Dr. H. Dorn
Institut für Physik
Humboldt-Universität

**Contents of the lectures "Conformal field theory",
SS 2014, Krippen, March 30 - April 04, block course of the
Research Training Group (GK 1504)**

Literature:

There is a huge set of papers related to the subject. At few places there will be references in the manuscript. A good starting point for diving into the literature are the following papers and references therein.

H.A. Kastrup, ”On the Advancements of Conformal Transformations and their Associated Symmetries in Geometry and Theoretical Physics”, arXiv:0808.2730

S. Rychkov, ”EPFL Lectures on Conformal Field Theory in D$\geq$ 3 Dimensions”, https://sites.google.com/site/slavarychkov/

Yu. Nakayama, ”A lecture note on scale versus conformal invariance”, arXiv:1302.0884

For two-dimensional conformal field theory one can start with the book

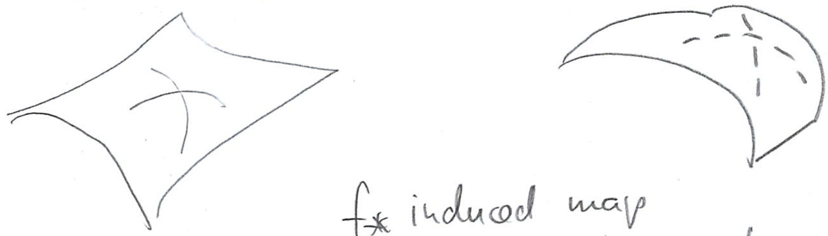
P. Di Francesco, P. Mathieu, D. Senechal: “Conformal Field Theory” Springer

or with one of the many good books or reviews related to string theory.

1. Geometrical aspects of conformal invariance

1.1. Local aspects, identification of conformal trafos

- $M, \hat{M}$ two differentiable manifolds with metrics $g, \hat{g}$
- f differentiable map $(M, g) \xrightarrow{f} (\hat{M}, \hat{g})$



f_* induced map of tangential vectors

f is conformal

$$\Longleftrightarrow \quad \hat{g}(f_* u, f_* v)\Big|_{f(p)} = \varrho(p)\, g(u,v)\Big|_p$$

$$\forall p \in M, \quad \forall u, v \in T_p(M)$$

Note: Use this concept also if it applies only to some open subset of M

Special case $\hat{M} = M$, $\hat{g} = g$, f diffeomorphism of M

f is isometry $\Longleftrightarrow$ $g(f_* u, f_* v) = g(u,v)$

f is conformally $\Longleftrightarrow$ $g(f_* u, f_* v) = \varrho \cdot g(u,v)$

in coordinates: $x \xmapsto{f} y$

$$u = u^\mu(x)\frac{\partial}{\partial x^\mu}\,,\quad f_* u = (f_* u)^\mu \frac{\partial}{\partial y^\mu} \quad \text{with } (f_* u)^\mu = \frac{\partial y^\mu}{\partial x^\nu} u^\nu(x)$$

i.e. <u>isometry</u> $g_{\mu\nu}(y)\frac{\partial y^\mu}{\partial x^\alpha} u^\alpha(x) \frac{\partial y^\nu}{\partial x^\beta} v^\beta(x) = g_{\alpha\beta}(x)\, u^\alpha v^\beta\,,\ \forall u, v$

$$\Leftrightarrow \boxed{g_{\mu\nu}(y)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} = g_{\alpha\beta}(x)} \qquad \text{(isometry)}$$

analogously for <u>conformal map</u>:

$$\boxed{g_{\mu\nu}(y)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} = \varrho(x)\, g_{\alpha\beta}(x)} \quad (*)$$

$\varrho(x)$ in $(*)$ is <u>not</u> independent: contract $(*)$ with $g^{\alpha\beta}$

$$\Rightarrow N\varrho = g_{\mu\nu}(y)\, g^{\alpha\beta}(x)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}$$

N dimension of M.

i.e.

$$\boxed{g_{\mu\nu}(y)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} = \frac{1}{N}\, g_{\kappa\lambda}(y)\, g^{\sigma\tau}(x)\frac{\partial y^\kappa}{\partial x^\sigma}\frac{\partial y^\lambda}{\partial x^\tau}\cdot g_{\alpha\beta}(x)}$$

(conformal)

Whether the nonlinear partial differential equations (iso) and (conformal) have solutions for the map $y(x)$ depends on the manifold under consideration.

- isometries preserve lengths
- conformal trafo preserve angles

$$\sphericalangle(u,v) := \frac{g(u,v)}{\sqrt{g(u,u)\, g(v,v)}}$$

Up to now we followed the

<u>active point of view:</u> x coordinates of original
y coordinates of image

Alternative: <u>passive point of view:</u>

no mapping of points, but change of coordinate system x original coordinate
y transformed coordinate

Then one talks about
isometric or conformal changes of coordinates.

<u>Analysis of the defining diff. eqs. for infinitesimal transformations</u>

$$y^\mu = x^\mu + \epsilon \cdot k^\mu(x)$$

<u>Isometry</u> $g_{\mu\nu}(x+\epsilon k)\,(\delta^\mu_\alpha + \epsilon \partial_\alpha k^\mu)(\delta^\nu_\beta + \epsilon \partial_\beta k^\nu) = g_{\alpha\beta}(x)$

balance linear in ϵ:

$$\boxed{g_{\alpha\nu}\partial_\beta k^\nu + g_{\mu\beta}\partial_\alpha k^\mu + k^\lambda \partial_\lambda g_{\alpha\beta} = 0}$$

Killing eq. for Killing vector fields $k(x)$

equivalent form for Killing equation:

$$\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0$$

or $\mathcal{L}_k g = 0$, ∇ covariant derivative

$\mathcal{L}_K$ Lie derivative in direction of k

<u>conformal trafo</u>

with $g = 1 + \epsilon \varphi(x)$

$$\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = \varphi g_{\alpha\beta} \quad \Big| \; g^{\alpha\beta}$$

$$\boxed{\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = \frac{2}{N} \nabla^\mu k_\mu \cdot g_{\alpha\beta}}$$

Conformal Killing equation for Conformal Killing vector fields (CKV's)

After having found solutions of Killing or conformal Killing, finite isometries or finite conformal transformations are found by integrating the flux of these vector fields.

Weyl transformations

manifold (M, g), now <u>no</u> mapping of points, but change of metric $g \to \hat{g} = \rho \cdot g$, ρ scalar function

In mathematics literature then $\hat{g}$ and g are called conformally equivalent.

<u>Note</u>: In a Weyl trafo the rescaling function ρ is arbitrary (smooth).
In a conformal trafo the rescaling function is fixed by the conformal map!

Alternative definition of conformal diffeomorphisms:

$f : M \to M$ is conformal

$\Leftrightarrow$ $f^* g$ and g are related by a Weyl trafo

Example $\mathbb{R}^N$:

Isometries: $\partial_\alpha k_\beta + \partial_\beta k_\alpha = 0$

$$\hookrightarrow \partial_\gamma \partial_\alpha k_\beta = -\partial_\gamma \partial_\beta k_\alpha = -\partial_\beta \partial_\gamma k_\alpha = +\partial_\beta \partial_\alpha k_\gamma =$$

$$= \partial_\alpha \partial_\beta k_\gamma = -\partial_\alpha \partial_\gamma k_\beta = -\partial_\gamma \partial_\alpha k_\beta$$

$\Rightarrow$ all 2nd and higher derivatives of k are zero

$\Rightarrow$ $k_\alpha = c_\alpha + \omega_{\alpha\beta} x^\beta$, inserting in Killing eq. $\Rightarrow \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$

$$\Rightarrow \boxed{k_\alpha = c_\alpha + \omega_{\alpha\beta} x^\beta, \text{ with } \omega_{\alpha\beta} = -\omega_{\beta\alpha}}$$

Number of parameters: from c_α: N; from $\omega_{\alpha\beta}$: $\binom{N}{2}$ $\Big\}$ $N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$

Algebra of Killing vectors of $\mathbb{R}^N$ = Lie algebra $so(N)$

After integration: Isometrie group $SO(N)$

$\mathbb{R}^{(N-1,1)}$:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \\ & & & & \ddots \end{pmatrix}$$

then again $\omega_{\alpha\beta} = -\omega_{\beta\alpha} \Rightarrow$ $\omega^0{}_n = -\omega_{0n} = \omega_{n0} = \omega^n{}_0$

$$\omega^m{}_n = \omega_{mn} = -\omega_{nm} = -\omega^n{}_m$$

General statement:

In a connected Riemannian space of dimension N the dimension of the space of Killing vector fields is $\leq \frac{N(N+1)}{2}$. Equality holds iff the space has constant curvature. (see Kobayashi, Nomizu)

Conformal maps in $\mathbb{R}^N$ or $\mathbb{R}^{(N-1,1)}$

$$\eta_{\alpha\beta} = \text{diag}(1,1,\dots) \text{ or } \text{diag}(-1,1,1,\dots)$$

$$\partial_\alpha k_\beta + \partial_\beta k_\alpha = \frac{2}{N} \partial^\mu k_\mu \eta_{\alpha\beta} \quad (*)$$

From this follows for $N \geq 3$: all 3rd and higher derivatives vanish

To show this, it is important, that one can choose 3 Indices to be unequal.

