Conformal invariance in quantum field theory

- Motivation and introduction

- Some history

- Geometry

- Energy momentum tensor, renormalization, trace anomaly

- Correlation functions

- Conformal bootstrap, c and a-theorem

- Continous symmetries $\Leftrightarrow$ conservation laws, theory more constrained
- $\exists$ spacetime & internal symmetries
- Fundamental relevance of Poincaré group (translations plus Lorentz)

  i.e. isometry trafos of Minkowski space

- elementary particles $\Leftrightarrow$ irreps of Poincaré group (mass, spin)
- discrete isometries T, P (time reversal, parity) relevant for physics too, CPT-theorem
- correlation functions, e.g. of scalar fields, depend on $(x_i - x_j)^2$ only

- Conformal group: spacetime trafos which leave angles unchanged, turns out to be Poincaré + dilatations + special conformal trafos

  (with 4 parameters)

Dilatation invariance requires absence of dimensionful parameters $\Rightarrow$

- candidate conformal theories should be massless,
- breaking of classical conformal invariance by renormalization effects

- Within the set of renormalizable QFT's, CFT's are a special subset (RG β-functions zero)

- RG flow of not conformal theories can end at CFT's in the IR or UV, example: asymptotic freedom of QCD
- treatment of not conformal theories as perturbations around CFT's

- Adding supersymmetry, one gets even more constrained theories, e.g. indications of integrability in planar $\mathcal{N} = 4$ super Yang-Mills

- Conformal symmetry plays a constitutive part in the AdS/CFT correspondence.

- Two-dimensional conformal symmetry plays a constitutive part in string theory. In particular it is responsible for the critical dimensions 26 and 10 respectively.

- In its Euclidean version, conformal symmetry shows up in statistical systems at criticality (infinite correlation length).

- $\exists$ several more reasons to be interested in conformal invariance.

Note: In two dimensions the conformal Lie algebra is infinite dimensional $\Rightarrow$ 2D conformal symmetry is then more constraining as for $D \geq 3$.

-Conformal mappings since old times in the toolkit of mathematicians

-Relevance for physics since 19$^{\text{th}}$ cent., in particular for electrodynamics

-Weyl trafo = position dependent scale trafo in GR, Weyl 1918

-boom in CFT in the 1960's and 70's (Wess,Kastrup,Callan,Coleman,Jackiw, Ferrara,Grillo,Gatto,Mack,Migdal,Polyakov,....)

in particular: conformal bootstrap program

- In same period start of applications to critical statistical systems
- 2D CFT for string theory starting in the 70's, systematic study up to the classification issue started with Lüscher,Mack 76 and mainly with Belavin,Polyakov,Zamolodchikov 84

-AdS/CFT since 1997, Maldacena,....

- Since 2008 renewed interest in conformal bootstrap, scale versus conformal inv. & analogs in $D \geq 3$ of Zamolochikovs c-theorem in 2D

Defining eq. of a conformal mapping in N-dim. Minkowski space $\mathbb{R}^{(1,N-1)}$:
$x^\mu \mapsto y^\mu$ such that for crossing curves $x_{(1)}(t)$, $x_{(2)}(t)$ at the crossing point

$$\frac{(\dot{x}_{(1)}, \dot{x}_{(2)})}{\sqrt{(\dot{x}_{(1)}, \dot{x}_{(1)})\ (\dot{x}_{(2)}, \dot{x}_{(2)})}} = \frac{(\dot{y}_{(1)}, \dot{y}_{(2)})}{\sqrt{(\dot{y}_{(1)}, \dot{y}_{(1)})\ (\dot{y}_{(2)}, \dot{y}_{(2)})}}$$

$$\Longleftrightarrow \quad \eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} = \rho(x)\ \eta_{\alpha\beta}\ , \quad N\rho(x) = \eta_{\mu\nu}\eta^{\alpha\beta} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta}$$

Find all solutions: - study infinitesimal cases $y^\mu = x^\mu + \epsilon\ k^\mu(x)$,

($k^\mu(x)$ conformal Killing vectors)

- integrate

Result: For $N \geq 3$ group of conformal trafo, continously connected to identity, is isomorph to $SO(2,N)$, i.e. a Lorentzgroup in a space with two more dimensions.

