Feedback from an asymmetric FP with lossless mirrors

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1 Summary

The reflectivity of a Fabry-Pérot resonator (FP) seems to be simple textbook matter. However, it is not easy to find explicite formulae. So I have reconsidered the following problem in the framework of traveling-wave equations (TWE) = optical field equation of the Maxwell-Bloch equations.

Calculate the field $\mathcal{E}_b(t)$ reflected from a passive FP if it is illuminated with an optical pulse $\mathcal{E}_{\text{in}}(t)$. Assume normal incidence (or a waveguide configuration), allow for mirrors having different reflectivities, and allow for absorption (scattering) losses in the FP.

Not knowing the general properties of a lossy mirror, I consider lossless mirrors only. Then, the solution is (derivation see section 2)

$$\boxed{\mathcal{E}_b(t) = r_{1_-} \sum_{n=0}^{\infty} \mathcal{R}^n \cdot \left[\mathcal{E}_{\text{in}}(t_n) - \frac{\mathcal{R}}{R_{1_-}}\,\mathcal{E}_{\text{in}}(t_{n+1})\right]} \qquad \begin{array}{rcl} \mathcal{R} &=& \sqrt{R_{1_+}R_{2_-}}e^{-\alpha L}e^{i\phi} \quad \text{round-trip factor} \\ \phi &=& \arg(r_1 r_2) + \omega_0\tau - 2L\text{Re}\beta(\omega_0) \\ \alpha &=& -2\text{Im}\beta(\omega_0) \quad \text{internal opt. losses} \\ t_n &=& t - n\tau, \ \text{with round-trip time } \tau = 2L/v_g. \end{array} \tag{1}$$

The parameters which enter are: r_1, r_2 amplitude reflectivities of the mirrors from inside the FP, r_{1_-} amplitude reflectivity of the illuminated mirror from outside and $R_{\ldots} = |r_{\ldots}|^2$, ω_0 reference frequency, $\beta(\omega_0)$ complex propagation constant at ω_0 within the resonantor, L resonator length, v_g group velocity in the resonator. Notice that incident and reflected field are taken just at the outer side of mirror 1. Inclusion of propagation to and from this mirror (latency time) is a trivial extension.

Remarks:

- $\mathcal{R}$ does not depend on the choice of ω_0 within the linear dispersion range of $\beta(\omega)$.
  Details see end of section 2.
- The round trip time τ is determined by the group velocity but not by the phase velocity.
- The feedback of T-periodic intensities disappears in case of the general resonance condition

  $$\mathcal{E}_{\text{in}}(t-T) = e^{i\psi}\mathcal{E}_{\text{in}}(t) \quad \text{and} \quad \tau = mT, \quad e^{im\psi}\frac{\mathcal{R}}{R_{1_-}} = 1. \tag{2}$$

  ψ is an arbitrary phase shift and m integer.
- It could be interesting to consider small deviations from resonance, $\tau = mT + \delta\tau$. In this case, the feedback becomes proportional to $\delta\tau\cdot$ actual intensity change, which can be used as control force.

2 Derivation

23.5.05 (1)

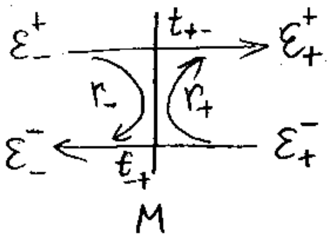
Feedback from FP

Full field: $\mathcal{E} = \mathrm{Re}\{\mathcal{E}^+ + \mathcal{E}^-\}$; $\mathcal{E}^\pm = E^\pm e^{i(\omega_0 t \mp k_0 z)}$

slowly var. Amplitude

In connection with TWE with a DFB section it must be $k_0 = \pi/\Lambda$.
But feedback must not depend on choice of ω_0 and k_0.

1. Boundary conditions for $\mathcal{E}^\pm$ at a „mirror" M



$$\mathcal{E}_+^+ = t_{+-}\mathcal{E}_-^+ + r_+\mathcal{E}_+^- \qquad (1)$$

$$\mathcal{E}_-^- = t_{-+}\mathcal{E}_+^- + r_-\mathcal{E}_-^+ \qquad (2)$$

This holds at any mirror position z_m.