$\Rightarrow$ $N=2$ needs separate treatment.

$$\Rightarrow k^\mu(x) = a^\mu + m^\mu{}_\alpha x^\alpha + w^\mu{}_{(\alpha\beta)} x^\alpha x^\beta$$

note: $( ) \Leftrightarrow$ symm.
$[ ] \Leftrightarrow$ antisymm.

$$\Rightarrow \partial_\mu k^\mu = m^\mu{}_\mu + 2 w^\mu{}_{(\mu\beta)} x^\beta$$

put this in $(*)$ $\Rightarrow$

$$m_{\beta\alpha} + m_{\alpha\beta} + 2 w_{\alpha(\beta\lambda)} x^\lambda + 2 w_{\beta(\alpha\lambda)} x^\lambda = \frac{2}{N} \eta_{\alpha\beta} \left(m^\mu{}_\mu + 2 w^\mu{}_{(\mu\lambda)} x^\lambda \right)$$

special $\alpha = \beta$ (underlined index means <u>no</u> sum)

$$2 m_{\underline{\alpha\alpha}} + 4 w_{\underline{\alpha}(\underline{\alpha}\lambda)} x^\lambda = \frac{2}{N} \eta_{\underline{\alpha\alpha}} \left(m^\mu{}_\mu + 2 w^\mu{}_{(\mu\lambda)} x^\lambda \right)$$

compare powers in x

$$\Rightarrow \quad m_{\underline{\alpha\alpha}} = \frac{\eta_{\alpha\alpha}}{N} m^{\mu}{}_{\mu} \quad (1)$$

$$w_{\underline{\alpha}(\underline{\alpha}\lambda)} = \frac{\eta_{\alpha\alpha}}{N} w^{\mu}{}_{(\mu\lambda)} \quad (2)$$

special $\alpha \neq \beta \Rightarrow$ r.h.s = zero $\Rightarrow$

$$m_{\alpha\beta} + m_{\beta\alpha} = 0 \quad (3) \qquad w_{\alpha(\beta\lambda)} + w_{\beta(\alpha\lambda)} = 0 \quad (4)$$

Now (1) & (3) $\Rightarrow$ $\boxed{m_{\alpha\beta} = \omega_{[\alpha\beta]} + s\,\eta_{\alpha\beta}}$

(4) $\Rightarrow$ (in case $\alpha \neq \beta$) $\underline{w_{\alpha(\beta\lambda)}} = -w_{\beta(\alpha\lambda)} = -w_{\beta(\lambda\alpha)} = +w_{\lambda(\beta\alpha)}$

$$= w_{\lambda(\alpha\beta)} = -w_{\alpha(\lambda\beta)} = -\underline{w_{\alpha(\beta\lambda)}}$$

$\Rightarrow$ $w_{\alpha(\beta\lambda)} = 0$

for α, β, λ pairwise unequal

(2) $\Rightarrow$ $w_{\underline{\alpha}(\underline{\alpha}\lambda)} = -\eta_{\underline{\alpha\alpha}}\, c_\lambda$ with $c_\lambda := -\frac{1}{N} w^{\mu}{}_{(\mu\lambda)}$

$\lambda \neq \alpha$: $w_{\lambda(\underline{\alpha\alpha})} \underset{(4)}{=} -w_{\alpha(\lambda\alpha)} = +\eta_{\underline{\alpha\alpha}}\, c_\lambda$

together $w_{\alpha(\beta\lambda)} = c_\alpha \eta_{\beta\lambda} - c_\lambda \eta_{\alpha\beta} - c_\beta \eta_{\alpha\lambda}$

$$\Rightarrow \quad k^{\mu}(x) = a^{\mu} + \omega^{\mu}{}_{\alpha} x^{\alpha} + s x^{\mu} + \left(c^{\mu}\eta_{\alpha\beta} - c_\alpha \delta^{\mu}{}_{\beta} - c_\beta \delta^{\mu}{}_{\alpha}\right) x^{\alpha} x^{\beta}$$

$$\boxed{k^{\mu}(x) = a^{\mu} + \omega^{\mu}{}_{\alpha} x^{\alpha} + s x^{\mu} + c^{\mu} x^2 - 2 x^{\mu}\, c \cdot x}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$

Counting parameters

	$N=4$	general
$a^\mu \hat{=}$ inf. translation	4	N
$\omega^\mu{}_\alpha \hat{=}$ inf. rotation, boost	6	$\frac{N(N-1)}{2}$
$s \hat{=}$ inf. dilatation	1	1
$c^\mu \hat{=}$ inf. special conformal	4	N
Σ:	15	$\frac{(N+1)(N+2)}{2}$

Again a general statement:

In a connected Riemannian space of dimension N the dimension of the space of conformal Killing vector fields is $\leq \frac{(N+1)(N+2)}{2}$ (again Kobayashi/Nomizu I but there no statement on $=$)

- Related finite trafo follow from integration.
- Alternative: guess finite trafo, check that infinitesimal version is OK.

Isometries (Poincare trafo): as well known.

Dilatations: $x^\mu \to y^\mu = e^s x^\mu = x^\mu + s x^\mu + \ldots$

finite special conformal trafo:

consider as an auxillary trick:

Inversion at unit sphere (for $\mathbb{R}^N$) or unit hyperboloid (for $\mathbb{R}^{N-1,1}$)

$$x^\mu \to y^\mu = \frac{x^\mu}{x^2} =: S(x)$$

<u>check of conformality:</u>

$$\frac{\partial y^\alpha}{\partial x^\mu} = \frac{\delta^\alpha_\mu x^2 - 2x^\alpha x_\mu}{(x^2)^2} \Rightarrow$$

$$\eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} = \eta_{\alpha\beta}\frac{(\delta^\alpha_\mu x^2 - 2x^\alpha x_\mu)(\delta^\beta_\nu x^2 - 2x^\beta x_\nu)}{(x^2)^4}$$

$$= \frac{(x^2)^2\eta_{\mu\nu} - 2\cancel{x_\mu x_\nu} x^2 - 2\cancel{x_\mu x_\nu x^2} + \cancel{4} x^2 x_\mu x_\nu}{(x^2)^4}$$

$$= \frac{1}{(x^2)^2}\eta_{\mu\nu} \qquad \square$$

S is a conformal trafo, but not smoothly connected with the identity.

Consider $\quad K_c := S T_c S \; ; \quad T$ translation by c

$$x^\mu \qquad S((x)^\mu) = \frac{x^\mu}{x^2}$$

$$(T_c S(x))^\mu = \frac{x}{x^2} + c^\mu$$

$$(S T_c S)^\mu = \frac{\frac{x}{x^2} + c^\mu}{\left(\frac{x}{x^2} + c\right)^2} = \frac{1}{x^2}\frac{x^\mu + c^\mu x^2}{\frac{1}{x^2} + 2\frac{cx}{x^2} + c^2}$$

i.e. $K_c : \boxed{x^\mu \longmapsto \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2 x^2}}$ infinitesimal version (linear in c^μ) as required.

Relation to the Lorentz groups in spaces with two more dimensions: $SO(N+1,1)$, $SO(N,2)$

Lie brackets of basis of CKV's

$$P_\mu := -i\partial_\mu$$

$$M_{\mu\nu} := -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$D := -i x^\alpha \partial_\alpha$$

$$K_\mu := -i(x^2 \delta_\mu^\nu - 2x_\mu x^\nu)\partial_\nu \quad \Rightarrow$$

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu)$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(\eta_{\mu\alpha} M_{\nu\beta} + \eta_{\nu\beta} M_{\mu\alpha} - \eta_{\mu\beta} M_{\nu\alpha} - \eta_{\nu\alpha} M_{\mu\beta})$$

$$[D, P_\mu] = i P_\mu$$

$$[D, K_\mu] = -i K_\mu, \quad [D, M_{\mu\nu}] = 0$$

$$[K_\mu, K_\nu] = 0, \quad [M_{\mu\nu}, K_\lambda] = i(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu)$$

$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu})$$

This is the conformal Lie algebra in all $\mathbb{R}^{(p,q)}$

Comparing the number of parameters motivates

<u>conjecture</u>: This is Lie-algebra of $SO(p+1, q+1)$

<u>check</u>: $\mu = 0, \dots, N-1$ here adapted to $\mathbb{R}^{(N-1,1)}$

$A = 0', 0, \dots N-1, N$

$$\eta_{AB} = \mathrm{diag}(0', 0, 1 \dots, N-1, N)$$

$M_{AB} := M_{\mu\nu}$ for $A, B \in (0, 1, \dots, N-1)$

$M_{0'N} := D$

$M_{\mu N} := 1/2\,(K_\mu - P_\mu)$

$M_{\mu 0'} := 1/2\,(K_\mu + P_\mu)$ $\Rightarrow$ conformal algebra takes the form

$$[M_{AB}, M_{RS}] = i(\eta_{AR} M_{BS} + \eta_{BS} M_{AR} - \eta_{AS} M_{BR} - \eta_{BR} M_{AS})$$

$\square$

1.2. Global aspects, compactification of $\mathbb{R}^{(N-1,1)}$

$\mathbb{R}^N$: special conformal trafo

$$x^\mu \to \frac{x^\mu x^2 + (x^2)^2 c^\mu}{(x+cx^2)^2}$$

is singular at just one point $(x+cx^2)^2 = 0$

$$\Leftrightarrow x^\mu = -\frac{c^\mu}{c^2}$$

i.e. special conf. trafo become globally one to one well defined after 1-point conformal compactification of $\mathbb{R}^N$ to S^N

conformal compactification of a non-compact space M means in general:

- map M conformally to a compact space $\hat{M}$
- in each compact subset of M the Weyl factor g is well defined
- infinity of M is mapped to points on $\hat{M}$ where the Weyl factor diverges

  $\hat{=}$ „conformal boundary of M"

(e.g. conformal infinity of $\mathbb{R}^N$ is mapped to north pole of S^N)

$$\mathbb{R}^{(N-1,1)}: \quad x^\mu \to \frac{x^\mu + c^\mu x^2}{c^2 \left(x + \frac{c}{c^2}\right)^2}.$$

Becomes singular not only at one point, but on a whole „critical light cone" centered at $-\frac{c^\mu}{c^2}$.

