# Locking Ideas

ede, 17.11.02

## 1 Notations

$\omega_{ext}$ = frequency of external signal, $\omega_{free}$ = frequency of free running self-pulsation, $\delta = \omega_{free} - \omega_{ext}$ detuning between both.

$\varphi$ = phase shift between the self-pulsation (pulses at the internal wavelength) and the externally injected pulses at an arbitrary but fixed position of the device, e.g., at one facet.

Hence, $\dot{\varphi}$ = instantaneous frequency of the internal self-pulsation (relative to $\omega_{ext}$).

## 2 Assumptions and Conclusions

**Assumption 1:** The instantaneous frequency $\dot{\varphi}$ is a unique function of $\varphi$: $\dot{\varphi} = \omega(\varphi)$.

This makes sense at least for sufficiently small detuning and signal intensity. It is obviously true in the limit of vanishing signal intensity, when holds $\omega(\varphi) = \delta$ =const. With increasing signal intensity, $\omega(\varphi)$ will show more and more variation.

**Conclusion 1:** Obviously, $\omega(\varphi)$ is a periodic function.

**Conclusion 2:** Locking appears if and only if $\omega(\varphi) = 0$ has a solution.

**Conclusion 3:** The locking dynamics is governed by the equation

$$\dot{\varphi} = \omega(\varphi). \quad \text{General solution:} \quad t = \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\omega(\varphi)}, \tag{1}$$

where $t$ is the time needed for the phase change from the initial value $\varphi_1$ to the final value $\varphi_2$.

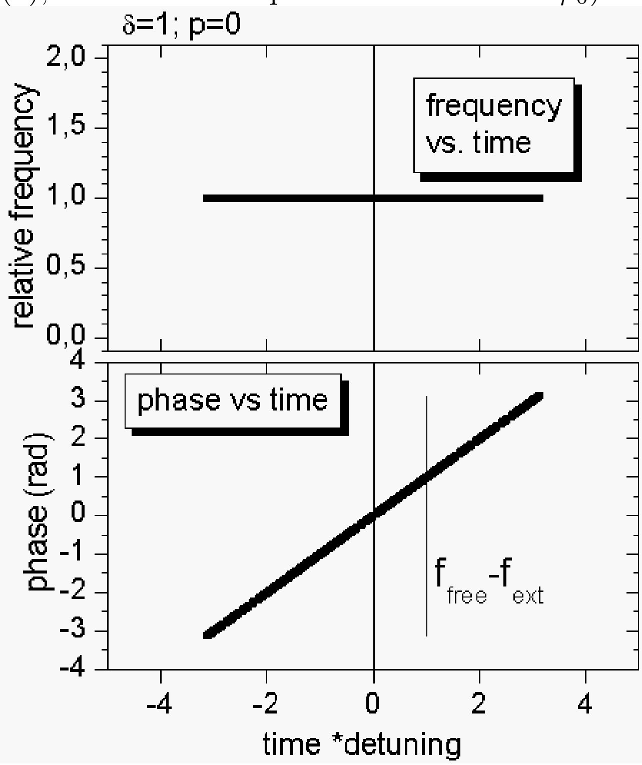
**Qualitative discussion:** If $\min(\omega(\varphi)) > 0$, there is no locking. But the phase $\varphi$ will change slowliest around the minimum of $\omega(\varphi)$. If this minimum is closely above 0, $\varphi$ will become nearly stationary for a comparatively long time. Such $\varphi$-plateaus did we observe in the numerics, they are a precurser of locking. The slope of the plateaus is given by the minimum of $\omega(\varphi)$. The border of the locking range is achieved when $\min(\omega(\varphi)) = 0$. Beyound, we have at least one zero of $\omega(\varphi)$, say at $\varphi_0$. It is approached exponentially fast with a time constant $1/\omega'(\varphi_0)$, which is very long at the border of the locking range but smaller in its interiour. To be concrete, I consider now a model.

$$\textbf{Model:} \quad \omega(\varphi) = \delta + p\sin(\varphi). \qquad \Longrightarrow \qquad \boxed{\dot{\varphi} = \delta + p\sin(\varphi)} \tag{2}$$

This seems to be Adler's equation, if I remember right. It can be solved analytically:

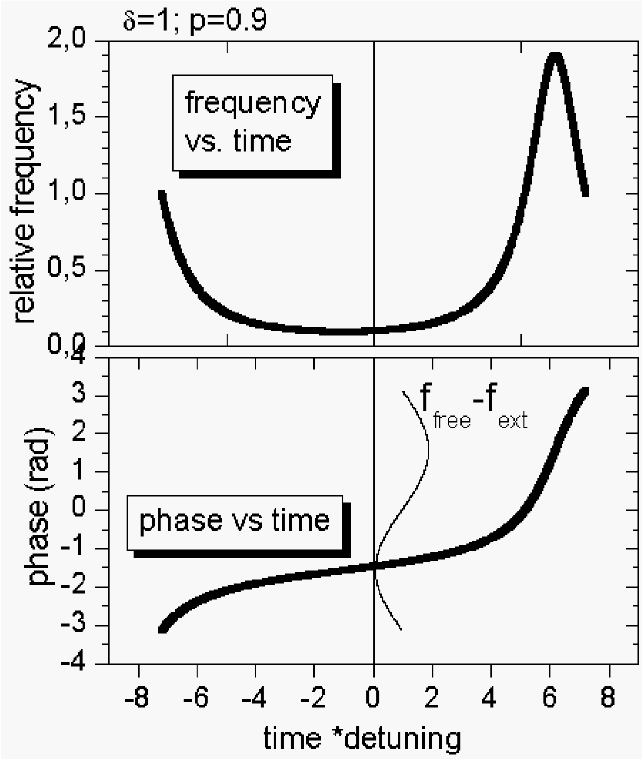
$$
\begin{aligned}
t &= \frac{2}{\sqrt{\delta^2 - p^2}} \arctan \frac{\delta \tan(\varphi/2) + p}{\sqrt{\delta^2 - p^2}} & (p < \delta) \quad (3)\\
t &= \frac{1}{\delta} \tan\left(\frac{\varphi}{2} - \frac{\pi}{4}\right) & (p = \delta) \quad (4)\\
t &= \frac{1}{\delta} \ln \left| \frac{\delta \tan(\varphi/2) + p - \sqrt{p^2 - \delta^2}}{\delta \tan(\varphi/2) + p + \sqrt{p^2 - \delta^2}} \right| & (5)
\end{aligned}
$$