Assumption: $\mathcal{E}^\pm$ are „power amplitudes", i.e. $|\mathcal{E}^\pm|^2$ = powers.

Nonabsorbing mirror: $P_{in} = P_{out}$, i.e.

$$|\mathcal{E}_-^+|^2 + |\mathcal{E}_+^-|^2 = |\mathcal{E}_+^+|^2 + |\mathcal{E}_-^-|^2$$

$$= \left[|t_{+-}|^2 + |r_-|^2\right]|\mathcal{E}_-^+|^2 + \left[|t_{-+}|^2 + |r_+|^2\right]|\mathcal{E}_+^-|^2$$

$$+ 2\,\mathrm{Re}\left\{\left[t_{+-}^* r_+ + r_-^* t_{-+}\right]\mathcal{E}_-^{+*}\mathcal{E}_+^-\right\}$$

Arbitrariness of $\mathcal{E}_-^+$ and $\mathcal{E}_+^-$ requires

$$|t_{+-}|^2 + |r_-|^2 = |t_{-+}|^2 + |r_+|^2 = 1, \quad t_{+-}^* r_+ + r_-^* t_{-+} = 0. \qquad (3)$$

Example: Index jump from n_- to n_+. Then

$$r_- = -r_+ = \frac{n_+ - n_-}{n_+ + n_-} \quad \& \quad t_{12} = t_{21} = \frac{2 n_+ n_-}{n_+ + n_-}. \qquad (4)$$

2. Boundary conditions for $E^\pm$ at z_m

Lik (1) and (2) with $r_\pm$ replaced by $\tilde{r}_\pm = r_\pm e^{\pm 2ik_0 z_m}$. (5)

Phase of reflectivities depends explicitely on mirror position.
Conservation laws (3) hold anyway.

3. Propagation in a passive resonator

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Is governed by the TW equations

$$(\partial_z \pm v_g^{-1}\partial_t) E^{\pm} = \mp i\delta E^{\pm}; \quad \delta = \beta(\omega_0) - k_0 \qquad (6)$$

Inserting ansatz $E^{\pm}(z,t) = f^{\pm}(z, t \mp z/v_g)$ yields

$$\partial_z f^{\pm}(z,t') = \mp i\delta f^{\pm}(z,t')$$

with solution

$$f^{\pm}(z,t') = f^{\pm}(z_{\pm},t') e^{\mp\delta\cdot(z-z_{\pm})}.$$

Choosing $z_+ = z_1$ (left mirror) and $z_- = z_2$ (right mirror), one arrives at

$$E^{\pm}(z,t) = G_{\pm}\cdot E^{\pm}\left(z_{\pm}, t \mp \frac{z-z_{\pm}}{v_g}\right); \quad G_{\pm} = e^{\mp i\delta\cdot(z-z_{\pm})}. \qquad (7)$$

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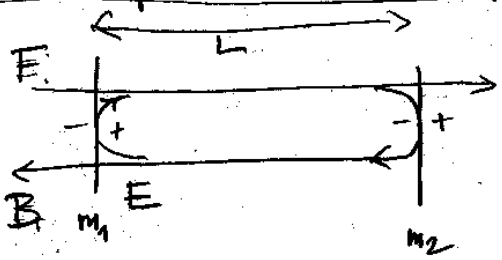
4. Resonator feedback with multiple reflections



$$F(t) = E^{+}(z_1^{-}, t)$$
$$B(t) = E^{-}(z_1^{-}, t)$$
$$E(t) = E^{-}(z_1^{+}, t)$$

$$B = t_{1-+}E + \tilde{r}_{1-}F \;; \qquad E = \left(t_{1+-}F_1 + \tilde{r}_{1+}E_1\right)\cdot G\,\tilde{r}_{2-}$$

where $G = e^{-2i\delta L}$ new notation: $f_n = f(t_n) = f(t - n\tau)\;;\; \tau = \frac{2L}{v_g}$. (8)

