Odd-number theorem: Optical feedback control at a subcritical Hopf bifurcation in a semiconductor laser

S. Schikora,* H-J. Wünsche, and F. Henneberger

Humboldt-Universität zu Berlin, Institut für Physik, Newtonstr. 15, D-12489 Berlin, Germany

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A subcritical Hopf bifurcation is prepared in a multisection semiconductor laser. In the free-running state, hysteresis is absent due to noise-induced escape processes. The missing branches are recovered by stabilizing them against noise through application of phase-sensitive noninvasive delayed optical feedback control. The same type of control is successfully used to stabilize the unstable pulsations born in the Hopf bifurcation. This experimental finding represents an optical counterexample to the so-called odd-number limitation of delayed feedback control. However, as a leftover of the limitation, the domains of control are extremely small.

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I. INTRODUCTION

Control of stability in dynamical systems, in particular chaos control [1], is a central topic in nonlinear science. Among numerous methods, delayed feedback control (DFC) suggested by Pyragas [2] has gained widespread acceptance as it acts self-adjusting and is implementable without great effort. In its simplest version, an unstable periodic orbit (UPO) is stabilized by measuring a suitable system variable \( s(t) \) and feeding back to the system the delayed difference

\[
F(t) = K[s(t - \tau) - s(t)]
\]

with appropriate control gain \( K > 0 \). The delay \( \tau \) is adjusted equal to the period of the target orbit. Thus, by construction, the control force \( F(t) \) vanishes for successful stabilization; i.e., the targeted UPO is recovered noninvasively. DFC has been implemented successfully in physics, chemistry, biology, and medicine [3,4].

Despite all its successes, DFC suffers from the so-called odd-number limitation stating that torsion-free UPOs, which have an odd number of real-positive Floquet multipliers, cannot be stabilized [5–9]. In particular, UPOs born in a subcritical Hopf bifurcation are of this type. It was generally accepted that alternative control schemes have to be applied [10–12], which have been implemented successfully in electronic circuit experiments [13,14]. However, the odd-number limitation is not universal. It has been shown in Ref. [15] that a control force

\[
F(t) = Ke^{i\varphi}[z(t - \tau) - z(t)]
\]

with nonzero control phase \( \varphi \) can stabilize the UPO when added to the normal form of a subcritical Hopf bifurcation in the complex plane of variable \( z \). The theoretical understanding for this surprising result is meanwhile well developed [16–21]. An experimental verification appeared only very recently [22], cloning directly the equations used in the theoretical study of Ref. [15] via an electronic circuit.

In our present paper, we examine the odd-number limitation by measurements in the optical domain on a practical device where the phase of light is intrinsically involved. A recently developed all-optical setup of DFC (Fig. 1) is applied where optical feedback from a Fabry-Perot interferometer (FP) provides the control force. The setup has already been used for stabilization of continuous-wave (CW) operation [23] as well as of UPOs with torsion [24] in semiconductor lasers. It is also an ideal candidate for controlling torsion-free UPOs because the control force is similar to type (2). The slowly varying complex amplitude \( E \) of the electric field emitted by the laser plays the role of the variable \( z \), and \( \tau \) is the round-trip time in the FP. Control phase \( \varphi \) and gain \( K \) are the optical phase shift and optical losses on the round trip between the laser and front mirror of the FP, respectively. Here \( \varphi \) is experimentally tunable by a position-stabilized piezo actor with high resolution (20 nm), and \( K \) can be adjusted by a variable gray filter.

Our approach extends previous work in several respects. The phase space of the multisection laser is much more complex than that of the electronic system in Ref. [22]. Real-world effects, in particular noise, play a significant role. Furthermore, the optical control force is qualitatively different from those investigated so far in the context of the odd-number limitation. It is neither a global one [19], nor is it tangent to the two-dimensional center manifold of the bifurcation as in Refs. [15–18,20–22]. It also corresponds to a generalization of the Pyragas-type control (1), namely the so-called extended time-delay autosynchronization (ETDAS) [25].

