Supporting Information for

A Universal Description of Stochastic Oscillators

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1. Theoretical derivations

Preliminary details about the derivations. To make this document self-contained, we briefly summarize the most important properties of the mathematical objects dealt with in the paper. Throughout the document we make the following assumptions:

1. We assume our stochastic oscillator can be described by a Langevin equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\xi(t), \qquad \mathbf{x} \in \mathbb{R} \tag{1}$$

where $\mathbf{f}$ is a is an n-dimensional vector, $\mathbf{g}$ is an $n \times n$ matrix, and ξ is n-dimensional white noise with uncorrelated components, so it satisfies, $\langle \xi_i(t)\xi_j(t')\rangle = \delta(t-t')\delta_{i,j}$. Moreover, we interpret Eq. (1) in the Itô sense for its mathematical convenience. We remark that choosing between the Itô or the Stratonovich interpretation will not change our framework, which is based on the uniquely defined Kolmogorov's forward and backward operator (1) that we introduce next.
The process described by Eq. (1) is a n-dimensional Markov process, that is uniquely determined by the transition probability density $P(\mathbf{x}, t \mid \mathbf{x}_0, s)$ (for $t > s$). This central statistics satisfies both the forward Kolmogorov (or "Fokker-Planck") equation (1, 2)

$$\frac{\partial P}{\partial t} = \mathcal{L}[P] = -\nabla_{\mathbf{x}}\cdot(\mathbf{f}(\mathbf{x})P) + \sum_{i,j}\frac{\partial^2}{\partial x_i x_j}\left(D_{ij}(\mathbf{x})P\right) \tag{2}$$

where $D = \frac{1}{2}gg^{\intercal}$. Here the functional $\mathcal{L}$ acts with respect to the $\mathbf{x}$ coordinates. $P(\mathbf{x}, t \mid \mathbf{x}_0, s)$ also obeys the backward Kolmogorov equation (1, 2)

$$-\frac{\partial P}{\partial s} = \mathcal{L}^{\dagger}[P] = \mathbf{f}(\mathbf{x}_0)\cdot\nabla_{\mathbf{x}_0}(P) + \sum_{i,j} D_{ij}(\mathbf{x}_0)\frac{\partial^2 P}{\partial x_{0,i}x_{0,j}} \tag{3}$$

where the operator $\mathcal{L}^{\dagger}$ acts with respect to the $\mathbf{x}_0$ coordinates. We note that $\mathcal{L}^{\dagger}$, which is also known as the generator of the Markov process, is the formal adjoint of the forward operator $\mathcal{L}$. Generally speaking, the domain of $\mathcal{L}$ is restricted to integrable (L_1) functions, while the domain of $\mathcal{L}^{\dagger}$ is restricted to bounded (L_∞) functions.*

2. We assume a discrete set of eigenvalues $\lambda = \mu_\lambda + i\omega_\lambda$ and a bi-orthogonal set of associated eigenfunctions of the forward ($\mathcal{L}$) and backwards ($\mathcal{L}^{\dagger}$) Kolmogorov operators

$$\mathcal{L}[P_\lambda] = \lambda P_\lambda, \qquad \mathcal{L}^{\dagger}[Q^*_\lambda] = \lambda Q^*_\lambda, \qquad \langle Q_{\lambda'} \mid P_\lambda\rangle = \int d\mathbf{x}\, Q^*_{\lambda'}(\mathbf{x})P_\lambda(\mathbf{x}) = \delta_{\lambda'\lambda}. \tag{4}$$

The smallest eigenvalue is λ_0, corresponds to the stationary state. Here $\langle Q_{\lambda'} \mid P_\lambda\rangle$ refers to the inner product of the backward and forward eigenfunctions $Q_{\lambda'}$ and P_λ, respective. This notation should not be confused with the ensemble average notation $\langle\cdot\rangle$ used later.

3. We assume that there is a pair of complex eigenvalues, referred to as $\lambda_1 = \mu_1 + i\omega_1, \lambda_1^* = \mu_1 - i\omega_1$, with least negative real part μ_1 (cf. main text for how this eigenvalue is related to the definition of a robustly oscillatory system); the corresponding eigenfunctions are denoted compactly by $Q_1^*(\mathbf{x})$ and $Q_1(\mathbf{x})$, respectively.