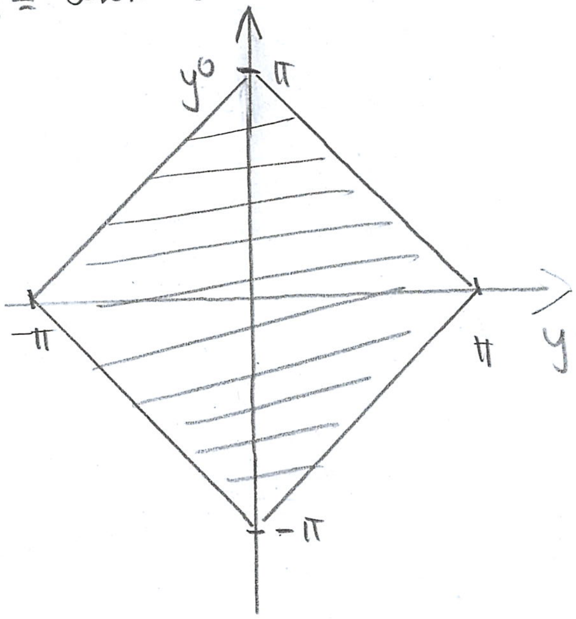
Interlude: Penrose diagrams

$N=2 \qquad u := x^1 + x^0 \qquad ds^2 = du\,dv$
$\qquad\qquad v := x^1 - x^0$

$\arctan u$ and $\arctan v \in (-\pi/2, \pi/2)$

$y^1 := \arctan u + \arctan v$
$y^0 := \arctan u - \arctan v$

$\Rightarrow$ whole $R^{(1,1)}$ mapped into shaded region of (y^0, y^1)-plane



$$dy_1^2 - dy_0^2 = d(y_1+y_0)\,d(y_1-y_0) = 4\,d(\arctan u)\,d(\arctan v)$$

$$= \frac{4}{(1+u^2)(1+v^2)}\,du\,dv = \frac{4}{(1+u^2)(1+v^2)}\,(dx_1^2 - dx_0^2)$$

$$\Rightarrow \quad dx_1^2 - dx_0^2 = \frac{\left(1+\left(\tan\frac{y^1+y^0}{2}\right)^2\right)\left(1+\left(\tan\frac{y^1-y^0}{2}\right)^2\right)}{4}\left(dy_1^2 - dy_0^2\right)$$

$$\boxed{dx_1^2 - dx_0^2 = \frac{1}{4}\,\frac{1}{\cos^2\frac{y^1+y^0}{2}\cdot\cos^2\frac{y^1-y^0}{2}}\left(dy_1^2 - dy_0^2\right)}$$

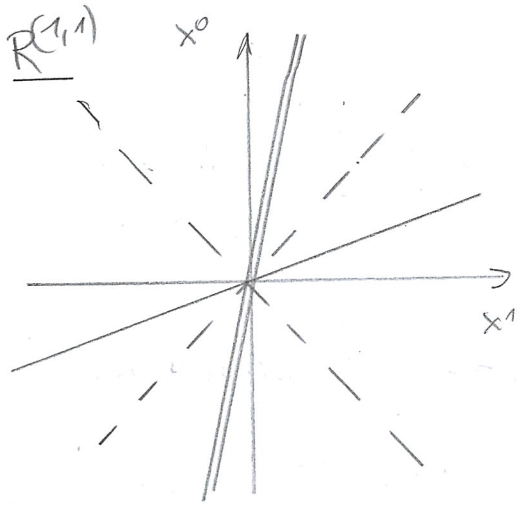
i.e. mapping is indeed conformal

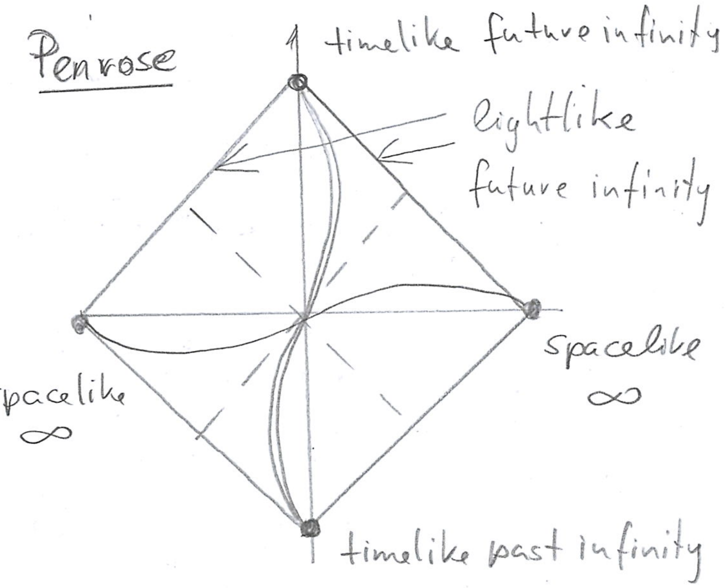
Weyl factor becomes singular on boundary of Penrose diagram ($\hat{=}$ image of infinity of $\mathbb{R}^{(1,1)}$).

<u>A crucial fact:</u>

Light-like geodesics with respect to a metric $g_{\mu\nu}(x)$ are also light-like geodesics with respect to a Weyl rescaled metric $\rho(x)\, g_{\mu\nu}(x)$.

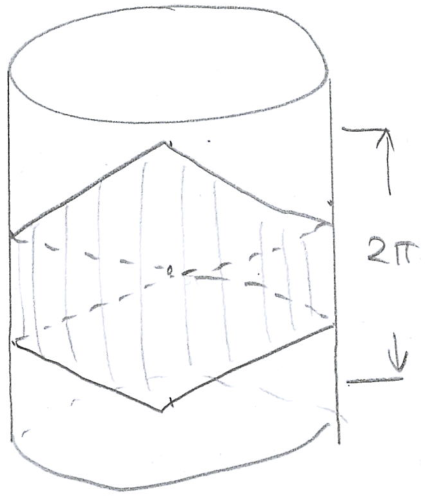
$\Rightarrow$ image of causal structure of $\mathbb{R}^{(1,1)}$ coincides with causal structure in Penrose diagram.





Equivalent realization:

Identifikation of points $y^1 = \pm\pi$
& mapping to cylinder



cylinder $\mathbb{R} \times S^1$
$= 2$ dim Einstein static univ. (ESU_2)

Higher dimensions: Conformal map $\mathbb{R}^{(N-1,1)} \to ESU_N$

$ESU_N := \mathbb{R} \times S^{N-1}$, S^{N-1} realized as embedding in $\mathbb{R}^N$ via

$$Z_1^2 + \dots + Z_{N-1}^2 + Z_N^2 = 1$$

Iterative definition of spherical coordinates:

$$(d\Omega_{N-1})^2 = \sum_{j=1}^{N} dZ_j^2 \underset{\substack{\uparrow \\ Z_N = \cos\vartheta,\ \sum_{j=1}^{N-1} Z_j^2 = \sin^2\vartheta}}{=} d\vartheta^2 + \sin^2\vartheta\,(d\Omega_{N-2})^2$$

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$\mathbb{R}^{(N-1,1)}$: $x^0, x^1, \dots x^{N-1}$: $r := \sqrt{x_1^2 + \dots + x_{N-1}^2}$, $0 \le r < \infty$

$u := r + x^0$
$v := r - x^0$

$$\left.\begin{aligned} y^0 &= \arctan(r+x^0) - \arctan(r-x^0) \\ \vartheta &= \arctan(r+x^0) + \arctan(r-x^0) \end{aligned}\right\} (*)$$

$$ds^2\big|_{\mathbb{R}^{(N-1,1)}} = -dx_0^2 + dr^2 + r^2 (d\Omega_{N-2})^2$$

$$ds^2\big|_{ESU_N} = -dy_0^2 + \sum_{j=1}^{N} (dz_j)^2$$

$$= -dy_0^2 + d\vartheta^2 + \sin^2\vartheta \,(d\Omega_{N-2})^2$$

Define map now by: equation (*) <u>and</u> identification of the unit spheres $S^{(N-2)}$ in both $\mathbb{R}^{(N-1,1)}$ and ESU_N.

$$r = 1/2 \left(\tan\frac{\vartheta+y^0}{2} + \tan\frac{\vartheta-y^0}{2}\right) \Rightarrow r^2 = \frac{1}{4}\,\frac{\sin^2\vartheta}{\left(\cos\frac{\vartheta+y_0}{2}\cos\frac{\vartheta-y_0}{2}\right)^2}$$

then, using the formulas from the 2D case

$$ds^2_{\mathbb{R}^{(N-1,1)}} = \frac{1}{4}\,\frac{1}{\left(\cos\frac{\vartheta+y^0}{2}\cos\frac{\vartheta-y^0}{2}\right)^2}\left(d\vartheta^2 - dy_0^2\right) + \frac{1}{4}\,\frac{\sin^2\vartheta}{(\quad)^2}(d\Omega_{N-2})^2$$

$$\boxed{ds^2\big|_{\mathbb{R}^{(N-1,1)}} = \frac{1}{4}\cdot\frac{1}{\cos^2\frac{\vartheta+y^0}{2}\cos^2\frac{\vartheta-y^0}{2}}\; ds^2\big|_{ESU_N}}$$

i.e. map is conformal !