The paper is structured as follows. The subcritical Hopf bifurcation of the experimental multisection laser is characterized in Sec. II. In particular, it is shown that intrinsic noise hides the hysteresis of the deterministic bifurcation. Noninvasive all-optical control is applied in Sec. III for two purposes. First, the hysteresis hidden by noise in the free-running laser is recovered by stabilizing the missing branches. Second, the odd-number UPO is stabilized in a simulation-aided experiment. Summary and conclusions are given in Sec. IV.

II. THE HOPF BIFURCATION OF THE FREE-RUNNING LASER

The device studied is a multisection semiconductor laser. The so-called active feedback laser (AFL) combines a single-mode distributed feedback laser with a passive waveguide and an amplifying section (Fig. 1). The three sections are integrated on one semiconductor chip and can be independently biased
indeed due to a subcritical Hopf bifurcation. what follows, we shall first give experimental evidence that mentally, a sudden onset of large-amplitude self-pulsations short-delay passive and active feedback [26,28,29]. Analysis of device-realistic deterministic models of lasers with this type of bifurcation has been predicted by the bifurcation analysis of device-realistic deterministic models of lasers with short-delay passive and active feedback [26,28,29]. Experimentally, a sudden onset of large-amplitude self-pulsations has been observed but not the requested hysteresis [26,30]. In what follows, we shall first give experimental evidence that this transition also occurs in our present device and that it is indeed due to a subcritical Hopf bifurcation.

To this purpose, the current $I_P$ on the passive section is used as bifurcation parameter. Under CW output, the omnipresent internal noise excites deviations from steady state, and the associated ROs are manifested in the rf spectrum by a solitary peak at about 8 GHz [lowest panel of Fig. 2(a)]. Area and width of this peak are taken as measure for oscillation amplitude and damping of the ROs, respectively. The linearly decreasing model in Fig. 2(b) reveals a Hopf bifurcation at the critical current $I_P = I_{Pc}^2$ obtained by extrapolating the damping rate to zero [30]. The sudden jump also seen in Fig. 2(b) to a stable large-amplitude self-pulsation (SP) is a fingerprint of the subcritical character. However, unlike the standard scenario, no hysteresis is found on tuning $I_P$ backward. On the other hand, the jump already occurs before the bifurcation, which also excludes the case of a degenerate Hopf bifurcation. The following detailed study of the transition will resolve this seeming contradiction.

Figure 2(a) gives an expanded view on the evolution of the power spectrum in the vicinity of the jump. The transition from ROs in the lowest panel to SPs in the upmost panel proceeds continuously. The SP peak at about 6.8 GHz grows at the expense of the RO peak at 7.9 GHz. The spectra in the middle panels are just linear superpositions of pure RO spectra and pure SP spectra with different relative weights. Thus, the two peaks originate from alternating RO and SP epochs during the sampling time of the spectrum analyzer. The 3-seconds-long sampling time exceeds the timescale of the intrinsic deterministic dynamics by more than eight orders of magnitude. Under these conditions, the peak areas reflect the mean residence times in the respective state. Both residence times change rapidly with $I_P$ in opposite directions and are comparable to each other only within the tiny interval between 21.1 and 21.2 mA [middle panels of Fig. 2(a)].

These experimental findings can be explained by noise-induced switching between the coexisting states of the underlying deterministic subcritical Hopf bifurcation. Figure 2(c) sketches a typical scenario using recent results on the stochastic Duffing–Van der Pol oscillator with linear damping parameter $\epsilon$ [31]. The deterministic UPO is born in $\epsilon = 0$ (H), bends back, and annihilates with a stable limit cycle (LC) in a saddle-node bifurcation at $\epsilon = -0.125$. The amplitudes of the two branches are given analytically by $a = 1/\sqrt{1 + 8\epsilon}$. Weak noise generally destroys the bistability of focus and LC because it induces transitions between the two states from time to time. The analytic approximation of the probability distribution of oscillation amplitudes [31]

$$p(a) \sim a \exp \left[ -\frac{a^2(a^4 - 3a^2 - 24\epsilon)}{48D} \right]$$

behaves qualitatively as observed in our experiment. Except for a tiny transition region with equal probability in both states (here close to $\epsilon = -0.1$), the oscillator spends nearly all the time either in a noisy LC or near the focus. Disappearance of hysteresis is thus just a phenotype of a subcritical Hopf bifurcation in the presence of substantial noise.
III. CONTROL EXPERIMENTS