The third equality in Eq. (4) implies a vanishing stationary mean value for $Q^*_\lambda(\mathbf{x})$ with $\lambda \neq 0$:

$$\langle Q^*_\lambda(\mathbf{x})\rangle = \int d\mathbf{x}\, Q^*_\lambda(\mathbf{x})P_0(\mathbf{x}) = 0, \quad (\lambda \neq 0). \tag{5}$$

and also allows us to write the probability density as

$$P(\mathbf{x}, t|\mathbf{x}_0, s) = P_0(\mathbf{x}) + \sum_{\lambda\neq 0} e^{\lambda(t-s)}P_\lambda(\mathbf{x})Q^*_\lambda(\mathbf{x}_0), \qquad \text{for } t > s. \tag{6}$$

Furthermore, we normalize the nontrivial backwards eigenfunctions $Q^*_\lambda(\mathbf{x})$ such that they satisfy

$$\left\langle |Q^*_\lambda(\mathbf{x}(t))|^2\right\rangle = \int d\mathbf{x}\, |Q^*_\lambda(\mathbf{x})|^2 P_0(\mathbf{x}) = 1, \quad (\lambda \neq 0). \tag{7}$$

We note that in the trivial case $\lambda = 0$, we have $Q^*_0(\mathbf{x}) \equiv 1$ and thus a non-vanishing mean value of one and a vanishing variance.
Specifically, if we perform the nonlinear transformation of the system's variable $\mathbf{x}(t)$ to our new variable, $Q^*_\lambda(\mathbf{x}(t))$, then, this new variable has zero mean and a variance of one.

*In the paper we focus on the case of stochastic processes of diffusion type, in which $\mathcal{L}$ and $\mathcal{L}^{\dagger}$ are second-order differential operators; in this case we further restrict their domain of action to twice-differentiable functions.

Another property of the eigenfunctions that will be repeatedly used here is the effect of the evolution operator on the forward eigenfunction (cf. (2), Eq. (6.30)):

$$e^{\mathcal{L}(\mathbf{x})\tau}P_\lambda(\mathbf{x}) = \left[I + \mathcal{L}(\mathbf{x})\tau + \frac{\mathcal{L}^2(\mathbf{x})\tau^2}{2} + ...\right] P_\lambda(\mathbf{x}), \; = \left[I + \tau\lambda + \frac{\lambda^2\tau^2}{2} + ...\right]P_\lambda(\mathbf{x}), \; = e^{\lambda\tau}P_\lambda(\mathbf{x}). \quad [8]$$

In this document we will use the following convention for the finite-time-window Fourier transform of a time (generally complex-valued) series $z(t)$

$$\tilde{z}(\omega) = \int_0^T dt \; z(t)e^{-i\omega t}. \quad [9]$$

The power spectrum of $z(t)$ and cross-spectra between two time series $z_1(t)$ and $z_2(t)$ are then given by

$$S_{zz}(\omega) = \lim_{T\to\infty} \frac{\langle|\tilde{z}|^2\rangle}{T}, \;\; S_{12}(\omega) = \lim_{T\to\infty} \frac{\langle\tilde{z}_1\tilde{z}_2^*\rangle}{T}. \quad [10]$$

In simulations, we cannot take the limit $T \to \infty$ but have to use a sufficiently long time window such that a further enlargement of it does not change the spectral densities anymore.

Derivation of the cross-correlation functions and cross-spectra of the backward eigenfunctions $Q_\lambda^*(\mathbf{x})$. Let us consider two eigenfunctions $Q_\lambda^*(\mathbf{x}(t+\tau))$, $Q_{\lambda'}(\mathbf{x}(t))$ for $\tau > 0$ in the stationary state. The correlation function depends only on the time difference τ and can be expressed by

$$C_{\lambda,\lambda'}(\tau) = \langle Q_\lambda^*(\mathbf{x}(\tau))Q_{\lambda'}(\mathbf{x}(0))\rangle = \int d\mathbf{x} \int d\mathbf{x}_0 \, Q_\lambda^*(\mathbf{x})Q_{\lambda'}(\mathbf{x}_0)P(\mathbf{x},\tau|\mathbf{x}_0,0)P(\mathbf{x}_0). \quad [11]$$

where $\langle\cdot\rangle$ denotes the ensemble average under stationary conditions. If we express in the conditional density as in Eq. (6), we find (for $\tau > 0$)

