Derivation of Lang-Kobayashi Equations

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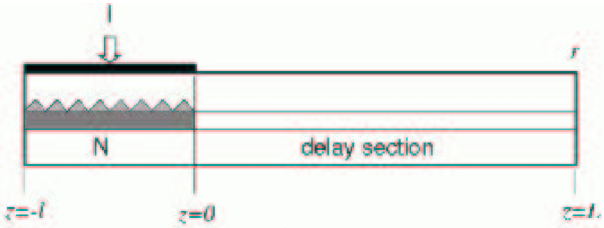
Abstract

This is a slightly updated version of the pamphlet from July 7, 2000.

1 Introduction

Resuming our yesterday's (seminar in July 2000) discussions I got some ideas how to derive the LKEs from the TWEs. I do it not generally but for a device with a short integrated reflector as sketched in the Figure. The LKEs to be derived are



device

Figure 1: *Scheme of the device. The single mode laser of length l is accomplished by a passive section of length L with reflectivity r. A throughgoing wave guide has to be assumed to ensure the validity of the TWE.*

$$\frac{dE}{dt} = (1 + i\alpha_H)NE + \eta e^{-i\Phi}E(t-1). \tag{1}$$

The delay time τ is here the unit of time and N is a properly scaled deviation of the carrier density from the threshold of the solitary laser.

It should be noted, that the relation between the feedback strength η within this equation and the parameters of the delay section (r, L) is not obviously defined. Indeed, the optical field is described by only one appropriate representative E, but the feedback section fixes the ratio between the incoming and outgoing amplitudes at $z = 0$, which are two different representatives of the field. One main result of my derivation will be an expression for η.

2 Restricting the TWE to the laser section only

The full TWE are

$$-i\partial_t\Psi = H\Psi \qquad \text{boundary conditions:} \qquad \Psi^+(-l,t) = 0,\ \Psi^-(L,t) = r\Psi^+(L,t). \tag{2}$$

In the delay section, there is no grating, hence[1]

$$-i(\partial_t \pm \partial z)\Psi^\pm = \beta\Psi^\pm \quad \text{with the general solutions} \quad \Psi^\pm(z,t) = e^{i\beta t}g^\pm(t \mp z). \tag{3}$$

[1]for simplicity I set $v_g = 1$ using either spatial coordinates for the time or reverse.

The two arbitrary functions $g^{\pm}$ are coupled by the b.c. at L:

$$g^{-}(t+L) = rg^{+}(t-L) \quad \Rightarrow \quad g^{-}(t) = rg^{+}(t-2L) \quad \Rightarrow \quad \Psi^{-}(0,t) = r\Psi^{+}(0,t-2L). \tag{4}$$

These are b.c. of the LK type at the rigth facet of the laser.

Conclusion:
the full TWE problem (2) can be replaced by the TWE for the laser only with the LK-type b.c. at its right facet $z = 0$.

3 An auxiliary problem

First I turn to the more simple problem

$$-i\partial_t\Psi = H\Psi \qquad \text{boundary conditions:} \qquad \Psi^{+}(-l,t) = 0,\ \Psi^{-}(0,t) = a(t) \tag{5}$$

with a given function $a(t)$ describing, e.g., an external injection.

I want to derive an equation for the amplitude of the main mode. Therefore, the modes have to be defined first.

3.1 Modes of the solitary laser

The LKE are originally used in the limit of small feedback $|r| \ll 1$. Then, the derivation of the density N from the threshold density N_s of the solitary laser can be expected as small. Hence, it is natural to use the modes belonging to N_s as basis. They solve

$$H_s\Phi_n(z) = \Omega_n\Phi_n(z) \quad \text{b.c.: } \Phi_n^{+}(-l,t) = 0,\ \Phi_n^{-}(0,t) = 0, \quad \text{norm: } (\Phi_n, \Phi_m) = \delta_{nm} \tag{6}$$

where $H_s = H(N_s)$ is the TWE operator at the threshold density of the solitary laser and (f,g) is the usual bilinear form according to which H is self-adjoint [1].