The control force of the all-optical DFC setup (Fig. 1) is given by the slowly varying field amplitude fed back into the AFL,

\[ E_{\delta}(t) = K e^{i\psi} \sum_{n=0}^{\infty} (Re^{i\psi})^n [e^{i\psi} E(t_n - \tau) - E(t_n)], \tag{4} \]

where \( E(t) \) is the laser output and \( t_n = t - \tau_l - n\tau \). There are some essential differences compared to the control force (2). The phase shift \( \psi \) in the FP cavity is an additional parameter. The summation over the number \( n \) of round trips in the FP corresponds to the well-established ETADAS version of TDFC [25], here with complex memory parameter \( Re^{i\psi} \). All time arguments \( t_n \) in the sum are shifted by the latency time \( \tau_l \). The role of \( \tau_l \) in overcoming the odd-number limitation has not been studied so far. However, for orbits with torsion and unstable foci it is known that too large a latency overrules \( \tau \), thus weakening the ability of control [32–34]. Stabilization here generally requires

\[ \tau_l \leq \lambda^{-1} - \frac{2}{\pi} \left( \frac{1 - R}{1 + R} \right), \tag{5} \]

where \( \lambda \) is the respective positive real part of the Lyapunov exponent of the uncontrolled state [34]. With increasing \( \tau_l \), control deteriorates nonmonotonically in this interval with the locally best performance at integer ratios \( \tau_l/\tau \). In our experiments, geometrical constraints set a lower limit of \( \tau_l = 60 \text{ ps} \), which is of the order of \( \tau \approx 130 \text{ ps} \). Therefore, we set the latency time to the optimum \( \tau_l \approx \tau \). Condition (5) is certainly met here because the working points are close to the Hopf bifurcation, where \( \lambda^{-1} \gg \tau \). Another important difference with previous studies is that the purely optical force (4) is not tangent to the center manifold of the Hopf bifurcation because the considered relaxation oscillations are due to the interplay of carrier inversion and optical intensity. This feature could degrade the control efficiency and keep the odd-number limitation in force.

In what follows, optical DFC is applied for two purposes. We first proof the coexistence of CW and SP emission on both sides of the amplitude discontinuity by stabilizing the missing branches of the hysteresis loop against noise-induced escapes in either case. Subsequently, an unstable pulsation (UPO) is stabilized in the same range of parameters, providing the experimental counterexample to the odd-number theorem. Figure 3 summarizes the main results of these measurements described in detail in the following.

Control is achieved in all cases by an iterative procedure. First, \( R, \tau_l, \) and \( K \) are set to an initial guess either according to previous work (CW: [23], SP: [24]) or according to the following numerical simulations (UPO). Second, \( \psi \) is varied over more than one period. Power spectra as well as the total power transmitted through the FP are recorded in each step. Third, an optimum \( \psi \) is selected, where the emission properties are compatible with the target state and the transmission through the FP is maximum. The further adjustment is as follows. Noninvasive control demands that the resonances of the FP exactly match the optical frequency comb of the stabilized state (see Fig. 4). Matching of the FP to the target is not feasible as the target’s frequency comb is unknown. Flexible tunability of the AFL device by the control currents and the operation temperature \( T \) is the key for solving this issue. The UPO parameters \( \omega_0 \) and \( T_0 \) are fine-tuned through \( I_A \) and \( T \) to full resonance with the FP, which is detectable by maximum transmission through the FP. The whole procedure is successful if the target state is stable and the feedback power reinjected in the laser is below a 10⁻³ fraction of the device output. Otherwise previous steps are repeated with slightly different parameters. This iteration yields accurate results provided the change of the device operation point does not alter significantly the bifurcation scenario. We ensure this by measuring for each pair \((I_A, T)\) used in the control demonstrations the complete amplitude-\( I_A \) characteristics of the free-running device in the absence of the FP. The scattered plot in Fig. 3(a) (small dots) pools these data for all operation points used in the control demonstrations. Lines: guide to the eye.