Iteration of E-equation:

$$E = aF_1 + bE_1 = aF_1 + b\left(aF_2 + b\left(aF_3 + b(\ldots\ldots)\right)\right)$$
$$= a\cdot\left(F_1 + bF_2 + b^2F_3 + b^3F_4 + \cdots\right)\;; \qquad a = t_{1+-}G\tilde{r}_{2-}$$
$$= a\sum_{n=1}^{\infty} b^{n-1}F_n \qquad b = \tilde{r}_{1+}\tilde{r}_{2-}G$$

$$\Rightarrow B = \tilde{r}_{1-}F + t_{1-+}t_{1+-}G\tilde{r}_{2-}\sum_{n=1}^{\infty}\left(\tilde{r}_{1+}\tilde{r}_{2-}G\right)^{n-1}F_n$$

From (3): $0 = |t_{+-}|^2 r_+ + r_-^* t_{+-}t_{-+} = (1-|r_-|^2)r_+ + r_-^* t_{+-}t_{-+}$

$$\rightarrow t_{+-}t_{-+} = -(1-|r_-|^2)\,r_+/r_-^*$$

$$\Rightarrow B = \tilde{r}_{1-}F + (1-|r_{1-}|^2)\frac{\tilde{r}_{1+}G\,r_{2-}}{r_-^*}\sum_{n=1}^{\infty}\left(\tilde{r}_{1+}\tilde{r}_{2-}G\right)^{n-1}F_n$$
$$= \tilde{r}_{1-}\left[F + \left(1 - \frac{1}{|r_{1-}|^2}\right)\sum_{n=1}^{\infty}\tilde{R}^n F_n\right]\;; \quad \tilde{R} = \tilde{r}_{1+}\tilde{r}_{2-}G$$
$$= \tilde{r}_{1-}\left[\sum_{n=0}^{\infty}\tilde{R}^n F_n - \sum_{n=0}^{\infty}\tilde{R}^{n+1}F_{n+1}/|r_{1-}|^2\right]$$

Thus,

$$\mathcal{B} = \tilde{r}_{1-} \sum_{n=0}^{\infty} \tilde{\mathcal{R}}^n \left(F_n - F_{n+1} \frac{\tilde{\mathcal{R}}}{R_{1-}} \right) ; \quad R_{1-} = |r_{1-}|^2 . \qquad (9)$$

Transforming back to full fields, $\mathcal{B} = \mathcal{E}_b e^{-i(\omega_0 t + k_0 z_1)}$ and $F = \mathcal{E}_{in} e^{-(i\omega_0 t - k_0 z_1)}$, yields

$$\boxed{\mathcal{E}_b(t) = r_{1-} \sum_{n=0}^{\infty} \mathcal{R}^n \left(\mathcal{E}_{in}(t_n) - \mathcal{E}_{in}(t_{n+1}) \frac{\mathcal{R}}{R_{1-}} \right)} \qquad (10)$$

$t_n = t - n\tau$, ~~and~~

$$\mathcal{R} = \sqrt{R_{1+} R_{2-}}\, e^{-\alpha L} \cdot e^{i\phi} ; \quad \alpha = -2\,\mathrm{Im}\,\beta(\omega_0) \qquad (11)$$

$$\phi = \arg(r_{1+} r_{2-}) - 2L\,\mathrm{Re}\,\beta(\omega_0) + \omega_0 \tau \qquad (12)$$

How does this depend on ω_0? Assume $\omega_0 = \omega_0' + \Delta$, Δ small.

$$\rightarrow \mathrm{Re}\,\beta(\omega_0' + \Delta) = \mathrm{Re}\,\beta(\omega_0') + \frac{\Delta}{v_g} = \mathrm{Re}\,\beta(\omega_0') + \frac{\Delta \tau}{2L} \qquad (13)$$

$$\rightarrow \phi = \arg(r_{1+} r_{2-}) - 2L\,\mathrm{Re}\,\beta(\omega_0') + \omega_0' \tau .$$

Thus, ϕ does not depend on the choice of ω_0 as long as linearisation (13) holds, i.e., group velocity dispersion is negligible.

With phase velocity $v_p = \omega_0 / \mathrm{Re}\,\beta(\omega_0)$ Eq. (12) reads

$$\phi = \arg(r_{1+} r_{2-}) + 2L\omega_0 \left(\frac{1}{v_g} - \frac{1}{v_p} \right).$$

The second term disappears if $v_g = v_p$ and (10) becomes the intuitive result. Only use of the long derivation is the proper appearence of group-velocity round-trip time.