Delay coupled phase oscillators

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I. INTRODUCTION

To find out how general our findings for the delay-coupled lasers are, I have tried to simplivy the model as much as possible. This is the result.

II. MODEL

I consider two coupled phase oscillators

$$\partial_t \varphi_k = \omega_k + K \sin\left(\varphi_l^\tau - \varphi_k\right) \tag{1}$$

with $k = 1, 2, l = 3 - k$. $\varphi(t)$ ist the actual phase of oscillator k and $\varphi_k^\tau(t) = \varphi_l(t - \tau)$ the phase of oscillator l before a delay time τ. This is just the model already assumed by Schuster and Wagner [1] who considered the synchronised states.

III. NUMERICAL SOLUTION

I use 2τ as time unit, i.e. $\tau = 1/2$. For clarity, I consider symmetrically detuned oscillators $\omega_2 = -\omega_1 = \Delta$. The detuning Δ/π is varied within ± 6. With $K = 2\pi$, the locking range without delay is exactly ± 2 (see below). This is well reproduced by the numerical solutions. With delay, the frequencies show the clearly the staircases already known from the lasers.

Conclusion: the staircases are the general and typical scenario for nearly locked delay-coupled oscillators.

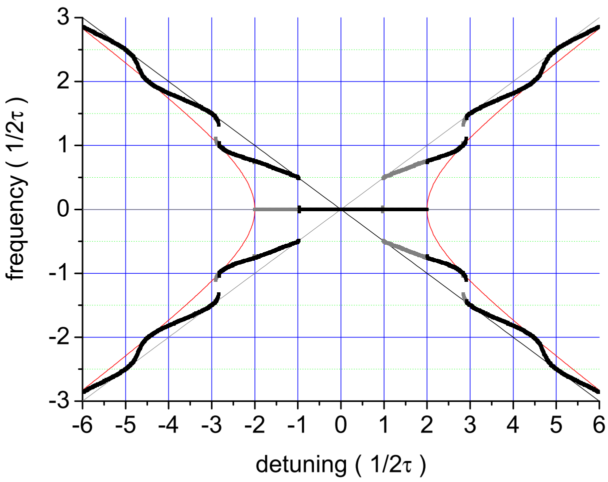

FIG. 1: Detuning scenario of the symmetrically detuned phase oscillators. Thick black: frequencies of both oscillators for increasing detuning. Thick grey: same for decreasing detuning. Thin lines: no coupling (grey) and no delay (red).

IV. ANALYSIS

We introduce phase difference $\Phi = (\varphi_2 - \varphi_1)/2$ and mean $\Psi = (\varphi_2 + \varphi_1)/2$ as new variables. With some algebra, Eqs. (1) transform to

$$\partial_t \Phi \; = \; \Delta - K \cos(\Psi - \Psi^\tau) \sin(\Phi + \Phi^\tau) \tag{2}$$
$$\partial_t \Psi \; = \; \Omega - K \sin(\Psi - \Psi^\tau) \cos(\Phi + \Phi^\tau). \tag{3}$$

These are the central equations of delayed phase synchronisation. In our example, the mean frequency Ω is zero.

A. delay free limit

With $\tau = 0$, these equations reduce to

$$\partial_t \Phi = \Delta - K \sin(2\Phi) \quad \text{and} \quad \partial_t \Psi = \Omega. \tag{4}$$

This is the well known Adler equation for the phase difference 2ϕ between the two oscillators.

1. locking

If $|\Delta| \leq K$, the right hand side approaches to zero at $2\Phi = 2\Phi_{\text{locked}}$ =acrsin(Δ/K), which is the constant phase difference of the locked oscillators. Since $\Psi = \Omega t$, it holds $\varphi_{1,2} = \mp\Phi_{\text{locked}} + \Delta \cdot t$.

2. unlocking

Still keeping $\tau = 0$, the unlocking of the two oscillators for $|\Delta| > K$ is also well understood. I suppress here the known analytic solutions of the equations in this regime.

Generic scenario: The two "frequencies" split beyond the locking range. The splitting $\delta\omega_0$ increases in a hyperbolic manner and approaches 2Δ (see Fig. 1). But: these frequencies are average frequencies, i.e. phase increments over a long time divided by this time. Just beyond the locking range, the phase difference Φ raises extremely nonuniformly. Long plateaus of nearly constant Φ, i.e. of a nearly locked state, are interrupted by short phase slips by 2π. Approximately, the plateau length T_{plateau} goes inversely with the frequency splitting, $T_{\text{plateau}} \approx 2\pi/\varphi\omega_0$.

Abbildung machen!!

B. nonzero delay

Hypothesis: the staircases are something like synchronisation between the phase-slips and the round trip frequency.

1. locking

This has been treated by Schuster and Wagner, I should check whether my formulae (2) yield the same. My fieling: they are simpler.

2. unlocking

Fig. 1 shows:

- the "plateaus" meander between the zero delay and zero coupling lines.
- zero coupling is touched exactly at odd multiples of the round-trip frequency ν_{rt}.
- zero delay is touched when the splitting equals even multiples of ν_{rt}.

- hysteresis is seemingly due to double-folds at smaller detunings.

I can explain the touching points analytically:
I restrict to symmetric detuning ($\Omega = 0$). In this case, Ψ =const. is always a solution of Eq. (3). Hence, the other equation simplifies to $\partial_t \Phi = \Delta - K \sin(\Phi + \Phi^\tau)$. It follows

$$\partial_t(\Phi + \Phi^\tau) = 2\Delta - K \left[\sin(\Phi + \Phi^\tau) + \sin(\Phi^\tau + \Phi^{2\tau})\right] .$$

If now $\Phi^{2\tau} = \Phi + m\pi$ with integer m, then

- the $[\ldots]$ vanishes for odd m, i.e. the oscillator frequencies equal the no coupling case
- $(\Phi + \Phi^\tau)$ obeys the Adler equation (4) for even m, i.e., the delay-free frequency.

V. CONCLUSION

[1] H. Schuster and P. Wagner, Progr. Theor. Phys. 81, 939 (1989).