![Figure 3](image_url)  
**FIG. 3.** (Color online) Control experiments at subcritical Hopf bifurcation: normalized SP amplitude versus distance to bifurcation. Cyan triangles: SPs stabilized with \( \tau = 145 \text{ ps} \), Green squares: CW emission stabilized with \( \tau = 39 \text{ ps} \). In both cases \( R = 0.5, K = 0.02 \). Red circles: UPO stabilized with \( \tau = 130 \text{ ps} \), \( R = 0.3, K = 0.06 \). The uncertainty of \( K \) is always about 15%. Black dots: Data of the free-running device, which are collecting measurements for all operation points used in the control demonstrations. Lines: guide to the eye.

![Figure 4](image_url)  
**FIG. 4.** (Color online) Conditions for noninvasive optical feedback from a FP cavity. Blue dashed curve: schematic reflectivity spectrum of an FP [from Fourier transform of Eq. (4)]. It has one resonance per period \( 2\pi/\tau \), where the reflectivity vanishes due to destructive interference between the wave returning from inside the cavity and the wave reflected by the first mirror. Vertical (black): the emission spectrum of a possible target state is a comb of equidistant lines separated by \( 2\pi/T_0 \). Optical frequency of a prominent line in the comb. \( T_0 \): period of the intensity pulsation. If \( \psi \) and \( \tau \) are integer multiples of \( 2\pi \) and \( T_0 \), respectively, all lines of the comb fit to zeros of the reflectivity, and the total reflected power vanishes.
A. Recovering hysteresis

The laser is first set in the range of CW operation, slightly left to the amplitude jump in Fig. 2(b). An etalon approximately resonant to the stable SP is chosen. Indeed, continuation of the SP branch in the region where it coexists with the CW states of the free-running device is accomplished with our iterative procedure (triangles in Fig. 3). Since the control is noninvasive, a corresponding orbit must exist in the phase space of the underlying deterministic system without feedback. In principle, this uncontrolled deterministic orbit could have become unstable on the way across the measured amplitude jump, say, due to a period doubling or torus bifurcation. However, there is no indication of other bifurcations. Therefore, we can conclude that the deterministic orbit remains stable and the jump is a result of the noise-induced escape we have described.

In order to complete the hysteresis, the free-running laser is set to the stable SP regime right from the amplitude jump in Fig. 2(b). As shown by the squares in Fig. 3, continuation of the CW branch succeeds in this case as well. Interestingly, though escapes from these CW states are suppressed by the feedback, the amplitude of the noise-induced RO is not reduced. Apparently, nonlocal effects in phase space like barrier enhancement or destruction of the SP orbit are responsible rather than an increasing damping of the local RO. Irrespective of these details, the results of both control measurements uncover the expected coexistence of CW and SP emission before the Hopf bifurcation.

After clarification of these points, we can regard the subcritical nature of the Hopf bifurcation as proven. Thus, there must branch off an UPO with an odd number of positive Floquet exponents. In the following, we focus on the noninvasive stabilization of this UPO. The difficulty is again that pulsation period and optical frequency of the UPO are unknown. Moreover, if at all, control is possible only in a very limited range of gain $K$ and phase $\phi$ [15] that has to be found for the present system. Extensive theoretical and computational efforts in the past enable us to answer these questions prior to experiment by the following highly reliable and device-specific simulations of the control setup.