Explicit characterization:

translations	$x^\mu \mapsto x^\mu + a^\mu$
Lorentztrafos	$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu \ , \quad \Lambda \in SO(1,N-1)$
dilatations	$x^\mu \mapsto \lambda x^\mu$
special conformal trafos	$x^\mu \mapsto \frac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2} \ = \ S\ T_c\ S$, with $S : x^\mu \mapsto \frac{x^\mu}{x^2}$

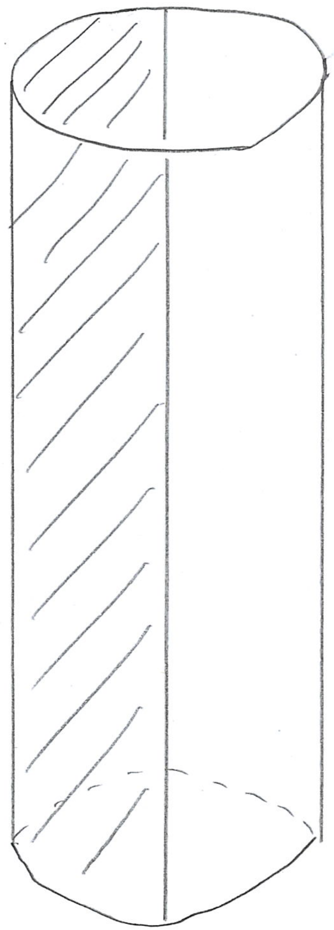
Special status of N=2: Lie algebra of conformal Killing vectors is infinite dimensional.

SO(2,N) is also the isometry group of AdS_{N+1}. This fact is crucial for the AdS/CFT correspondence.

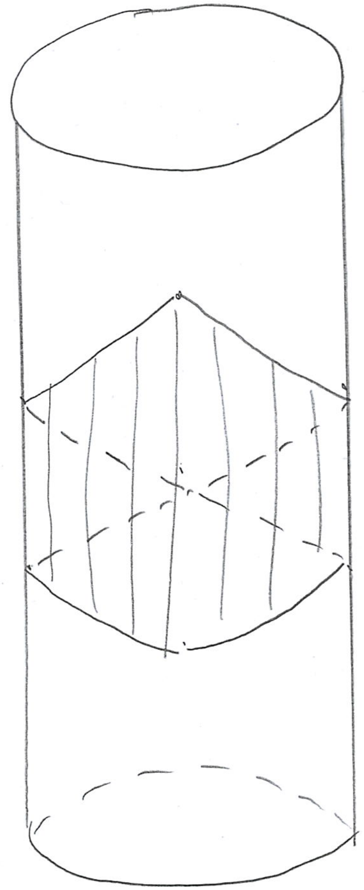
Geometry

$$
\begin{tikzcd}[column sep=huge]
\mathbb{R}^{(1,N-1)} \arrow[r, "\text{isometries}"] \arrow[dr, "\text{conformal}"] & \text{Poincaré}_N \supset SO(1,N-1) \\
 & \boxed{SO(2,N)} \\
AdS_{N+1} \arrow[ur, "\text{isometries}"] \arrow[dr, "\text{conformal}"] & \\
 & SO(2,N+1)
\end{tikzcd}
$$

$\exists$ conformal maps of AdS_2 and $\mathbb{R}^{(1,1)}$ to $ESU_2 = \mathbb{R} \times S^1$

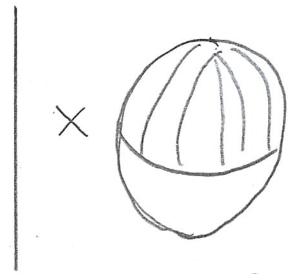
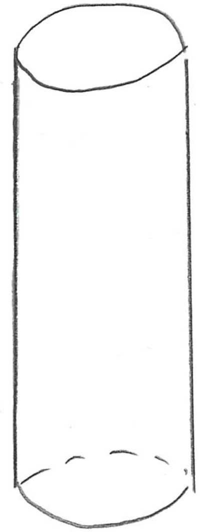


AdS_2 conformal to $1/2$ of ESU_2



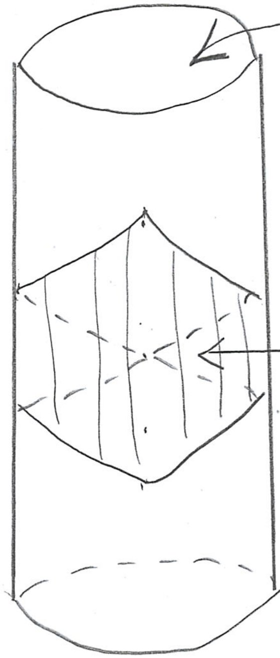
$Minkowski_2$ conformal to a certain patch of ESU_2 (Penrose diamond)

Visualization of the geometry for AdS_3 / CFT_2:

AdS_3 conformal to $\frac{1}{2}\, ESU_3$ =  = 

$\mathbb{R} \times \frac{1}{2} S^2$

the whole interior of this cylinder



interior of cylinder conformal to AdS_3

part of the boundary of AdS_3 conformal to $Minkowski_2$

The quantities conserved due to Poincaré invariance (energy, momentum, angular momentum) are given as suitable “pieces” of the energy-momentum tensor, expect the same for additional conformal quantities.

