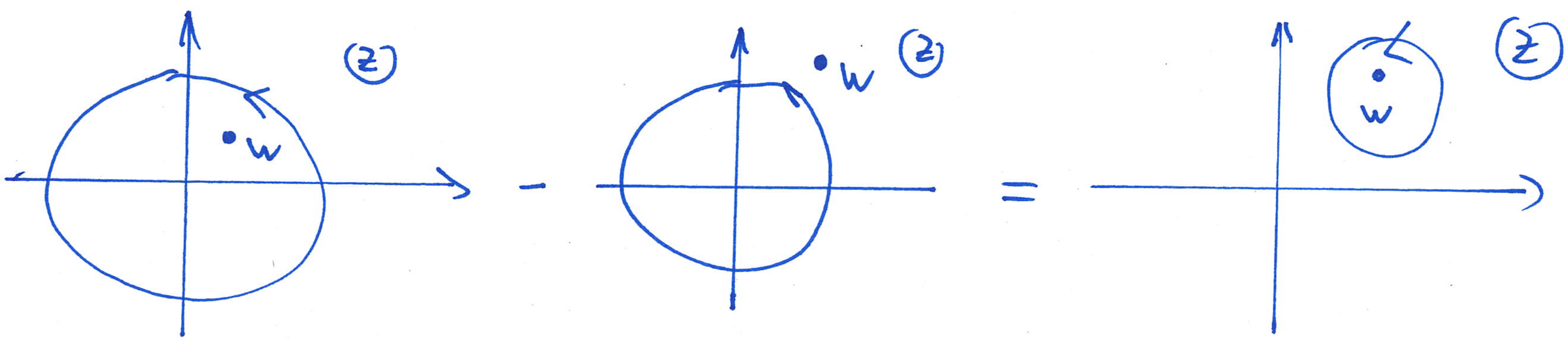
$a(z)$, $b(z)$ holomorph

$$A := \oint a(z)\,dz, \quad B := \oint b(z)\,dz,$$

A & B conserved
$\hat{=}$ indep. of integr. path

$$[A,B] = \oint_{|z|>|w|} dz \oint dw \; a(z)\, b(w) - \oint_{|z|<|w|} dz \int dw \; b(w)\, a(z)$$

$$= \left(\oint_{|z|>|w|} dz \oint dw - \oint_{|z|<|w|} dz \oint dw \right) \left(R\left(a(z)\, b(w)\right)\right)$$



$$\boxed{[A,B] = \oint_0 dw \oint_w dz \; R\big(a(z)\, b(w)\big)}$$

Applied to $T(z)$:

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n \quad \longleftrightarrow \quad L_n = \frac{1}{2\pi i} \oint dz \; z^{n+1} T(z)$$

$$\bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{z}^{-n-2} \bar{L}_n \quad \longleftrightarrow \quad \bar{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \; \bar{z}^{n+1} \bar{T}(\bar{z})$$

$\longleftrightarrow$ corresponds to Fourier exp. in

$\mathbb{R} \times S$

$$[L_m, L_n] = -\frac{1}{4\pi^2} \oint_0 dw\, w^{n+1} \oint_w dz\, z^{m+1}\, R(T(z)\,T(w))$$

OPE T·T was understood for use in corr. fcts.

i.e. refers to R-ordered products

$$\Rightarrow [L_m, L_n] = -\frac{1}{4\pi^2} \oint_0 dw\, w^{n+1} \oint_w dz\, z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) + \text{regul.} \right)$$

$\vdots$

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m (m^2-1)\, \delta_{n+m,0}}$$

Virasoro algebra

similar for $[\bar{L}_m, \bar{L}_n]$ & $[L_m, \bar{L}_n] = 0$

Compare with Witt algebra for $\ell_n := -z^{n+1}\partial$

generators for conf. trafo

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n}$$

- Virasoro from Witt via adding a central extension: $\frac{c}{12} m(m^2-1)\,\delta_{n+m,0} \cdot \mathbb{1}$
- Virasoro contains $SL(2,\mathbb{C})$ subalgebra, gen. by L_0, L_1, L_{-1}