All Ω_n but one have positive imaginary parts. The exception is the lasing mode of the solitary laser, say $n = 0$. For brevity we use $\Omega_0 = 0$ without loss of generality by asuming that the central optical frequency behind the TWE is set just to the optical frequency of the solitary laser.

3.2 Dynamic equation for the mode amplitude

It is clear, that the solution Ψ of the auxiliary problem (5) can not be represented in the space spanned by the solitary laser modes Φ_n, because no element Φ of this space can fulfill a b.c. $\Psi^{-}(z=0) \neq 0$. But we can use the ansatz

$$\Psi = f(t)\Phi_0 + \sum_{m\neq 0} f_m\Phi_m + F \quad \text{with the remainder fulfilling } (\Phi_m, F) = 0 \text{ for all } m. \tag{7}$$

Inserting this into the auxiliary problem (5), multiplying from the left $(\Phi_0, \cdot$, one arrives at

$$-i\dot{f} = \beta' n f + (\Phi_0, H_s F), \qquad \text{with } \beta' = \frac{\partial\beta(N)}{\partial N}\Big|_{N=N_s}, \quad n = N - N_s. \tag{8}$$

The second term can be expressed by $a(t)$:

$$
\begin{aligned}
(\Phi_0, H_s F) &= -i \int_{-l}^{0} dz\, (\Phi_0^- \partial_z F^+ - \Phi_0^+ \partial_z F^-) + (\Phi_0, M_s F) \qquad \Big|_{\text{partial integration}} & (9) \\
&= -i \left[\Phi_0^- F^+ - \Phi_0^+ F^-\right]_{-l}^{0} + (H_s \Phi_0, F) & (10) \\
&= i\Phi_0^+(0) F^-(0) = i\Phi_0^+(0) a(t). & (11)
\end{aligned}
$$

Inserting this into (8) yields finally

$$-i\dot{f} = \beta' n f + i\Phi_0^+(0) a(t). \qquad (12)$$

This is a nice intermediate result for the description of external injection into a single mode laser.

Why no coupling to other modes? Because a spatially constant carrier density variation does not couple the modes of a single laser.

4 Consequences for the TWE with LK-type b.c.

First, we replace $a(t)$ simply by $r\Psi^+(0, t-\tau)$ with $\tau = 2L$.

Assumptions:

- other modes $m \neq 0$ have negligible amplitudes.
- $F^+(z=0) = 0$. This assumption is very reasonable because F has to be orthogonal to *all* modes $(\Phi_m, F) = 0$. To my opinion, this means, F can be nonzero only at those points where all modes vanish, i.e., outgoing amplitudes at the facets $F^+(0)$ and $F^-(-l)$. This means, we set $a(t) = r\Psi^+(0, t-\tau) = rf(t-\tau)\Phi_0^+(0)$, yielding finally

$$\boxed{\dot{f}(t) = -iv_g \beta' n(t) f(t) + \frac{\eta}{\tau} f(t-\tau)} \quad \text{Coupling coefficient:} \quad \boxed{\eta = -r\, \frac{L}{l}\, \frac{\Phi^+(0)^2 l}{(\Phi_0, \Phi_0)}} \qquad (13)$$

$$|\eta| = |r| \frac{L|\Phi^+(0)|^2}{< \Phi_0, \Phi_0 >} \sqrt{K_z} \qquad (14)$$

This is the LKE with the mode amplitudes as representatives for the field and an explicite expression (in physical units) for the coupling factor.

It is very satisfying that the coupling factor contains information about the outcoupling efficiency of the solitary laser mode.

References

[1] H. Wenzel, U. Bandelow, H.J. Wünsche and J. Rehberg, "Mechanisms of fast self pulsations in two-section DFB lasers", *IEEE J. Quantum Electron.*, **32**, pp. 69-79, 1996.