Let us denote by $k^{(a)}(x)$ $(a = 1, ..., 15)$ one of the independent solutions of the conformal Killing equation

$$\partial_\mu k_\nu + \partial_\nu k_\mu \ = \ \frac{2}{N} \partial^\alpha k_\alpha \ \eta_{\mu\nu} \ .$$

Then for $j^{(a)}_\mu := \ k^{(a)\nu} T_{\mu\nu}$ one gets, using $T_{\mu\nu} = T_{\nu\mu}, \quad \partial_\mu T^{\mu\nu} = 0$

$$\partial^\mu j^{(a)}_\mu \ = \ \frac{1}{N} \, T^\mu{}_\mu \, \partial^\nu k^{(a)}_\nu \ .$$

See:

$$T^\mu{}_\mu \ = \ 0 \qquad \Longrightarrow \qquad \text{conformal invariance}$$

The opposite direction is also true for unitary theories (with some technical assumptions). In particular the related question:

scale invariance $\Longrightarrow$ full conformal invariance ?

has a long history and is subject of a lot of recent activities.

<u>CFT's as a subset of renormalizable QFT's:</u>

$$T^{\mu}{}_{\mu}(x) \;=\; \sum_j \beta_j(g)\,\mathcal{O}_j(x) \;+\; \text{mass terms}$$

e.g. for QCD:

$$T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g^2}\,F^{a}_{\mu\nu}F^{a,\mu\nu} \;+\; (1+\gamma_m(g))\,m\bar{\psi}\psi$$

For theories, which are conformal invariant on the classical level, the term with the RG β-functions is due to the breaking of the symmetry by renormalization effects: trace anomaly.

Not conformal inv. QFT's can flow under the change of the RG-scale to CFT's in the IR or UV.

Most prominent example in particle physics: asymptotic freedom of QCD.

Further contribution to trace anomaly for QFT's in curved spacetimes:
- relevant for gravitational physics (e.g. Hawking radiation of black holes)
- but also relevant for flat space physics, Why ?

In general the canonical energy momentum tensor needs several "improvement" steps (Belinfante, conformal). Final result can be obtained in equivalent way:

couple in suitable manner to gravity and use

$$T^{\mu\nu}(x) \;=\; \frac{-2}{\sqrt{-g}}\,\frac{\delta S}{\delta g_{\mu\nu}(x)}\,|_{g_{\alpha\beta}=\eta_{\alpha\beta}}\;.$$

$$\langle 0|T(x)T(y)|0\rangle \propto \;\frac{\delta}{\delta g(x)}\frac{\delta}{\delta g(y)}\int d\varphi\; e^{iS[\varphi,g]}$$

Since

$$\frac{\delta R}{\delta g_{\mu\nu}} \neq 0 \qquad \text{in the flat limit,}$$

pure curvature dependent terms in the trace anomaly leave footprints in flat two and higher point functions of $T_{\mu\nu}$.

Curved space trace anomaly of CFT's:

$$T^{\mu}_{\ \mu} = \frac{c}{24\pi} R \qquad\qquad N = 2$$

$$T^{\mu}_{\ \mu} = c\, W^2 - a\, E \qquad\qquad N = 4$$

(W Weyl tensor, E Euler density, both certain combinations out of the curvature tensor)

The numbers c and a belong to the set of data characterizing a CFT.

For $N = 2$, c turns out to be the central charge of the Virasoro algebra, it has also an immediate physical consequence:

$$E_0 = \frac{\pi}{6}\frac{c}{L} \text{ for a CFT on a strip of width } L.$$

Transformation law for quasiprimary operators

$$U(K)\ \varphi_j(x')\ U^{-1}(K)\ =\ \left(\det\frac{\partial x'}{\partial x}\right)^{-d/N}\ D_j{}^l(R)\ \varphi_l(x)$$

with R a Lorentztrafo, related to the conformal trafo $K: \quad x \mapsto x'$

by $R^\mu{}_\alpha := \left(\det\frac{\partial x'}{\partial x}\right)^{-1/N} \frac{\partial x'^\mu}{\partial x^\alpha}$, $\qquad d$ scaling dim.