Commutators of L_n's with primaries

$$[L_n, \varphi(w,\bar{w})] = \frac{1}{2\pi i} \oint_0 dz\, z^{n+1} [T(z), \varphi(w,\bar{w})]$$

$$= \frac{1}{2\pi i} \oint_w dz\, z^{n+1} \left(\frac{1}{z-w} \partial \varphi(w,\bar{w}) + \frac{h_+}{(z-w)^2} \varphi(w,\bar{w}) + \text{regul.} \right)$$

$$\boxed{[L_n, \varphi(w,\bar{w})] = (n+1) h_+ w^n \varphi(w,\bar{w}) + w^{n+1} \partial\varphi}$$

descendent fields, conformal families

$$T(z)\,\phi(w,\bar{w}) = \sum_{k=0}^{\infty} (z-w)^{-2+k}\,\phi^{(k)}(w,\bar{w})$$

had already $\phi^{(0)} = h_+\phi$, $\phi^{(1)} = \partial\phi$

in general

$$\phi^{(k)}(w,\bar{w}) = \underbrace{\frac{1}{2\pi i}\oint dz\,(z-w)^{-k+1}\,T(z)}_{=:\,L_{-k}(w)}\,\phi(w,\bar{w})$$

up to now we had $L_{-k} = L_{-k}(0)$

i.e. $\phi^{(k)}(w,\bar{w}) = L_{-k}(w)\,\phi(w,\bar{w})$

generate further descendents by OPE of $\phi^{(k)}$ with T ....

$$\phi^{(k_1,k_2,\dots)}(w,\bar{w}) := L_{-k_1} L_{-k_2} \cdots \phi(w,\bar{w}) \; ; \quad k_j \geq 1$$

<u>find:</u> conf. dim. (i.e. coeff. of $(z-w)^{-2}$ in OPE with T)

of $\phi^{(k_1,k_2,\dots)}$ is $h_+ + k_1 + k_2 + \dots$

primary & all of its descendents = conf. family

Operator - state correspondence

$$|h, \bar{h}, \{k\}, \{\bar{k}\}\rangle := \lim_{z \to 0} \phi^{(\{k\}, \{\bar{k}\})}(z, \bar{z}) |0\rangle$$

specification of vacuum $|0\rangle$

want to preserve as much as possible of symmetry

$$L_n |0\rangle = 0 \, , \quad \forall n \; ?$$

$\rightarrow$ contradiction to Virasoro $\left(0 = \frac{c}{12} m (m^2 - 1) \delta_{n+m,0} \right)$

consistent choice:

$$L_n |0\rangle = 0$$
$$\bar{L}_n |0\rangle = 0 \quad , \quad \forall\, n \geq -1$$

Some properties of these states $|h, \bar{h}, \{k\}, \{\bar{k}\}\rangle$:

$$L_0 |h, \bar{h}\rangle = \lim_{z \to 0} L_0 \,\phi(z, \bar{z}) |0\rangle = \lim_{z \to 0} [L_0, \phi] |0\rangle$$

$$= h \lim_{z \to 0} \varphi |0\rangle \quad \Rightarrow$$

use OPE result from page 28

$$L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle$$

furthermore: $\quad L_n |h, \bar{h}\rangle = 0$ for $n \geq 1$

$$L_0 |h, \bar{h}, \{k\}, \{\bar{k}\}\rangle = (h + k_1 + k_2 + \dots) |h, \bar{h}, \{k\}, \{\bar{k}\}\rangle$$

Consider now $|h, \{k\}\rangle$ generated by

a family of a holomorphic primary

$=:$ Verma-modul $V(c, h)$

State space of a 2d-CFT:

$$\mathcal{H} = \bigoplus_{h, \bar{h}} N_{h, \bar{h}} \, V(c, h) \otimes V(c, \bar{h})$$

- each Verma modul defines a representation of the Virasoro algebra
- $N_{h,\bar{h}}$ integer number, multiplicity

rational CFT's: Only a <u>finite</u> sum in state space repres. by Verma modules