Adler Equation with Delay

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I study the generalized Adler equation derived in the coupled laser paper for delay coupled phase synchronization with symmetric detuning.

I consider the equation

$$\dot{\phi} = \Delta - K\left[\sin(\phi) + \sin(\phi^{\tau})\right], \tag{1}$$

which generalizes the delay-free Adler equation. First, I shall consider in detail the special solutions corresponding to effectively no coupling or no delay. Then I use this knowledge in order to construct a numerical solution schema for the general case.

SPECIAL FREQUENCY SOLUTIONS

Solutions with frequencies being integer multiples of $1/2\tau$ define representative points in the system dynamics. Here, because of $\phi = \phi^{\tau} + m\pi$, the coupling for ϕ (not φ_k) is either zero (m odd) or delay-free (m even). With increasing Δ, the system meanders between those limits (Fig. 5d of PRL). Branches with negative slope are unstable, providing the staircase feature. It is exactly this kind of self-organization in the phase relation between the oscillators that governs the synchronization.

zero-coupling frequencies

The solutions in this case are simply

$$\phi_m(t) = \Delta_m \cdot t, \qquad \Delta_m = m \cdot \frac{\pi}{\tau}, \qquad (m \text{ odd}). \tag{2}$$

zero-delay frequencies

Here we arrive at the orginial Adler equation in the form

$$\dot{\phi} = \Delta - 2K\sin(\phi). \tag{3}$$

The solution can be obtained by variable separation. It is in general

$$\phi(t) = 2\ \arctan\left(\frac{\omega\tan(\omega t/2) + 2K}{\Delta}\right) + 2\pi\ \mathrm{int}\left(\frac{\omega t - \pi}{2\pi}\right), \qquad \omega = \sqrt{\Delta^2 - 4K^2}. \tag{4}$$

Here the principal value of arctan is to be taken and the int-term ensures continuous increase of the phase.

The zero-delay case appears at

$$\omega = \omega_m := m\frac{\pi}{\tau} \qquad \text{i.e. } \Delta = \Delta_m := \sqrt{\omega_m^2 + 4K^2}. \tag{5}$$

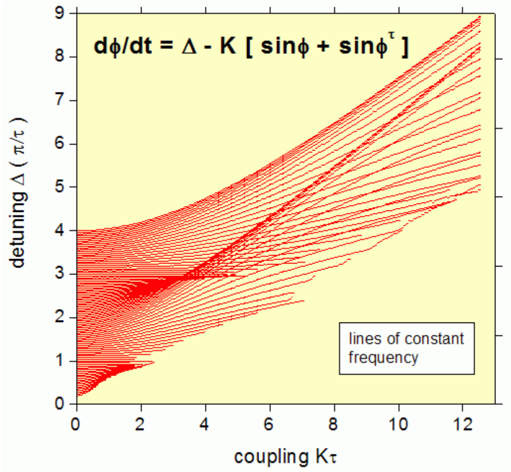
$$\phi_m(t) = 2\ \arctan\left(\frac{m\tan(m\pi t/2\tau) + 2K\tau/\pi}{\sqrt{m^2 + (2K\tau/\pi)^2}}\right) + 2\ \mathrm{int}\left(\frac{mt - \tau}{2\tau}\right). \tag{6}$$

NUMERICS: STRETCHING ALGORITHM

The idea is to calculate solutions having a given frequency f_0 by adjusting Δ. An embedding is used increasing K from 0 to a maximal value in small steps. In each step, Δ is adjusted until the given frequency f_0 is reached.

The figshows resulting lines of constant f_0 in the Detuning-coupling plane. The lines start at $K = 0$ from $\Delta = 2\pi f_0$. They keep horicontal for odd $\Delta\tau/\pi = 2\tau f_0$ exhibit a maximum bend upwards for even ones, as expected.

Ending of lines before 4π is due to unstability of either the solution itself or of the algorithm used. From this point of view, this approach is not suitable.



Calculated with t.050302.r in directory `D:\verb\ede\dfb\coupledlaser\oscis\adlerdelay`.

NUMERICS: PATHFOLLOWING FROM SPECIAL POINTS