I've plotted these solutions for some characteristic cases, assuming $\delta = 1$. Abscissa is always the dimensionless time $\delta t$. Lower panels: thick: $\varphi$ versus $\delta t$, thin: $\varphi$ vs. $\omega(\varphi)$, i.e., the model (2), to indicate the position of the zeros $\varphi_0$).



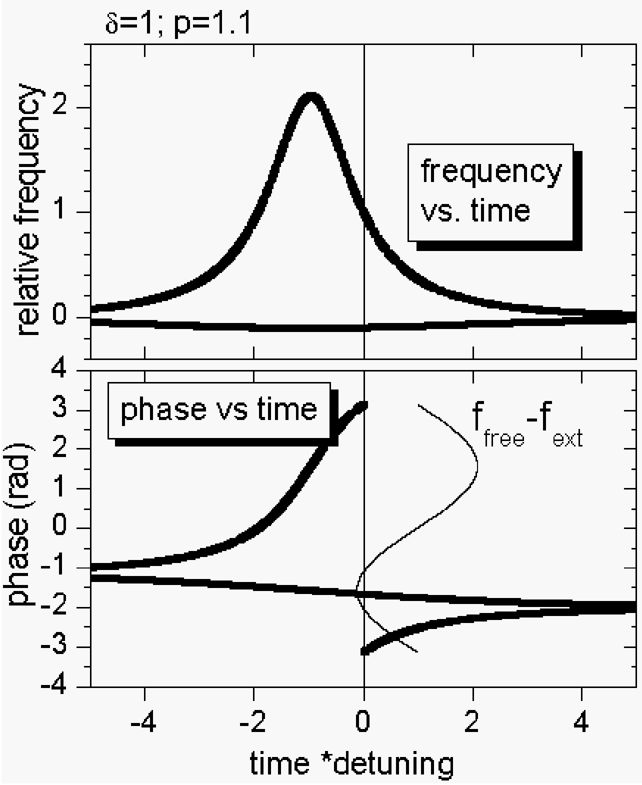

**Case p=0.**

Without signal, the frequency remains constant, the phase raises linearly. The time for one phase period (change by $2\pi$) is just $2\pi/\delta$.



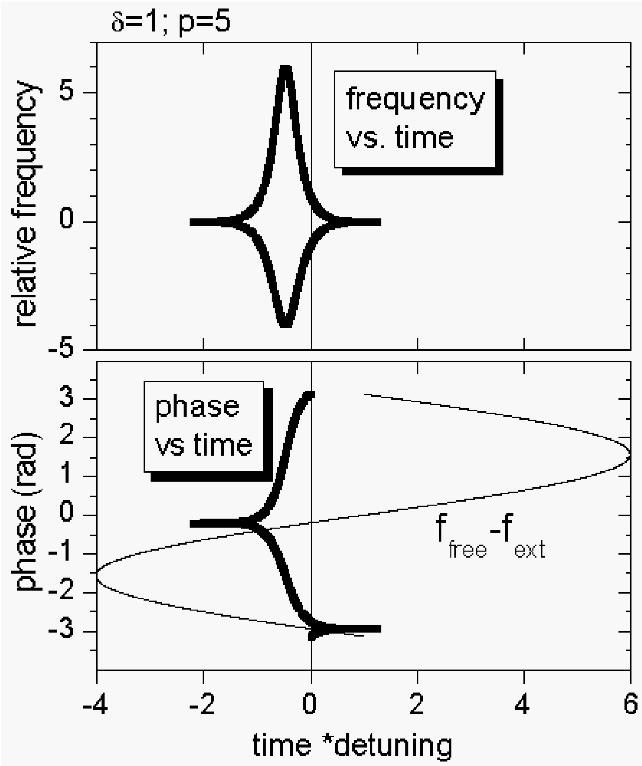

**Case p=0.9.**

There is no zero yet of (2), we are still outside the locking range. However, the minimum $\min(\omega(\varphi)) = \delta - p = 0.1$ is already close to zero (cf. thin line in the lower panel). Hence, we clearly observe a range with stagnating phase, the precursor of locking known from the numerics with LDSL. Note that the time period is enhanced now to approximately $14/\delta$ due to this stagnation.




**Case p=1.1**

Now we have two zeros of $\omega(\varphi)$. This one with a negative slope is the stable one, which the phase approaches to. Depending on the starting phase, the time to lock is comparable with the extension of the time axis, roughly $10/\delta$. We are still close to the border of the locking range.




**Case p=5**

Now we are more in the middle of the locking range, the time needed is only roughly $1/\delta$ yet.