$$\begin{aligned} -\infty < x^0 < +\infty \\ 0 \leq r < \infty \end{aligned} \quad \hat{=} \quad \begin{aligned} -\pi < y^0 < +\pi \\ 0 < \vartheta < \pi - |y^0| \end{aligned}$$

Explicit formula for inverse map

$$x^0 = \frac{1}{2}\left(\tan\frac{\vartheta+y^0}{2} - \tan\frac{\vartheta-y^0}{2}\right) = \frac{\sin y^0}{\cos y^0 + \cos\vartheta} = \frac{\sin y^0}{\cos y^0 + Z_N}$$

$$r = 1/2\left(\tan\frac{\vartheta+y^0}{2} + \tan\frac{\vartheta-y^0}{2}\right) = \frac{\sin\vartheta}{\cos y^0 + \cos\vartheta} = \frac{\sqrt{\sum_{j=1}^{N-1} Z_j^2}}{\cos y^0 + Z_N}$$

i.e.
$$\boxed{\begin{aligned} x^0 &= \frac{\sin y^0}{\cos y^0 + Z^N} \\ x^j &= \frac{Z_j}{\cos y^0 + Z^N}, \quad j = 1, \dots, N-1 \end{aligned}} \qquad (*)$$

$$-\pi < y^0 < +\pi$$

$$Z^N + \cos y^0 > 0 \qquad (\hat{=}\ 0 < \vartheta < \pi - |y^0|)$$

<u>We see:</u>

$$(ESU_N)_{per} := S^1 \times S^{N-1}$$

$\hat{=}$ identification of y^0 and $y^0 + 2\pi$

$$\boxed{\mathbb{R}^{(N-1,1)} \underset{\text{1 to 1, conformally}}{\longleftrightarrow} 1/2\,(ESU_N)_{per}}$$

<u>Antipode map on $(ESU_N)_{per}$:</u> $(y^0, Z) \to (y^0 + \pi, -Z)$

Then with $(*)$, used also for $Z^N + \cos y^0 < 0$, we find, that antipodal points on $(ESU_N)_{per}$ are mapped to the same point in $\mathbb{R}^{(N-1,1)}$!

i.e. $$\boxed{\mathbb{R}^{(N-1,1)} \underset{1 \text{ to } 1, \text{ conformally}}{\longleftrightarrow} (ESU_N)_{per} \Big/ \text{Antipode map}}$$

Back to special conformal transformations

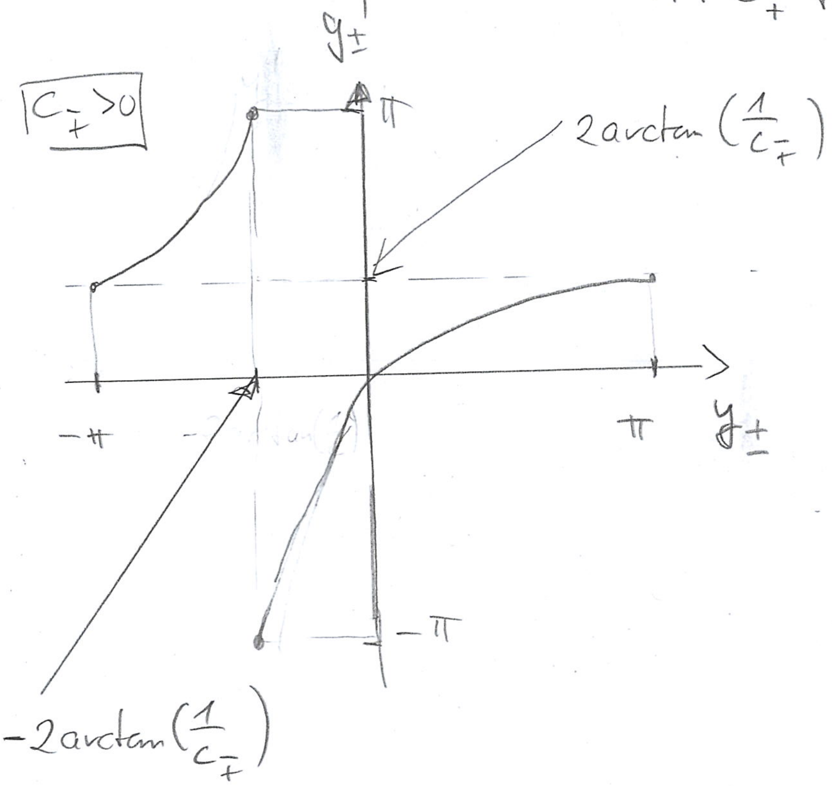
<u>N=2</u> $\quad x^2 = u \cdot v \qquad (u = x^1 + x^0,\ v = x^1 - x^0)$

$$u \to \frac{u + C_+ u \cdot v}{1 + C_+ v + C_- u + C_+ C_- u v} = \frac{u}{1 + C_- u}$$

$$v \to \frac{v}{1 + C_+ v} \qquad \text{i.e. } u \,\&\, v \text{ transform independently}$$

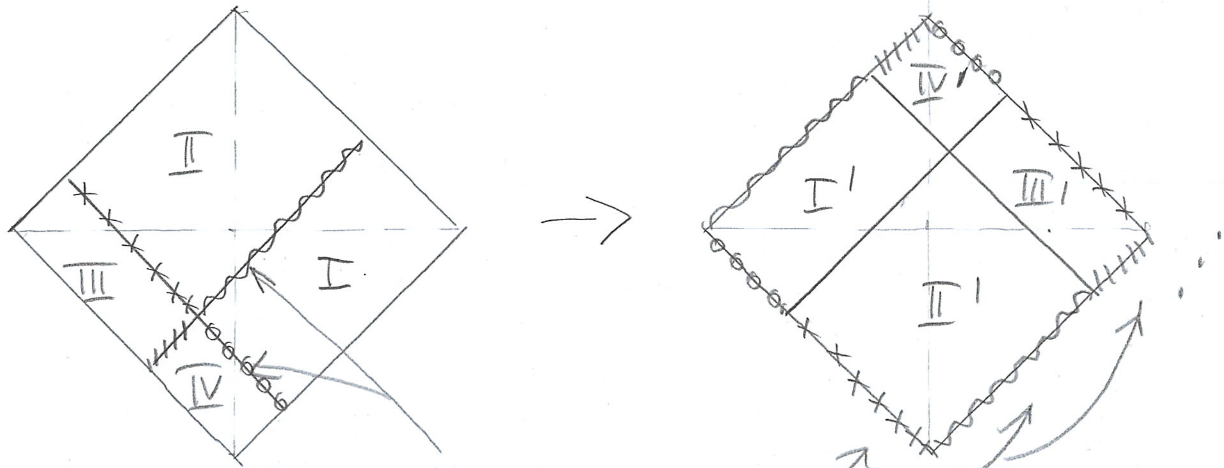
The related mapping in the Penrose diagram is:

$$y_\pm \to y_\pm' = 2 \arctan \left(\frac{\tan \frac{y_\pm}{2}}{1 + C_\mp \tan \frac{y_\pm}{2}} \right)$$



$$\boxed{2 \arctan(\pm\infty) = \pm\pi}$$

case $c_+, c_- > 0$



critical light cone
and its image

and images of four parts of original
Penrose diagram

We see: special conformal trafos become
well defined 1 to 1 maps if one
identifies opposite edges of the Penrose
diagram.

For generic N:

Take as conformal compactification of $\mathbb{R}^{(N-1,1)}$
$(ESU_N)_{per}/_{\text{Antipode map}}$) then special conf. trafos
are globally 1 to 1 well defined.

While this use of a compactification is satisfactory from a pure mathematical point of view, it has drawbacks for physics:

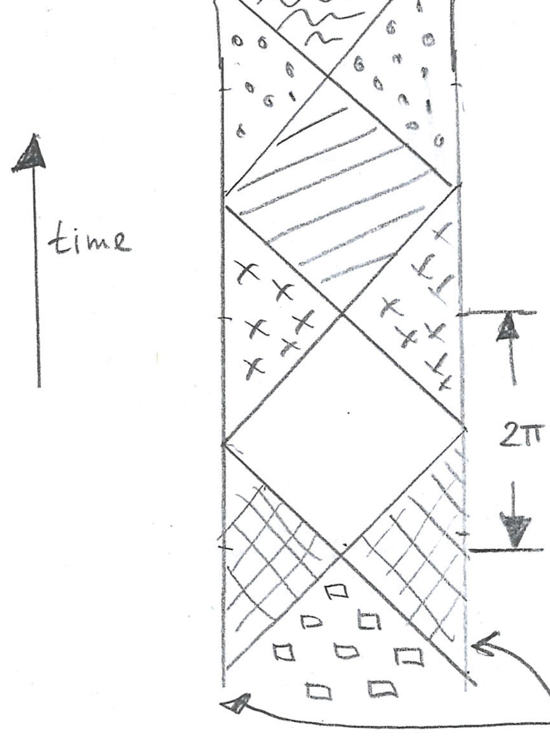
⟶ this compactification introduces closed timelike & lightlike geodesics, i.e. ~~causality~~

<u>Way out</u>: (see Lüscher, Mack: Comm. Math. Phys. 41 (1975) 203)

- Work with $\underline{ESU_4 = \mathbb{R} \times S^3}$. This can carry infinite many conformal copies of $\mathbb{R}^{(3,1)}$.
- Then local QFT's invariant w.r.t. the <u>universal covering of $SO(4,2)$</u> can be realized.

≙ replace rotations in $(0',4)$ plane by translations on $\mathbb{R}$.