<u>Note:</u> $\quad$ det $= 1$ for all Poincaré trafo,

$= \lambda^N$ for dilatations,

$= (1 + 2cx + c^2x^2)^{-N}$ for special conformal trafo.

This fixes two and three-point functions up to constants.

e.g. for scalars:

$$\langle \varphi_1(x_1)\ \varphi_2(x_2)\rangle \;=\; c_{12}\left((x_1-x_2)^2\right)^{-\ \frac{d_1+d_2}{2}}, \qquad c_{12}=0 \text{ if } d_1 \neq d_2$$

$$\langle \varphi_1(x_1)\ \varphi_2(x_2)\ \varphi_3(x_3)\rangle = c_{123}\ ((x_1-x_2)^2)\ ^{a_{12}}((x_2-x_3)^2)\ ^{a_{23}}((x_1-x_3)^2)\ ^{a_{13}}$$

with $a_{12} = (d_3 - d_1 - d_2)/2$ etc.

For 4 and more points one can form conformal invariants out of the $(x_j - x_l)^2$, the cross ratios:

$$\frac{(x_i-x_k)^2}{(x_i-x_l)^2}\,\frac{(x_j-x_l)^2}{(x_j-x_k)^2}\ .$$

The dependence on these conformal invariants is not fixed, of course.

Conformal Ward identities:

for all quasiprimaries: $$\sum_{j=1}^{n}(d_j + x_j\frac{\partial}{\partial x_j})\ \langle\varphi_1(x_1)\ldots\varphi_n(x_n)\rangle = 0,$$

for scalars: $$\sum_{j=1}^{n}\left(2(x_j)_\mu(x_j\frac{\partial}{\partial x_j}+d_j)-x_j^2\frac{\partial}{\partial x_j^\mu}\right)\ \langle\varphi_1(x_1)\ldots\varphi_n(x_n)\rangle = 0.$$

Of interest can be also anomalous conformal Ward identities

(taking care of symmetry breaking by a regularization).

e.g. in $\mathcal{N}=4$ SYM such identities, applied to dimensionally regularized

Wilson loops on null polygons, can be used to derive the so called

BDS structure (after Bern, Dixon, Smirnov).

Via Wilson loop - scattering amplitude correspondence then also for amplitudes.

For progress with higher point function use

operator product expansion (OPE).

Naive understanding of OPE as a tool in renormalizable QFT:

Look at example of composite operators formed out of a free field & and use Wick's theorem.

<u>In general:</u>

$$A(x)\ B(y) \ = \ c_0(x-y)1 \ + \ \sum_{j=1}^{p} c_j(x-y)\ \mathcal{O}_j(y) \ + \ \mathcal{N}(A(y)B(y)) \ + \ldots,$$

valid as an asymptotic expansion for $x \to y$.

OPE in CFT's:

- convergent
- conformal invariance fixes relative weights of contributions of a quasiprimary and its derivatives (descendents).

Altogether, a CFT can be defined as
a set of quasiprimaries and their descendents, closed under OPE.
The data fixing the CFT are:
- the set of scaling dimensions of the quasiprimaries = spectrum of dilatation operator,
- the set of representations (scalar, spinor,...),
- the coefficients in the trace anomaly (a, c in case of $N = 4$),
- the structure constants in the OPE.

The c-theorem was proven for 2-dim. CFT by Zamolodchikov in 1986, after a long history, only in 2011 Komargodski and Schwimmer found an accepted proof of an analog in 4 dim.

c-theorem in 2D:

For unitary QFT's exists a function of the couplings $c(g_1, g_2, \dots)$, which is monotonic decreasing along an RG flow (in the IR direction).
At RG fixpoints (β-fct = 0) it is equal to the central charge of the corresponding CFT.
If the flow connects CFT's in the UV and IR, one has

$$c_{\text{UV}} > c_{\text{IR}} .$$

This fits to the picture, that c counts in some sense the degrees of freedom.