B. Stabilizing the UPO: numerical simulation

The simulation is done with the software LDSL-tool [35] based on the well-established traveling-wave model of multisection lasers. In the framework of the Maxwell-Bloch equations, this model describes the coupled dynamics of polarization, inversion, and bidirectional propagating optical waves with spatial resolution along the longitudinal coordinate. The model equations with incorporation of feedback (4) are explicitly given in Ref. [23] and need not be repeated here (see also section 21.5 of Ref. [4]). The parameters have been adapted to the present experimental AFL configuration.\(^\text{1}\)

\(^\text{1}\)Parameters of device simulations differing from Ref. [23] (notations as there): section length $[L_1, L_P, L_L] = [220, 500, 220] \mu$m, complex Bragg coupling $\kappa = [0.0, 1] \times (250 + 6i) \text{ cm}^{-1}$, internal optical losses $\gamma = [25, 25, 25] \text{ cm}^{-1}$, static wavenumber at transparency $\delta = [22 147, \text{variable}, 19 126] \text{ m}^{-1}$, gain compression factor $\varepsilon = [1, 0.1] \times 10^{-2} \text{ m}^3$, injection currents $I = [9, 0, 80] \text{ mA}$, Triple values are given according to [A, P, L]. Central wavelength $\lambda_0 = 2\pi c/\omega_0 = 1.54 \mu\text{m}$. Latency time $\tau_l = 0$.\n
FIG. 5. Simulations of the control scenario. (a) Pulsation amplitude and (b) frequency versus internal phase shift in the AFL. Solid gray: stable states of the free-running laser [RO in (b): damped relaxation oscillations toward CW], black dotted: UPO stabilized with $\phi/2\pi = 0.84, R = 0.3, K = 0.06, \tau_l = 0$.

First, the subcritical Hopf bifurcation of the free-running AFL is numerically reproduced. The internal phase shift $\varphi_p$ in the passive section, tuned in the experiment by current $I_p$, is changed in small steps ($10^{-3} \pi$). The attractor reached in each step serves as initial state for the next one. A clear hysteresis loop with bifurcation point $H$ is obtained by changing $\varphi_p$ up and down [solid gray lines in Fig. 5(a, b)]. Since noise is absent, the RO frequencies in panel b are determined from the transients toward CW. It is important that the SPs are slower than the ROs because a negative drift of the oscillation frequency with increasing amplitude is a prerequisite for circumventing the odd-number limitation [15].

In the next step, the UPO is uncovered. Therefore, the laser is set in a CW state; i.e., $\varphi_p$ is chosen slightly left of $H$ and coupled to a FP that is initially in resonance with the frequency spectrum of the weakly damped RO representing a good guess for the actual UPO. We search for control parameters $K$ and $\varphi$, where stable small-amplitude pulsations exist. Then, iteratively adapting the FP, the feedback power is made to decline. We stop when remaining below a $10^{-5}$ fraction of the 10-mW AFL output. At this level, the internal laser dynamics is surely not affected by the feedback signal ensuring noninvasiveness. Zero power would require infinitely many iterations. The UPO is then traced by moving phase $\varphi_p$ farther away from $H$, always readjusting the FP to maintain noninvasiveness. This yields the dotted lines in Fig. 5(a, b). Beyond a certain distance from the bifurcation, noninvasive control is lost due to the increased instability of the UPO.

In the last step, the domain of control is evaluated. In Fig. 6, the control phase $\varphi$ is changed at the point marked by the dashed arrow in Fig. 5. Various states appear, among them CW...
FIG. 6. Simulations of phase tuning. (a) Pulsation amplitude and (b) frequency versus control phase \( \phi \) at \( \phi /2\pi = 0.3785 \) for otherwise identical parameters as in Fig. 5. (c) Feedback power reinjected into the laser when stopping iterations. Shaded area: range for noninvasive control of UPO. Feedback power is here below \( 10^{-5} \) of laser output.

and SP. In the hatched window, the feedback power becomes negligible, demonstrating that the small-amplitude pulsation residing here represents the noninvasively stabilized UPO. At the left border of this window, control is lost due to the onset of a very slow amplitude modulation. No amplitude is drawn here. At the right border, the amplitude suddenly starts to increase, leading back to the large-amplitude SPs at the end of the phase period. This feature is indicative of the transcritical bifurcation breaking the odd-number limitation [15,16,18–21]. The domain of control found in these simulations is very small. The control phase covers only a few-percent fraction of a period, and the control gain is even restricted to \( 0.05 < K < 0.07 \), making experimental demonstrations quite difficult.

