Lösung stat. Gleichung

$$\partial_z \ln \frac{P_0}{P_1} = g\varepsilon(P_0 - P_1)$$

Sättigung: $P_s = P_0 + P_1$

$\rightarrow P_0 = \frac{P_s}{2}(1+X)$; $P_1 = \frac{P_s}{2}(1-X)$

neue Unbekannte X

$$\ln\left(\frac{1+X}{1-X}\right)' = g\varepsilon P_s X$$

z bzw. t-Skalierung

$z \rightarrow z g \varepsilon P_s$

$$\boxed{\ln\left(\frac{1+X}{1-X}\right)' = X}$$

universelle Gleichung

- Weg a

$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ neue Variable

$$\rightarrow e^{2y} = \frac{1+x}{1-x} \rightarrow (1-x)e^{2y} = 1+x \rightarrow e^{2y} - 1 = (e^{2y}+1)x$$

$$\rightarrow x = \frac{e^{2y} - 1}{e^{2y}+1} = \tanh(y)$$

also $\boxed{y' = \frac{1}{2}\tanh(y)}$

- Separation: $\frac{1}{2}dt = \frac{dy}{\tanh y} = \coth(y)dy$

$$\rightarrow \frac{t}{2} = \ln(\sinh(y))\Big|_{y_0}^{y}$$

$y_0 > 0$

$$\rightarrow \ln \sinh y = \theta \; ; \qquad \boxed{\theta = \frac{t}{2} + \ln \sinh(y_0)}$$

$$\boxed{\sinh y = e^{\theta}}$$

$$e^{\theta} = \frac{1}{2}\left\{\sqrt{\frac{1+x}{1-x}} - \sqrt{\frac{1-x}{1+x}}\right\} = \frac{1}{2}\,\frac{1+x-(1-x)}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$$

$\cancel{e^{2\theta}(1-x^2)=}$

Vorher: $e^{y}-e^{-y}=2e^{\theta}$ Probe:

$\frac{d}{dt}$: $(e^{y}+e^{-y})y' = 2e^{\theta}\cdot\theta' = e^{\theta}$

$2\cosh(y)\,y' = \sinh(y) \Rightarrow y' = \frac{1}{2}\tanh(y)$ ✓

$e^{\theta} = \frac{x}{\sqrt{1-x^2}} \rightarrow e^{2\theta}(1-x^2) = x^2 \rightarrow \frac{e^{2\theta}}{1+e^{2\theta}} = x^2$

$$x = \frac{e^{\theta}}{\sqrt{1+e^{2\theta}}} = \frac{1}{\sqrt{1+e^{-2\theta}}} \xrightarrow[\theta\gg1]{} 1-\frac{1}{2}e^{-2\theta}$$

$\Rightarrow$ $\cancel{1-x}$ $\frac{P_1}{P_0} \simeq \frac{1-x}{2} \sim \frac{1}{4}e^{-2\theta}$ ✓

$$x'\left(\frac{1}{1+x}+\frac{1}{1-x}\right) = x$$

$$\frac{2x'}{1-x^2} = x$$

$x = 1-\varepsilon$; $\varepsilon \ll 1$

$\frac{-2\varepsilon'}{2\varepsilon} = 1 \rightarrow \varepsilon' = -\varepsilon$

$\rightarrow \varepsilon =$

$$\frac{2xx'}{1-x^2} = x^2 \rightarrow \frac{(x^2)'}{x^2(1-x^2)} = 1 \qquad x^2 = \varsigma$$

$\rightarrow$ $\frac{d\varsigma}{\varsigma(1-\varsigma)} = dt$ | $\frac{1}{\varsigma(1-\varsigma)} = \frac{1}{\varsigma}+\frac{1}{1-\varsigma}$

$= d(\ln\varsigma - \ln(1-\varsigma)) = d\ln\left(\frac{\varsigma}{1-\varsigma}\right)$

$\rightarrow \ln\left(\frac{1}{\varsigma}-1\right) = C' - t$

$\rightarrow \frac{1}{\varsigma}-1 = Be^{-t}$

$$\frac{1}{\zeta} = 1 + Be^{t}$$

$$\zeta = \frac{1}{1+Be^{-t}} = x^2$$

$$\boxed{x = \left(1+Be^{-t}\right)^{-\frac{1}{2}}}$$

$$x_0 = \frac{1}{\sqrt{1+B}}$$

$$x_0^2 = \frac{1}{1+B}$$

$$\frac{1}{x_0^2} = 1 + B$$

$$\to B = \frac{1}{x_0^2} - 1$$

- $$x = \frac{x_0}{\sqrt{x_0^2(1-e^{-t}) + e^{-t}}} = \frac{x_0}{\sqrt{x_0^2 + (1-x_0^2)e^{-t}}}$$

$$\boxed{x = \left(1+e^{-(t-t_0)}\right)^{-\frac{1}{2}}}$$

$$\boxed{t_0 = \ln\left(\frac{1-x_0^2}{x_0^2}\right)}$$

$$e^{t_0} = B$$

$$t_0 = \ln B = \ln\left(\frac{1-x_0^2}{x_0}\right)$$

$$x = \frac{P_0 - P_1}{P_0 + P_1}$$

- ~~bzw~~

$$R := \ln\left(\frac{P_1}{P_0}\right) = \ln\left(\frac{1-x}{1+x}\right) = -2y = -\operatorname{arcsinh}\left(e^{\theta}\right)$$

$$y = \operatorname{arcsinh} e^{\theta} \Leftrightarrow e^{\theta} = \sinh y = \frac{e^{y} - e^{-y}}{2} \to 2e^{\theta} = e^{y} - e^{-y}$$

$$e^{2y} - 2e^{y}e^{\theta} - 1 = 0 \Rightarrow e^{y} = e^{\theta} \pm \sqrt{e^{2\theta}+1}$$

$$y = \ln\left(e^{\theta} + \sqrt{1+e^{2\theta}}\right) = \theta + \ln\left(1+\sqrt{1+e^{-2\theta}}\right)$$