For 4-dim. unitary QFT's now it is established, that if RG flow connects CFT's in UV and IR

$$a_{\mathrm{UV}} \;>\; a_{\mathrm{IR}}\;.$$

With related techniques at present several papers available on properties of RG flows and the issue

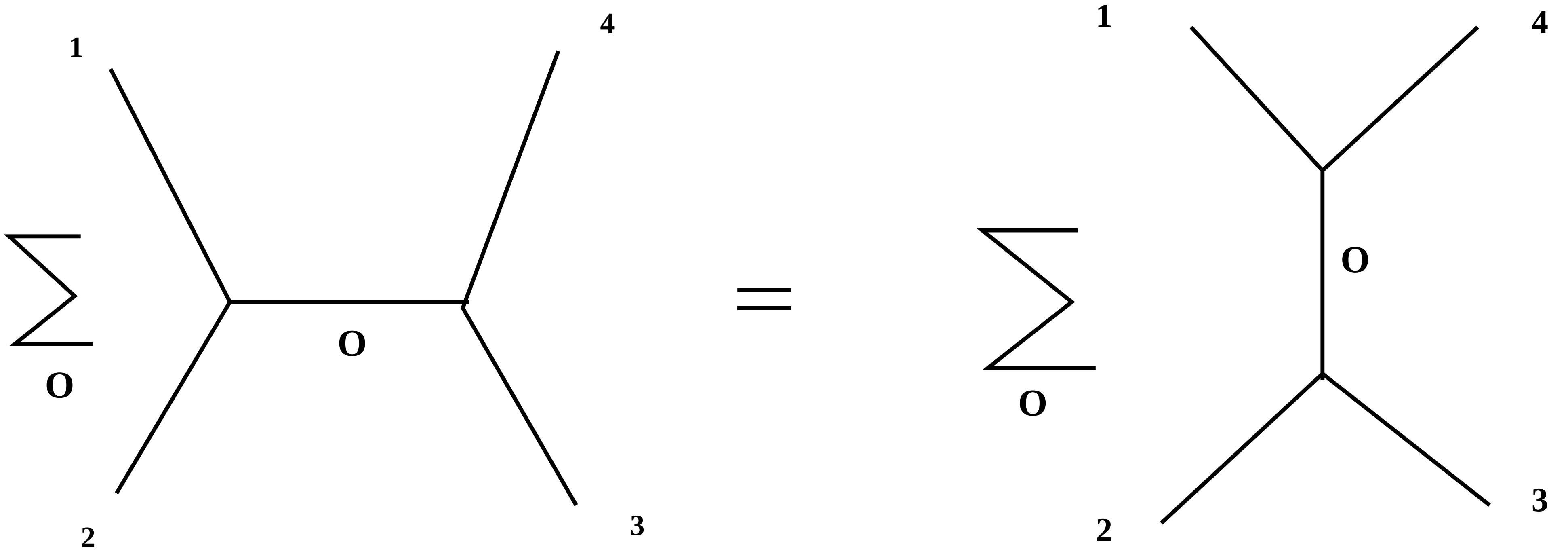
scale invariance $\Longrightarrow$ full conformal invariance ?

Conformal bootstrap:

- program started in beginning of 70's
- succesfully implemented in 2-dim case (stronger symmetry constraints), up to partial classification
- renaissance in 4-dim case in recent years

Basic idea:

- Use in n-point correlation functions OPE of two operators to represent it as an infinite sum of (n-1)-point functions with (in principle) fixed coefficient functions,
- continue up to ending with 3-point functions,
- require associativity
- hope that associativity (crossing) for 4-point functions is sufficient.


1
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One of the recent results:

Universal bounds on the scaling dimensions of the first operator appearing in the OPE of two operators with given dimensions.

e.g. in Rychkov et al 1203.6064: 3-dim Ising model saturates the bound

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Contents of the lectures "Conformal field theory", SS 2014, Krippen, March 30 - April 04, block course of the Research Training Group (GK 1504)

Literature:

There is a huge set of papers related to the subject. At few places there will be references in the manuscript. A good starting point for diving into the literature are the following papers and references therein.

H.A. Kastrup, "On the Advancements of Conformal Transformations and their Associated Symmetries in Geometry and Theoretical Physics", arXiv:0808.2730

S. Rychkov, "EPFL Lectures on Conformal Field Theory in D$\geq$ 3 Dimensions", https://sites.google.com/site/slavarychkov/

Yu. Nakayama, "A lecture note on scale versus conformal invariance", arXiv:1302.0884

For two-dimensional conformal field theory one can start with the book

P. Di Francesco, P. Mathieu, D. Senechal: "Conformal Field Theory" Springer

or with one of the many good books or reviews related to string theory.