<u>Illustration</u>: The infinite many copies of $\mathbb{R}^{(1,1)}$ on ESU_2



to be identified ≙ cylinder

1.3. Explicit relation to linear representation of $SO(N,2)$

We have seen that the Lie algebra of the conformal group $Conf(\mathbb{R}^{(N-1,1)})$ is equal to $so(N,2)$. The group $SO(N,2)$ is the Lorentz group in $\mathbb{R}^{(N,2)}$ and therefore has the familiar linear realization. Now we are interested in an explicit relation of the corresponding $SO(N,2)$-matrices to the finite conformal trafos (isometries in $\mathbb{R}^{(N-1,1)}$, dilatations, special conformal trafos) For the subset of Lorentz trafo in $\mathbb{R}^{(N-1,1)}$ this is trivial, for translations, dilatations and spe. conf. trafo some work is necessary.

(The constructions goes back in history at least to P. A. M. Dirac, Ann. Math. 37 (1936) 429)

Start with light cone in $\mathbb{R}^{(N,2)}$:

$$-W_{0'}^2 - W_0^2 + W_1^2 + \dots + W_N^2 = 0$$

and define $\mathbb{R}P^{N+1}_{(N,2)}$ (a projective space) as the space of equivalence classes in $\mathbb{R}^{(N,2)} \setminus \{0\}$ for the equivalence relation

$$W \sim V \Leftrightarrow W = \lambda V, \ \lambda \in \mathbb{R} \setminus \{0\}$$

Then points of the light cone in $\mathbb{R}P^{N+1}$ are mapped to $\mathbb{R}^{(N-1,1)}$ by

$$\left\| \begin{aligned} W^\mu &= \lambda x^\mu, \quad \mu = 0,1,\dots,N-1 \\ W^{0'} &= 1/2\, \lambda (1 + x^\mu x_\mu) \\ W^N &= 1/2\, \lambda (1 - x^\mu x_\mu) \end{aligned} \right. \qquad \lambda \neq 0, \text{ real}$$

(back 21)

the inversion is

$$\left\| \quad x^0 = \frac{W^0}{W^{0'} + W^N}, \quad x^j = \frac{W^j}{W^{0'} + W^N}, \quad \text{for } W^A W_A = 0 \right.$$

Check of conformality:

$$ds^2\Big|_{\mathbb{R}^{(N-1,1)}} = \sum_j \left(d\left(\frac{W^j}{W^{0'} + W^N} \right) \right)^2 - \left(d\left(\frac{W^0}{W^{0'} + W^N} \right) \right)^2$$

$$= \sum_{j=1}^{N-1} \left(\frac{dW^j}{W^{0'} + W^N} - \frac{W^j (dW^{0'} + dW^N)}{(W^{0'} + W^N)^2} \right)^2 - \left(\frac{dW^0}{W^{0'} + W^N} - \frac{W^0 (dW^{0'} + dW^N)}{(W^{0'} + W^N)^2} \right)^2$$

$$= \frac{1}{(W^{0'} + W^N)^2} \left(\sum_j (dW^j)^2 - (dW^0)^2 \right)$$

$$+ \frac{2}{(W^{0'} + W^N)^3} \left(\overbrace{W^0 dW^0 - \sum_j W^j dW_j}^{\overset{\text{cone}}{=} -W^{0'} dW^{0'} + W^N dW^N} \right) (dW^{0'} + dW^N)$$

$$+ \frac{1}{(W^{0'} + W^N)^4} \left(\underbrace{\sum_j (W^j)^2 - (W^0)^2}_{\overset{}{\underset{\text{cone}}{=}} (W^{0'})^2 - (W^N)^2} \right) (dW^{0'} + dW^N)^2$$

$$= \frac{1}{(w^{0'}+w^N)^2} \left(\sum_j (dw_j)^2 - (dw_0)^2 \right)$$

$$+ \frac{1}{(w^{0'}+w^N)^3} \left(\underbrace{2\left(w^N dw^N - w^{0'} dw^{0'}\right)\left(dw^{0'}+dw^N\right) + \left(w^{0'}-w^N\right)\left(dw^{0'}+dw^N\right)^2}_{=\,(w^{0'}+w^N)\left((dw^N)^2-(dw^{0'})^2\right)} \right)$$

$$\Rightarrow (ds^2)\Big|_{\mathbb{R}^{(N-1,1)}} = \frac{1}{(w^{0'}+w^N)^2} \left(\sum_{j=1}^{N-1} (dw_j)^2 + (dw_N)^2 - (dw_0)^2 - (dw_{0'})^2 \right)$$

$$= \frac{1}{(w^{0'}+w^N)^2} (ds^2)\Big|_{\text{cone in } \mathbb{R}^{(N,2)}} \qquad \square$$

Let us consider now

$$\text{Minkowski } \mathbb{R}^{(N-1,1)} \longrightarrow \text{cone in } \mathbb{RP}^{N+1} \xrightarrow{SO(N,2)}$$

$$\text{cone in } \mathbb{RP}^{N+1} \longrightarrow \text{Minkowski } \mathbb{R}^{(N-1,1)}$$

Λ matrix $\in SO_e(N,2)$

$$\hat{x}^\mu = \frac{\hat{w}^\mu}{\hat{w}^{0'}+\hat{w}^N} = \frac{\Lambda^\mu{}_A w^A}{\Lambda^{0'}{}_A w^A + \Lambda^N{}_A w^A}$$

$$= \frac{\Lambda^\mu{}_\nu x^\nu + \Lambda^\mu{}_{0'} \frac{1+x^2}{2} + \Lambda^\mu{}_N \frac{1-x^2}{2}}{\Lambda^{0'}{}_\nu x^\nu + \Lambda^{0'}{}_{0'} \frac{1+x^2}{2} + \Lambda^{0'}{}_N \frac{1-x^2}{2} + \Lambda^N{}_\nu x^\nu + \dots}$$

(*)
$$\boxed{\hat{x}^\mu = \frac{\Lambda^\mu{}_\nu x^\nu + \frac{1}{2}\left(\Lambda^\mu{}_{0'} + \Lambda^\mu{}_N\right) + \frac{x^2}{2}\left(\Lambda^\mu{}_{0'} - \Lambda^\mu{}_N\right)}{\left(\Lambda^{0'}{}_\nu + \Lambda^N{}_\nu\right)x^\nu + \frac{1}{2}\left(\Lambda^{0'}{}_{0'} + \Lambda^{0'}{}_N + \Lambda^N{}_{0'} + \Lambda^N{}_N\right) + \frac{x^2}{2}\left(\Lambda^{0'}{}_{0'} + \Lambda^N{}_{0'} - \Lambda^{0'}{}_N - \Lambda^N{}_N\right)}}$$

- $X \to \hat{X}$ is by construction conformally
- Λ and $-\Lambda$ correspond to the same trafo in Minkowski.
- Let denote by $SO_e(N,2)$ the subgroup of $O(N,2)$ with $\det \Lambda = +1$ and with Λ continously connected to the unity. (i.e. the exponentiation of $so(N,2)$)

  Then if $\Lambda \in SO_e(N,2)$: $-\Lambda \in SO_e(N,2)$ for N even
  $\notin SO_e(N,2)$ for N odd.

- Using the $SO_e(N,2)$ conditions on the matrix elements of Λ one can show, that backward each conformal trafo of Minkowski space $\mathbb{R}^{(N-1,1)}$ fixes uniquely an element $\in SO_e(N,2)$ (perhaps up to $\pm$)

i.e.: $$\boxed{Conf_e\left(\mathbb{R}^{(N-1,1)}\right) \text{ isomorph to } \begin{cases} SO_e(N,2)/\{\pm 1\}, & N \text{ even} \\ SO_e(N,2), & N \text{ odd.} \end{cases}}$$

(numbering $0,1,\dots,N-1;0',N$)

<u>Lorentztrafo in $\mathbb{R}^{(N-1,1)}$</u>:

$$\Lambda^A{}_B = \left(\begin{array}{c|c} \Lambda^\mu{}_\nu & 0 \\ \hline 0 & 1_1 \end{array}\right)$$

Translation in $\mathbb{R}^{(N-1,1)}$

$$\hat{x}^\mu = x^\mu + a^\mu$$

eq. (*) on page 23 $\Longrightarrow$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$$

$$\Lambda^\mu{}_{0'} - \Lambda^\mu{}_N = 0$$

and:

$$\Lambda^\mu{}_{0'} + \Lambda^\mu{}_N = 2a^\mu$$

$$\eta^{KL} = \Lambda^K{}_A \eta^{AB} \Lambda^L{}_B \qquad \Lambda^{0'}{}_\nu + \Lambda^N{}_\nu = 0$$

$$\Lambda^{0'}{}_{0'} + \Lambda^N{}_{0'} - \Lambda^{0'}{}_N - \Lambda^N{}_N = 0$$

$$\Lambda^{0'}{}_{0'} + \Lambda^{0'}{}_N + \Lambda^N{}_{0'} + \Lambda^N{}_N = 2$$

$$\Rightarrow \Lambda^A{}_B = \begin{pmatrix} \mathbb{1} & a^\alpha & a^\alpha \\ +a_\beta & 1+\frac{a^2}{2} & \frac{a^2}{2} \\ -a_\beta & -\frac{a^2}{2} & 1-\frac{a^2}{2} \end{pmatrix} \quad \Big\|$$

and analogously:

special conformal: $\hat{x}^\mu = \dfrac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2}$

$$\Rightarrow \Lambda^A{}_B = \begin{pmatrix} \mathbb{1} & c^\alpha & -c^\alpha \\ c_\beta & 1+\frac{c^2}{2} & -\frac{c^2}{2} \\ c_\beta & \frac{c^2}{2} & 1-\frac{c^2}{2} \end{pmatrix} \quad \Big\|$$

dilatations in $\mathbb{R}^{(N-1,1)}$

$$\hat{x} = e^{-g} x$$

$$\Rightarrow \quad \Lambda^A{}_B = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \cosh g & \sinh g \\ 0 & \sinh g & \cosh g \end{pmatrix} \quad \Big\|$$

