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Exercises for the course "Conformal Field Theory"
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1. Show that the group $O(4,2)$ has four components characterized by
$$\det \Lambda \ = \ \pm 1 \ , \qquad \left| \Lambda^0{}_0 \Lambda^{0'}{}_{0'} - \Lambda^0{}_{0'} \Lambda^{0'}{}_0 \right| \ \geq 1 \ .$$
Starting with $\mathrm{Conf}_e(\mathbb{R}^{(3,1)}) = SO_e(4,2)/Z_2$ substantiate that the *full* conformal group of four-dimensional Minkowski space has the four components
$$SO_e(4,2)/Z_2, \ \ T\ SO_e(4,2)/Z_2, \ \ P\ SO_e(4,2)/Z_2, \ \ \text{and}\ TP\ SO_e(4,2)/Z_2 \ ,$$
where T and P denote the $O(4,2)$ representatives of time inversion and parity. Discuss in this pattern also the role of the inversion on the unit hyperboloid
$S: \ \ x \mapsto x/x^2$.

2. Consider the map from the lightcone in $\mathbb{R}P^{N+1}$ to $\mathbb{R}^{(N-1,1)}$
$$x^\mu \ = \ \frac{W^\mu}{W^{0'} + W^N} \ , \quad \mu = 0, 1, \ldots N-1$$
and show
$$(x-y)^2 \ = \ \frac{(W-V)^2}{(W^{0'}+W^N)(V^{0'}+V^N)} \ ,$$
as well as for the cross ratios of four points $x_{(1)}, \ldots x_{(4)}$
$$\frac{(x_{(1)}-x_{(3)})^2 \ (x_{(2)}-x_{(4)})^2}{(x_{(1)}-x_{(4)})^2 \ (x_{(2)}-x_{(3)})^2} \ = \ \frac{(W_{(1)}W_{(3)}) \ (W_{(2)}W_{(4)})}{(W_{(1)}W_{(4)}) \ (W_{(2)}W_{(3)})} \ .$$

3. Show that for solutions of the conformal Killing equation in $\mathbb{R}^N$, $\ N>2$ all third and higher derivatives vanish.

4. Show that the stereographic projection of S^N to $\mathbb{R}^N$ is a conformal map.

5. Consider a pseudo-Riemannian space with metric $g_{\mu\nu}$. Show that each null geodesic $x^\mu(t)$ with respect to the corresponding Levi-Civita connection is also null geodesic with respect to the Levi-Civita connection for the Weyl rescaled metric $\hat{g}_{\mu\nu}(x) = \rho(x) g_{\mu\nu}(x)$.

6. Show that in two-dimensional Riemann spaces around each point there is an open set in which, after a suitable coordinate transformation, the metric takes the form $g_{\mu\nu}(x) \;=\; \rho(x)\,\delta_{\mu\nu}$.

7. To find the special conformal transformations in $\mathbb{R}^{(1,1)}$ perform the explicit integration of the conformal Killingvector field $k^\mu(x) \;=\; c^\mu x^2 \;-\; 2x^\mu\, cx$.

8. Consider the often used bijection between the flat plane $\mathbb{R}\times\mathbb{R}$ and the flat cylinder $\mathbb{R}\times S^1$:

$$t \;=\; e^\tau\;\cos\sigma\;, \qquad x \;=\; e^\tau\;\sin\sigma\;, \qquad t,\;x,\;\tau\;\in\mathbb{R},\quad 0\le\sigma<2\pi\;.$$

Show that this map is conformally in the Euclidean case, but *not* conformally in the Lorentzian case.
Furthermore, show that in the Lorentzian case the bijection between $\mathbb{R}\times\mathbb{R}$ and the torus $S^1\times S^1$:

$$t\pm x \;=\; i\;\frac{1-e^{i(\tau\pm\sigma)}}{1+e^{i(\tau\pm\sigma)}}\;, \qquad t,\;x\in\mathbb{R}\;,\quad \tau\pm\sigma\in[0,2\pi)$$

is conformally.

