Model Equations Scatterer Laser

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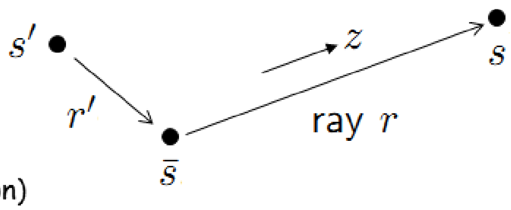
basic model equations



$E_r(z,t) = \mathcal{E}_r(z,t) G(k,z) e^{-i\omega_0 t} + \text{c.c.}$

$G(k,r) = \exp(ikr + i\pi/4)/\sqrt{8\pi k r}$ (2D Green function)

$$\left(\partial_z + \frac{1}{c}\partial_t\right)\mathcal{E}_r(z,t) = \left[\frac{1-i\alpha_H}{2}(g(z,t)-\bar{g}) - \frac{\alpha_0}{2}\right]\mathcal{E}_r(z,t) \quad (1)$$

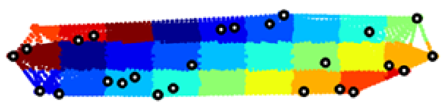
$$\tau_n \frac{d}{dt} g_d(t) = g_0 - g_d(t)\left[1 + S_d(t)\right], \quad (2)$$

boundary condition: $$\mathcal{E}_r(0,t) = \sum_{r' \in r_{\text{in}}(r)} A_{rr'} \left[\mathcal{E}_{r'}(l',t) G(k,l') + \beta_{\text{spont}}\right] \quad (1')$$

scattering amplitude

gain: piecewise constant in certain spatial domains d

$g(z,t) = g_d(t)$ if$z \in d$



explanations:

$E_r(z,t)$ is the full optical amplitude along ray r. Each ray has its own local space coordinate $z \in [0,l]$.

ω_0: reference frequency (freely choosable in the emission band of the laser, typical value $2\pi \cdot 10^{15}$ s^{-1}), $k = k_0 - i\bar{g}/2$ with $k_0 = \omega_0/c$, c: velocity of light in the semiconductor, $\bar{g}$: reference gain (amplification per length).

It holds $\bar{g} \ll \omega_0/c$ (typically $\bar{g} = 10^4$ m^{-1} and $k_0 = 10^7$ m^{-1}).

$g(z,t) = g_d(t)$ if $z \in d$. α_0: back ground absorption. (Both of same order of magnitude as $\bar{g}$). α_H: so called "alpha-factor" modeling amplitude-phase coupling (typical $\alpha_H = 0$ to 5).

$\mathcal{E}(z,t)$ varies slow (time scale: ps; space scale 0.1 mm) compared to $G(k,z)e^{-i\omega_0 t}$ (fs; 0.1 μm). $g_d(t)$ varies even slower (time scale: τ_n = some 10^2 ps).

$A_{rr'}$: scattering strength from ray r' into ray r. $r_{\text{in}}(r)$: set of rays arriving at r. l: ray length.

β_{spont}: small stochastic source modeling spontaneous emission.

$S_d(t) = \int_{\vec{r}\in d} |E(\vec{r},t)|^2 d^2 r$: mean intensity in domain d. It is approximated in the program by the mean intensity impinging to the scatters in the domain.

g_0: unsaturated gain (solution of (2) for $S_d = 0$).

instantaneous modes:

In general, optical modes are solutions of the homogeneous optical equations with time-independent coefficients, i.e. (1) with g independent of t and with $\beta_{\mathsf{spont}} = 0$ in (1').

Since $g_d(t)$ is slow, it is "nearly" constant, and Equ. (1) can be approximately solved by the exponential ansatz $\mathcal{E}_r(z,t) = e^{-i\int \Omega(t)dt}\psi_d(z,t)$. Neglecting the temporal derivative of ψ_d yields

$$\partial_z \psi_r(z,t) - \left[\frac{1 - i\alpha_H}{2}(g(z,t) - \bar{g}) - \frac{\alpha_0}{2}\right]\psi_r(z,t) = i\frac{\Omega}{c}\psi_r(z,t) \tag{3}$$

with boundary conditions (1') with $\beta_{\mathsf{spont}} = 0$. This is the eigenvalue problem that defines what we call instantaneous light modes. In general, the eigenvalues Ω are complex. Re$(\omega_0 + \Omega)$ defines the optical frequencies (wavelengths) where in expermiment we see peaks in the optical spectra, and -Im$(\Omega$ is the damping (decay constant) of the corresponding mode.

Now I turn over to the exponential random matrix that we discussed. For simplicity, I assume a constant g (as it is at threshold) and $\alpha_0 = \alpha_H = 0$. Then, Eq.(3) has a true exponential solution. Inserting into (1'), using $G(k,r) = e^{\bar{g}/2}G(k_0,r)$, yields

$$\psi_r(0) = \sum_{r'} A_{rr'}G(\tilde{k}, l_{r'})\psi_{r'}(0) \qquad \text{with } \tilde{k} = k_0 + \frac{\Omega}{c} - i\frac{g}{2}. \tag{4}$$

This homogeneous system for the initial amplitudes on the rays has nontrivial solutions only if

$$\det\left(\delta_{rr'} - A_{rr'}G(\tilde{k}, l_{r'})\right) = 0. \tag{5}$$

This is equivalent to the nonlinear eigenvalue equation of a random laser considered in our PRL paper. Note: $A_{rr'} \neq 0$ only if the end point of r' agrees with the start point of r.

In my numerical calculations I considered point scatterers. In this case all nonvanishing elements of $A_{rr'} = A$ are identical to each other, which allows a considerable simplification. We write (4) in the form

$$\psi_{s\bar{s}}(0) = A\sum_{s'} G(\tilde{k}, l_{\bar{s}s'})\psi_{\bar{s}s'}(0), \tag{6}$$

where s and $\bar{s}$ are end and start scatterer of ray r, and $\bar{s}$ and s' accordingly of r'. The sum is the total amplitude $\phi_{\bar{s}}$ arriving at scatterer $\bar{s}$. Multiplying the whol equation with $G(\tilde{k}, l_{s\bar{s}})$ and summing over $\bar{s}$ yields

$$\phi_s = A\sum_{\bar{s}} G(\tilde{k}, l_{s\bar{s}})\phi_{\bar{s}}, \tag{7}$$

yielding the nonlinear eigenvalue equation

$$\det\left(\delta_{s\bar{s}} - AG(\tilde{k}, l_{s\bar{s}})\right) = 0. \tag{8}$$

I think, the last case is simpler and interesting enough.