Up to now we have:

$$\mathbb{R}^{(N-1,1)} \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} \text{light cone in } \mathbb{R}P^{N+1}_{(N,2)}$$

$$\mathbb{R}^{(N-1,1)} \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} (ESU_N)_{per} / \text{Antipode map}$$

$$\Rightarrow (ESU_N)_{per} / \text{Antipode map} \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} \text{light cone in } \mathbb{R}P^{N+1}_{(N,2)}$$

The correspondig direct mapping formulas are:

$$\tan y^0 = \frac{W^0}{W^{0'}}$$

$$Z^j = \frac{W^j}{\sqrt{W_0^2 + W_{0'}^2}} \; ; \; j = 1, \dots, N$$

<u>inversion:</u> $W^{0'} = \lambda \cos y^0 \quad W^j = \lambda Z^j$
$W^0 = \lambda \sin y^0$

There is still another point of view, related to the remarks on page 20. (An early refrence to the two options for conf. comp. of $\mathbb{R}^{(3,1)}$ one finds in L. Castell, Nucl. Phys. B13 (69) 231)

$$\mathbb{R}^{(N-1,1)} \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} 1/2 \; ESU_N$$

If one works for the $SO(N,2)$ implementation not with $\mathbb{R}P^{N+1}_{(N,2)}$, but with $\left(\mathbb{R}P^{N+1}_{(N,2)}\right)$ oriented

Rs 26 →

i.e. the <u>oriented projective space</u>, where the equivalence classes in $\mathbb{R}^{(N,2)}$ are defined via

$$w \sim v \iff w = \lambda v , \lambda > 0 \quad (\text{instead of } \lambda \neq 0 \text{ real})$$

Then we find

$$\mathbb{R}^{(N-1,1)} \xleftrightarrow[\text{conformal}]{\text{bijective}} \tfrac{1}{2}\left(\mathbb{R}P^{N+1}_{(N,2)}\right)_{\text{oriented}}$$

with mapping formula as on page (22)

<u>and</u> the restriction $w^{0'} + w^N > 0$

The other half $w^{0'} - w^N$ carries a second conformal copy of $\left(\mathbb{R}P^{N+1}_{(N,2)}\right)$ oriented.

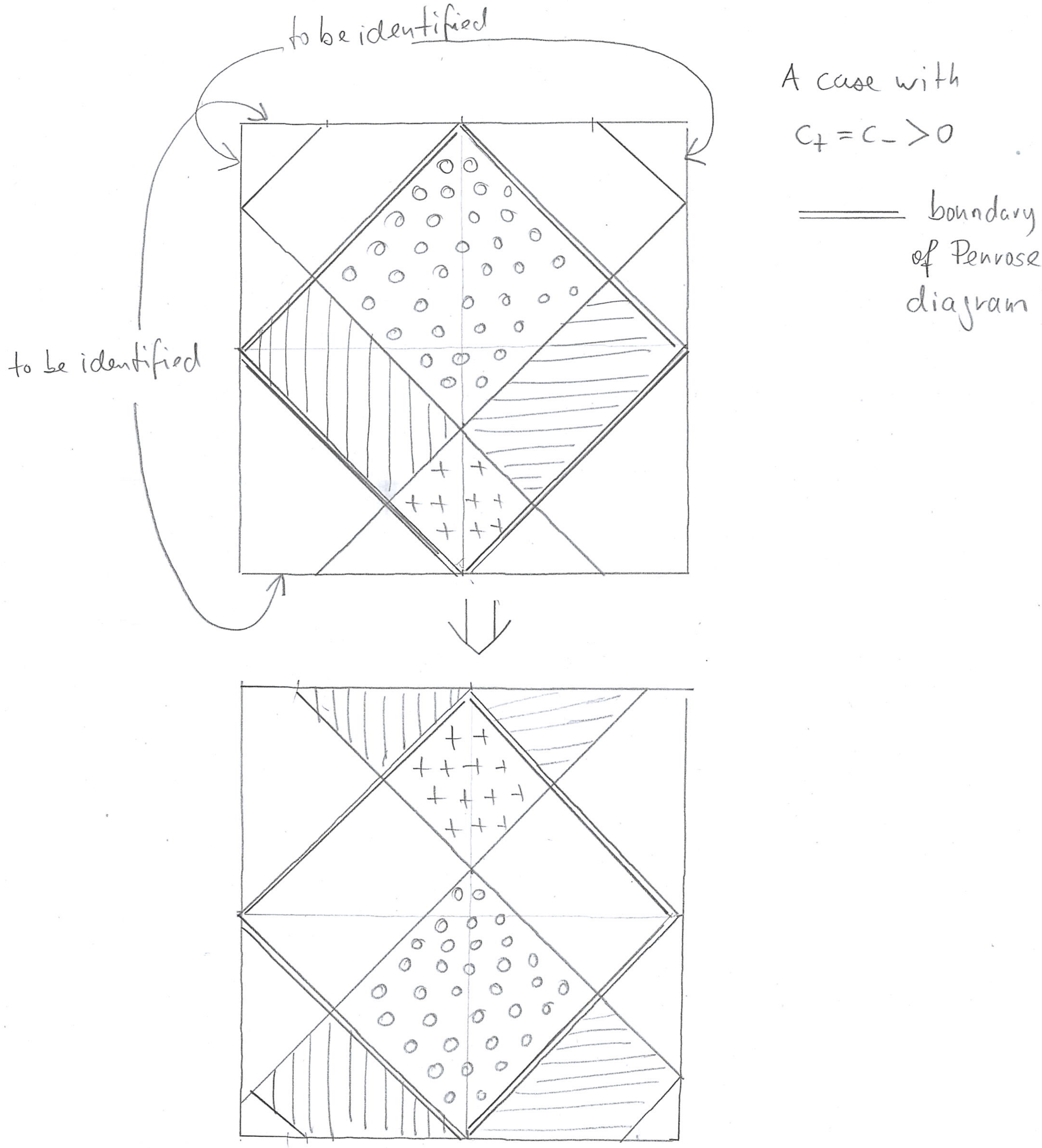
A further consequence is

$$(ESU_N)_{per} \xleftrightarrow[\text{conformal}]{\text{bijective}} \left(\mathbb{R}P^{N+1}_{(N,2)}\right)_{\text{oriented}} ,$$

and the conformal group of $(ESU_N)_{per}$ is then

$SO(N,2)$, without any reason to divide out $\pm \mathbb{1}$.

Closing this section we give a sketch of the action of special conformal trafos (in the Minkowski sense) on $(ESU_2)_{per}$ (compare pages 18, 19)



The image of the Penrose diagram is partly located outside (in the second copy of $\mathbb{R}^{1,1}$ on ESU_2). After identification of antipodes on ESU_2 one gets back the picture on page 19.

1.4. Anti-de Sitter and de Sitter-spaces, their isometries and conformal trafo

Local characterization: spacetimes with constant curvature

$$R_{\mu\nu\alpha\beta} = \frac{1}{N(N-1)} \, Ric \left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha} \right), \quad Ric: \text{ constant scalar curvature}$$

$R > 0$ de Sitter

$R < 0$ anti de Sitter

realization as embedding in

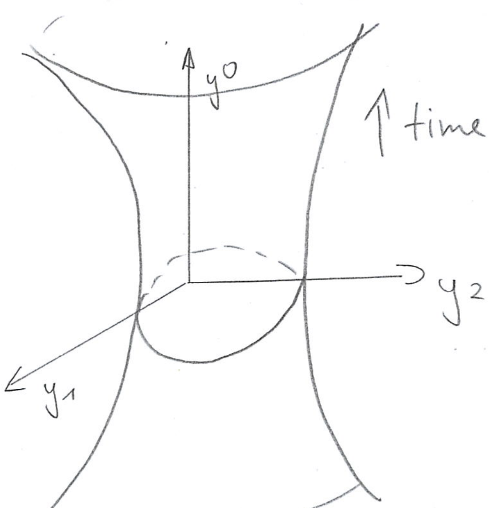
$\mathbb{R}^{(N,1)}$ dS

$\mathbb{R}^{(N-1,2)}$ AdS

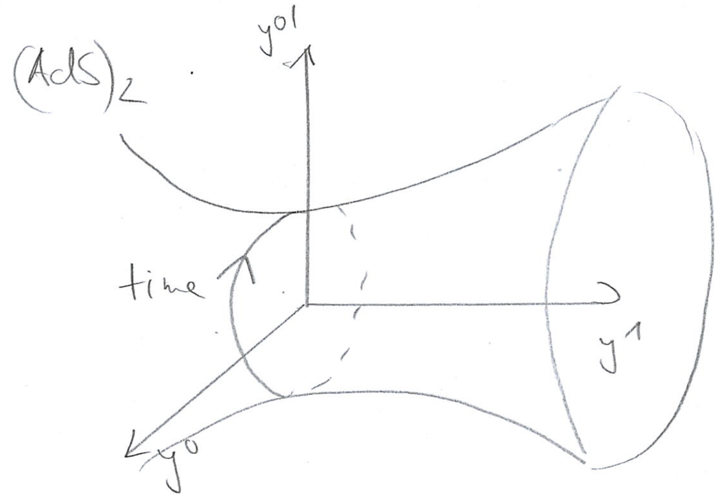
$$dS: \; y_1^2 + \dots + y_N^2 - y_0^2 = R^2 \qquad \Rightarrow Ric = \# \cdot \frac{1}{R^2}$$

$$AdS: \; y_1^2 + \dots + y_{N-1}^2 - y_0^2 - y_{0'}^2 = -R^2 \qquad \Rightarrow Ric = -\# \frac{1}{R^2}$$