C. Stabilizing the UPO: experiment

Neither \( \varphi \) nor \( K \) can be directly derived from the experimental setup with the required accuracy. In order to correlate these parameters with piezo voltage and filter attenuation, the following overall measurements are carried out first. The free-running AFL is set to CW emission just left of the amplitude jump in Fig. 2(b) (\( I_p = 21 \text{ mA} \)) and resonance with the FP is adjusted. The filter attenuation is then decreased stepwise and the piezo voltage varied at each step such that \( \varphi \) changes more than a period. rf spectra of the laser output and the power transmitted through the FP are recorded for each \( \varphi \). Comparing these experimental series with the simulations enables us to identify the range where control is expected. This laborious approach finally leads to a point of noninvasive control. Figure 7(a) shows the series of rf spectra measured in the last iteration loop when tuning the piezo voltage \( U_{\text{piezo}} \) with otherwise converged parameters. Single-peak spectra are either noise-induced ROs or the target UPO and spectra with a second harmonics belong to large-amplitude SPs. The experimental behavior in the shaded voltage domain is fully reminiscent of the simulations in Fig. 6(a) and 6(b), where noninvasive stabilization occurs just below the onset of the SP. Indeed, various facts provide direct evidence that control of an UPO is also achieved experimentally. First, the peak frequency is below that of noise-induced ROs but equals the free spectral range of the FP \( f_{\text{FP}} = \tau^{-1} \) [Fig. 7(c)]. Second, the rf line narrows considerably when approaching the control region. Third, transmission through the FP is maximum, \( T_{\text{FP}} = 0.98 \) [Fig. 7(b)]. Thus, the relative control power is below \( (1 - T_{\text{FP}})K^2 = 0.72 \times 10^{-4} \). It cannot vanish completely because noise permanently causes deviations from the target state, but it is sufficiently small to conclude that the laser is in a state already present without control. We have also checked that feedback on this level from a single mirror does not change laser states noticeably [24]. For all these reasons,
FIG. 8. (Color online) (a) Power spectra of the stabilized UPO at largest distance to bifurcation (arrow in left panel) and of the RO peak without the FP. (b) rf line width of UPO (red circles) as well as RO after removing FP (stars) and (c) frequency difference between RO peak without the FP. (b) rf line width of UPO (red circles) as well as RO at largest distance to bifurcation (arrow in left panel) and of the rf feature in the shaded region of Fig. 7 is an unstable small-amplitude pulsation of the AFL and represents thus the targeted and stabilized UPO. The series of Fig. 7 has been measured without repeatedly readjusting resonance to the FP. Due to a small parameter drift during the measurement, the series is therefore not exactly periodic, and control is lost in the second period around $V_{\text{piezo}} = 3.8 \text{ V}$.

Stabilization of the UPO can also be carried out successfully when moving the AFL operation point ($I_p$) farther away from the bifurcation, establishing the negative-slope branch in Fig. 3 (circles). With increasing distance, amplitude as well as frequency separation from the RO peak [Fig. 8(c)] grow, whereas the line width [Fig. 8(b)] decreases. Fig. 8(a) demonstrates how distinctly different the rf spectra with and without FP become at the point beyond which control can no longer be achieved. Closer to bifurcation, the rf features of UPO and noise-driven ROs are so similar that a distinction is impossible.

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[35] LDSL-tool is maintained at the Weierstrass Institute; see [http://www.wias-berlin.de/software/ldsl/].