9. Show that the following action

$$S \;=\; -\frac{1}{2}\int d^Nx\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\varphi\;\partial_\nu\varphi \;+\; \frac{N-2}{4(N-1)}R\;\varphi^2\right)$$

is invariant under the Weyl transformation

$$g_{\mu\nu}(x)\to(\lambda(x))^2\;g_{\mu\nu}(x)\;, \qquad \varphi(x)\to(\lambda(x))^{\frac{2-N}{2}}\;\varphi(x)\;.$$

Calculate the corresponding energy-momentum tensor

$$T^{\mu\nu}(x) \;=\; \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}\;.$$

10. Show that in a CFT the transformation formula for quasi-primary operators

$$U(K)\;\varphi_j(x')\;U^{-1}(K) \;=\; \left|\det\frac{\partial x'}{\partial x}\right|^{-d/N}\;D_j^{\;l}(R)\;\varphi_l(x)$$

implies for infinitesimal $K: x' = x+\epsilon k(x),\;\; k(x)$ conformal Killing, $D_j^{\;l}(R(K)) \;=\; \delta_j^l + \epsilon\omega_j^{\;l}$ and with $\mathcal{K}$ denoting the quantum representative of the generator of the infinitesimal trafo K:

$$[\mathcal{K}\;,\varphi_j(x)] \;=\; i\Big(\frac{d}{N}(\partial^\mu k_\mu)+k^\mu\partial_\mu\Big)\varphi_j(x)\;\;-i\omega_j^{\;l}\varphi_l(x).$$

11. Calculate both the canonical as well as the Belinfante energy momentum tensor for the Maxwell action

$$S = -\frac{1}{4}\int d^N x\; F_{\mu\nu}F^{\mu\nu} \ .$$

Show that the general transformation formula for quasiprimaries, given in the lectures, implies (with scaling dimension $d = 2$ for the field strength tensor $F$and $\rho^{N/2}$ =det$\frac{\partial x'}{\partial x}$)

$$F'^{\mu\nu}(x') = \rho^{-2}\,\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\,F^{\alpha\beta}(x) \ , \quad \text{but} \;\; F'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\,F_{\alpha\beta}(x) \ .$$

12. Show that

$$G_n(\lambda x_j, g, \mu) = \lambda^{-n d_{cl}}\; G_n(x_j, \bar{g}(\lambda), \mu)\; \exp\left(n\int_g^{\bar{g}(\lambda)} \frac{\gamma(g')}{\beta(g')}\,dg'\right) \ ,$$

with $\lambda\frac{d\bar{g}(\lambda)}{d\lambda} = -\beta(\bar{g}(\lambda)), \;\; \bar{g}(1) = g$ solves the differential equation

$$\left(\beta(g)\frac{\partial}{\partial g} + \lambda\frac{d}{d\lambda} + n(d_{cl} + \gamma(g))\right) G_n(\lambda x_j, g, \mu) = 0 \ .$$

13. In a CFT the two point function for two scalar quasi-primaries $\varphi_1, \;\varphi_2$ with scaling dimensions $d_1, \;d_2$ has the form

$$\langle\varphi_1(x)\varphi_2(y)\rangle = \frac{c_{12}}{\left((x-y)^2\right)^{\frac{d_1+d_2}{2}}} \ , \qquad \text{with} \;\; c_{12} = 0 \;\; \text{if } d_1 \neq d_2 \ .$$

Prove this statement using Poincare invariance and the differential equation of the special conformal Ward identity.

14. Show that for special conformal transformations in $\mathbb{R}^{(1,N-1)}$

$$x'_{\mu} = \frac{x_{\mu} + c_{\mu}x^2}{1 + 2cx + c^2x^2}$$

the determinant is given by

$$\det\left(\frac{\partial x'}{\partial x}\right) = (1 + 2cx + c^2x^2)^{-N} \ .$$

15. Show that
a) For correlation functions of n quasiprimaries $\varphi_1, \ldots, \varphi_n$ in $\mathbb{R}^{1,N-1}$ with the scaling dimensions $d_1, \ldots, d_n$ one has due to dilatation invariance

$$\langle\varphi_1(\lambda x_1)\ldots\varphi_n(\lambda x_n)\rangle = \lambda^{-\sum_j d_j}\,\langle\varphi_1(x_1)\ldots\varphi_n(x_n)\rangle \ .$$

b) This implies for the Fourier transform defined by [1] $\langle\varphi_1(x_1)\ldots\varphi_n(x_n)\rangle = \int \frac{d^N k_1}{(2\pi)^N}\ldots\frac{d^N k_{n-1}}{(2\pi)^N}\,\Gamma_n(k_1,\ldots$

$$\Gamma_n(\lambda k_1,\ldots,\lambda k_n) \;=\; \lambda^{-(n-1)N+\sum_j d_j}\,\Gamma_n(k_1,\ldots,k_n)\;.$$

c) In particular for the two-point function this means

$$\Gamma_2(k,-k) \;=\; \frac{c_{12}}{(k^2)^{\frac{N-\sum_j d_j}{2}}}\;.$$

16.

17.

18. .....

[1] Keep in mind that due to translation invariance $\sum_{j=1}^{n} k_j = 0$ and therefore integration is over $(n-1)$ momenta only.