$(dS)_2$ $\qquad\qquad$ $(AdS)_2$



$(AdS)_2$



- $\exists$ closed timelike geodesics
- can be avoided by timelike $S^1 \to \mathbb{R}$

i.e. $AdS_N \to \widehat{AdS_N}$ universal cover

<u>Isometry</u>

AdS_N: $SO(N-1,2)$

dS_N: $SO(N,1)$

Conformal map of AdS_N to $(ESU_N)_{per}$

Start with global coordinates for AdS_N

$$y^0 = R\cosh\rho \sin\tau$$
$$y^{0'} = R\cosh\rho\cos\tau$$
$$y^j = R\sinh\rho\, \omega^j \quad , \quad \sum_{j=1}^{N-1} (\omega^j)^2 = 1$$

$$\Rightarrow \quad ds^2 = \sum (dy^A)^2 = R^2\Big(-(\sinh\rho\sin\tau\, d\rho + \cosh\rho\cos\tau\, d\tau)^2$$
$$-(\sinh\rho\cos\tau\, d\rho - \cosh\rho\sin\tau\, d\tau)^2$$
$$+\sum_j(\omega^j\cosh\rho\, d\rho + \sinh\rho\, d\omega^j)^2\Big)$$

$\sum \omega_j^2 = 1\,,\quad \sum \omega_j d\omega_j = 0$

$$ds^2 = R^2\left(d\varrho^2 + \sinh^2\varrho\,(d\Omega_{N-2})^2 - \cosh^2\varrho\, d\tau^2\right)$$

$0 \leq \varrho < \infty$

$0 \leq \tau < 2\pi$ $\quad$ ($\tau \in \mathbb{R}$ for $\widehat{AdS_N}$)

with $\sinh\varrho = \tan\vartheta$

$(0 \leq \vartheta < \pi/2)$

$$ds^2\big|_{AdS_N} = \frac{R^2}{\cos^2\vartheta}\left(d\vartheta^2 + \sin^2\vartheta\,(d\Omega_{N-2})^2 - d\tau^2\right)$$

compare with page 16 $\Rightarrow$

$$ds^2\big|_{AdS_N} = \frac{R^2}{\cos^2\vartheta}\, ds^2\Big|_{(ESU_N)\,per}\,, \quad \text{since } 0 \leq \vartheta < \pi/2$$

we find

$$\boxed{AdS_N \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} \tfrac{1}{2}\,(ESU_N)\,per}$$

and for the universal cover

$$\widehat{AdS_N} \longleftrightarrow \tfrac{1}{2}\,(ESU)_N$$

with (note: $\tau = y_0$ of page 16)
$\uparrow$ small

$$\tan\tau = \frac{y^0}{y^{0'}}$$

$$Z^j = \frac{y^j}{\sqrt{y_0^2 + y_{0'}^2}} \qquad j = 1,\dots,N-1$$

(for $R=1$)

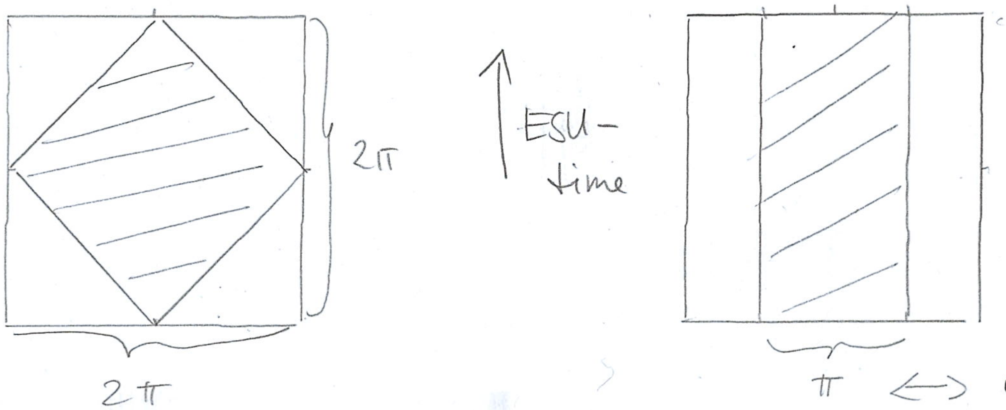
$$Z^N = \frac{R}{\sqrt{y_0^2 + y_{0'}^2}}$$

Comparison of maps of $\mathbb{R}^{(N-1,1)}$ and AdS_N to halves of $(ESU_N)_{per}$, or ESU_N

Illustration for $N=2$, note that then $\Omega^{N-2} = \{+1,-1\}$

maps to $(ESU_2)_{per}$

$\underline{R^{(1,1)}}$ $\underline{AdS_2}$

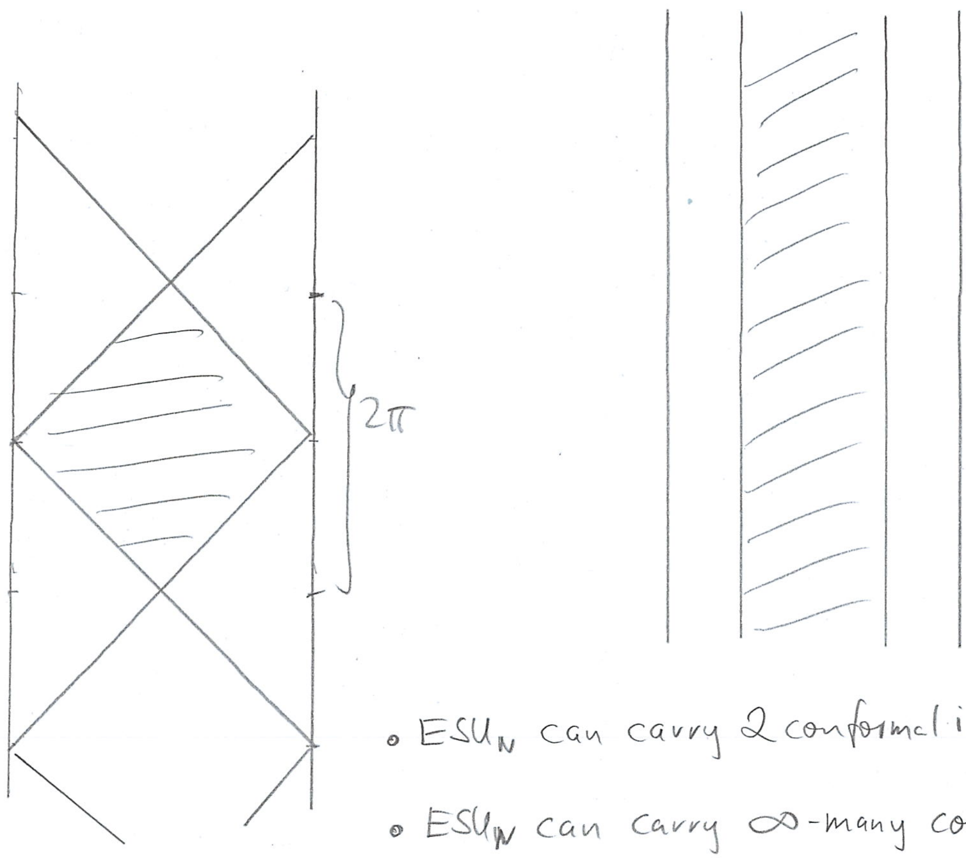


$\pi \longleftrightarrow 0 \leq \vartheta < \pi/2$ &

$\Omega^0 = \{+1,-1\}$

and maps to ESU_2

$\underline{R^{(1,1)}}$ $\underline{\widehat{AdS_2}}$



- ESU_N can carry 2 conformal images of $\widehat{AdS_N}$
- ESU_N can carry ∞-many conf. images of $\mathbb{R}^{N-1,1}$

The conformal group of AdS_N

Consider in $\left(\mathbb{R}P^{N+1}_{(N,2)}\right)_{\text{oriented}}$ again the

cone $\quad -W_0^2 - W_{0'}^2 + W_1^2 + \dots + W_N^2 = 0$

$$y^A = \frac{W^A}{W^N}, \quad A = 0', 0, 1, \dots, N-1, \quad \text{with } W^N > 0 \qquad (R=1)$$

maps half of $\left(\mathbb{R}P^{N+1}_{(N,2)}\right)$ oriented one to one on the AdS_N-hyperboloid in $\mathbb{R}^{(N-1,2)}$

$$\left(\text{Inverse map is: } \begin{array}{l} W^N = \lambda > 0 \\ W^A = \lambda y^A \end{array} \right)$$

$$\sum_A dy^A dy_A = \sum_A \left(\frac{dW^A}{W^N} - \frac{W^A dW^N}{(W^N)^2} \right)\left(\frac{dW_A}{W^N} - \frac{W_A dW^N}{(W^N)^2} \right)$$

$$= \frac{1}{W_N^2} \left(\sum_A dW^A dW_A + (dW^N)^2 \right)$$

use: $\sum_A W^A W_A + W_N^2 = 0$

$\sum_A W^A dW_A + W^N dW^N = 0$

$\Rightarrow$ map is conformal

$$\Rightarrow AdS_N \to \left(\mathbb{R}P^{N+1}_{(N,2)}\right)_{\text{oriented}} \xrightarrow{\text{Isometry}} (\quad) \longrightarrow AdS_N$$

gives the conformal trafo of AdS_N.

i.e. $\boxed{\mathrm{Conf}_e(AdS_N) = SO(N,2)}$

with

$$y^A \to \hat{y}^A = \frac{\Lambda^A{}_B y^B + \Lambda^A{}_N}{\Lambda^N{}_B y^B + \Lambda^N{}_N}, \quad A,B = 0,0',1,\dots,N-1$$

In the case of $\mathbb{R}^{(N-1,1)}$ (without compactification) isometries <u>and</u> dilatations were globally well defined.

How is the situation here?

$$1 = \eta^{NN} = \Lambda^N{}_N \eta^{NN} \Lambda^N{}_N + \Lambda^N{}_A \eta^{AB} \Lambda^N{}_B$$

$$1 = (\Lambda^N{}_N)^2 + \Lambda^N{}_A \Lambda^{NA}$$

<u>case a)</u> $\Lambda^N{}_A = 0, \ \forall A = 0,0',1,\dots,N-1$

$\Rightarrow \Lambda^N{}_N = 1$ & $y^A = \Lambda^A{}_B y^B$ i.e. isometry of AdS_N

<u>case b)</u> $\Lambda^N{}_A \neq 0$ (as a vector)

Then denominator of trafo-formula vanishes on a codimension 1 hyperplane in $\mathbb{R}^{(N-1,2)}$.
Such a plane has always common points with the AdS-hyperboloid

$\Longrightarrow$ without compactification <u>only</u> isometries of AdS are globally well defined conformal trafos.

1.5. Comments on the geometry of AdS/CFT

$$\mathbb{R}^{(N-1,1)} \xrightarrow{\text{Isometries}} \text{Poincaré}_N \supset SO(N-1,1)$$

$$\mathbb{R}^{(N-1,1)} \xrightarrow{\text{conformal}} SO(N,2)$$

$$AdS_{N+1} \xrightarrow{\text{Isometr.}} SO(N,2)$$

$$AdS_{N+1} \xrightarrow{\text{conformal}} SO(N+1,2)$$

<u>as well as</u>

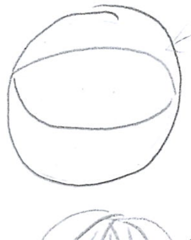
$$AdS_{N+1} \underset{\text{conformal}}{\overset{\text{bijective}}{\longleftrightarrow}} \tfrac{1}{2}\left(ESU_{N+1}\right)_{per} = \tfrac{1}{2}\left(S^1 \times S^N\right)$$

$$\partial\left(AdS_{N+1}\right) \longleftrightarrow S^1 \times S^{N-1} = \left(ESU_N\right)_{per}$$

$\uparrow$
Rand

$$\boxed{\text{carries two conformal copies of } \mathbb{R}^{(N-1,1)}}$$

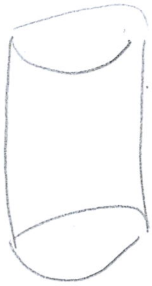
<u>Visualization for AdS_3</u>

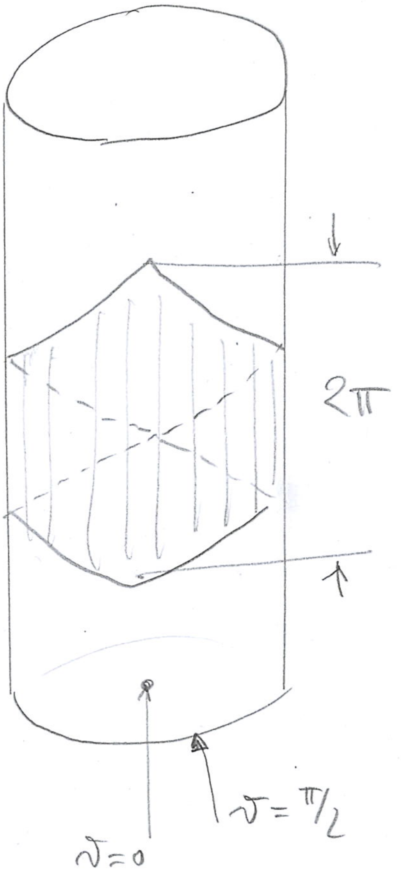
$(ESU_3)_{per} = S^1 \times$ 

$AdS_3 \Longleftrightarrow S^1 \times$  only upper half-sphere

$$\partial(AdS_3) \leftrightarrow S^1 \times S^1 = (ESU_2)_{per}$$

Plotting for the universal cover

$\widehat{AdS_3} \leftrightarrow \mathbb{R} \times$  =  interior of the cylinder



|||| conf. copy of $\mathbb{R}^{(1,1)}$

Comment on Poincaré coordinates for AdS_{N+1}

With y^A denoting the coordinates in $\mathbb{R}^{(N,2)}$ where AdS_{N+1} is embedded as a hyperboloid:

$$y^\mu = \frac{x^\mu}{r}, \quad \mu = 0,1,2,N-1$$

$$y^{0'} + y^N = \frac{1}{r}, \quad y^{0'} - y^N = \frac{r^2 + x^\mu x_\mu}{r}$$

$$\Rightarrow dy^A dy_A = \frac{1}{r^2}\left(dx^\mu dx_\mu + dr^2\right)$$

- See: Poincaré patch with $r>0$ covers half of AdS_{N+1} ($y^{0'}+y^N > 0$)
- The other half $y^{0'}+y^N < 0$ is covered with $r<0$

$r=0 \Leftrightarrow$ conformal boundary of AdS_{N+1}

$r \to +0 \triangleq$ one conformal copy of $\mathbb{R}^{(N-1,1)}$ on $(ESU_{N+1})_{per}$
$r \to -0 \triangleq$ second conformal copy of $\mathbb{R}^{(N-1,1)}$ on —"—

1.6. The case N=2

conformal Killing eq.

$$\partial_\alpha k_\beta + \partial_\beta k_\alpha = \partial^\mu k_\mu \cdot \begin{cases} \delta_{\alpha\beta} : \mathbb{R}^2 \\ \eta_{\alpha\beta} : \mathbb{R}^{(1,1)} \end{cases}$$

i.e.

$$\left.\begin{aligned} \partial_1 k_2 + \partial_2 k_1 &= 0 \\ 2\partial_1 k_1 &= \partial_1 k_1 + \partial_2 k_2 \\ 2\partial_2 k_2 &= \partial_1 k_1 + \partial_2 k_2 \end{aligned}\right\} \Leftrightarrow \begin{aligned} \partial_1 k_2 &= -\partial_2 k_1 \\ \partial_1 k_1 &= \partial_2 k_2 \end{aligned} \qquad \boxed{\mathbb{R}^2}$$

i.e. Cauchy-Riemann

$$\left.\begin{aligned} \partial_0 k_1 + \partial_1 k_0 &= 0 \\ 2\partial_1 k_1 &= \partial_1 k_1 - \partial_0 k_0 \\ 2\partial_0 k_0 &= -(\partial_1 k_1 - \partial_0 k_0) \end{aligned}\right\} \Leftrightarrow \begin{aligned} \partial_0 k_1 &= -\partial_1 k_0 \\ \partial_1 k_1 &= -\partial_0 k_0 \end{aligned} \qquad \boxed{\mathbb{R}^{(1,1)}}$$

$$\Leftrightarrow \begin{aligned} (\partial_0 + \partial_1)(k_0 + k_1) &= 0 \\ (\partial_1 - \partial_0)(k_1 - k_0) &= 0 \end{aligned}$$

$\Rightarrow$ $\boxed{\mathbb{R}^2}$: $k_\mu(x)$ CKV $\Leftrightarrow$ $\exists$ holomorphic $f(z)$, $z := x_1 + i x_2$

with $k_1(x_1, x_2) = \mathrm{Re}\, f(z)$

$k_2(x_1, x_2) = \mathrm{Im}\, f(z)$

$\boxed{\mathbb{R}^{(1,1)}}$: $k_\mu(x)$ CKV $\Leftrightarrow$ $\exists$ differentiable f_+, f_-

with

$$k_+ := k_1(x_0, x_1) + k_0(x_0, x_1) = f_+(x_0 + x_1)$$
$$k_- := k_1(x_0, x_1) - k_0(x_0, x_1) = f_-(x_1 - x_0)$$

i.e. in both cases the Algebra of CKV's is infinite dimensional

$\boxed{\mathbb{R}^2}$: Functions $f(z)$ which are holomorphic in the whole complex plane generically <u>do not</u> define a 1 to 1 map.

Globally well defined (after compactification to S^2) are only

$$z \to \frac{az+b}{cz+d} \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\mathbb{Z}_2$$

Möbiusgruppe

<u>note:</u> This is isomorphic to $SO_e(3,1)$

$\boxed{\mathbb{R}^{(1,1)}}$: $x_\pm := x_1 \pm x_0$

$x_\pm \to f_\pm(x_\pm)$ defines a global bijective map if both f_+ and f_- are monotonic on the full real axis $\mathbb{R}$.

$$\Rightarrow \boxed{\mathrm{Conf}(\mathbb{R}^{(1,1)}) = \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})}$$

since $\mathrm{Diff}(\mathbb{R}) = \underset{\substack{\uparrow \\ f'>0}}{\mathrm{Diff}_+(\mathbb{R})} \cup \underset{\substack{\uparrow \\ f'<0}}{\mathrm{Diff}_-(\mathbb{R})}$

this group has four disconnected components, and

$$\mathrm{Conf}_e(\mathbb{R}^{(1,1)}) = \mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$$