$$\begin{aligned} C_{\lambda,\lambda'}(\tau) &= \int d\mathbf{x} \int d\mathbf{x}_0 \, Q_\lambda^*(\mathbf{x})Q_{\lambda'}(\mathbf{x}_0)P(\mathbf{x},\tau|\mathbf{x}_0,0)P_0(\mathbf{x}_0), \\ &= \int d\mathbf{x} \int d\mathbf{x}_0 \; Q_\lambda^*(\mathbf{x})Q_{\lambda'}(\mathbf{x}_0)\Big(\sum_{\bar\lambda=0} Q_{\bar\lambda}^*(\mathbf{x}_0)P_{\bar\lambda}(\mathbf{x})e^{\bar\lambda\tau}\Big)P_0(\mathbf{x}_0), \\ &= \sum_{\bar\lambda=0} e^{\bar\lambda\tau} \int d\mathbf{x} \; Q_\lambda^*(\mathbf{x})P_{\bar\lambda}(\mathbf{x}) \int d\mathbf{x}_0 \; Q_{\bar\lambda}^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x}_0)P_0(\mathbf{x}_0), \\ &= \langle Q_\lambda^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x}_0)\rangle \, e^{\lambda\tau} \end{aligned} \quad [12]$$

Note that in the third equality we have used the biorthogonality condition $\langle Q_{\lambda'} \mid P_\lambda\rangle = \delta_{\lambda,\lambda'}$ in Eq. (4) to get rid of the sum and then we simply express the second integral as $\langle Q_\lambda^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x}_0)\rangle$, thus showing it corresponds exactly with the co-variance between Q_λ and Q'_λ. For negative times, $\tau < 0$, we can use the following expression

$$C_{\lambda,\lambda'}(\tau) = \langle Q_\lambda^*(\mathbf{x}(\tau))Q_{\lambda'}(\mathbf{x}(0))\rangle = \int d\mathbf{x} \int d\mathbf{x}_0 \, Q_\lambda^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x})P(\mathbf{x},0|\mathbf{x}_0,\tau)P(\mathbf{x}_0). \quad [13]$$

After performing similar computations, we obtain

$$\begin{aligned} C_{\lambda,\lambda'}(\tau) &= \int d\mathbf{x} \int d\mathbf{x}_0 \; Q_\lambda^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x})\Big(\sum_{\bar\lambda=0} Q_{\bar\lambda}^*(\mathbf{x}_0)P_{\bar\lambda}(\mathbf{x})e^{-\bar\lambda\tau}\Big)P_0(\mathbf{x}_0), \\ &= \sum_{\bar\lambda=0} e^{-\bar\lambda\tau} \int d\mathbf{x} \; Q_{\lambda'}(\mathbf{x})P_{\bar\lambda}(\mathbf{x}) \int d\mathbf{x}_0 \; Q_{\bar\lambda}^*(\mathbf{x}_0)Q_\lambda^*(\mathbf{x}_0)P_0(\mathbf{x}_0), \\ &= \langle Q_\lambda^*(\mathbf{x}_0)Q_{\lambda'}(\mathbf{x}_0)\rangle \, e^{-\tau\lambda'^*} \end{aligned} \quad [14]$$

where in the second equality we used $Q_{\lambda'} = Q_{\lambda'^*}^*$. For the autocorrelation function, we obtain due to the unit variance of the eigenfunctions (cf. Eq. (7)), a simple exponential function; specifically, for the eigenfunction $Q_1^*(\mathbf{x}(t))$ the above formulas yield the correlation function, Eq. [12] in the main text.

With the expressions for the cross-correlation, we use the Wiener-Khinchin theorem to obtain the cross-spectrum

$$S_{\lambda,\lambda'}(\omega) = \int_{-\infty}^{\infty} C_{\lambda,\lambda'}(\tau)e^{-i\omega\tau}d\tau = \kappa_\lambda \int_{-\infty}^{0} e^{-(\lambda'^*+i\omega)\tau}d\tau + \kappa_\lambda \int_0^{\infty} e^{(\lambda-i\omega)\tau}d\tau = -\kappa_\lambda\left(\frac{1}{\lambda'^*+i\omega} + \frac{1}{\lambda-i\omega}\right). \quad [15]$$

Fig. S1 illustrates this result by means of the cross-spectra between the first and the second (middle panel) and the first and the third (right panel) backward eigenfunction for one of our example systems, the noisy SNIC system, for two different parameter

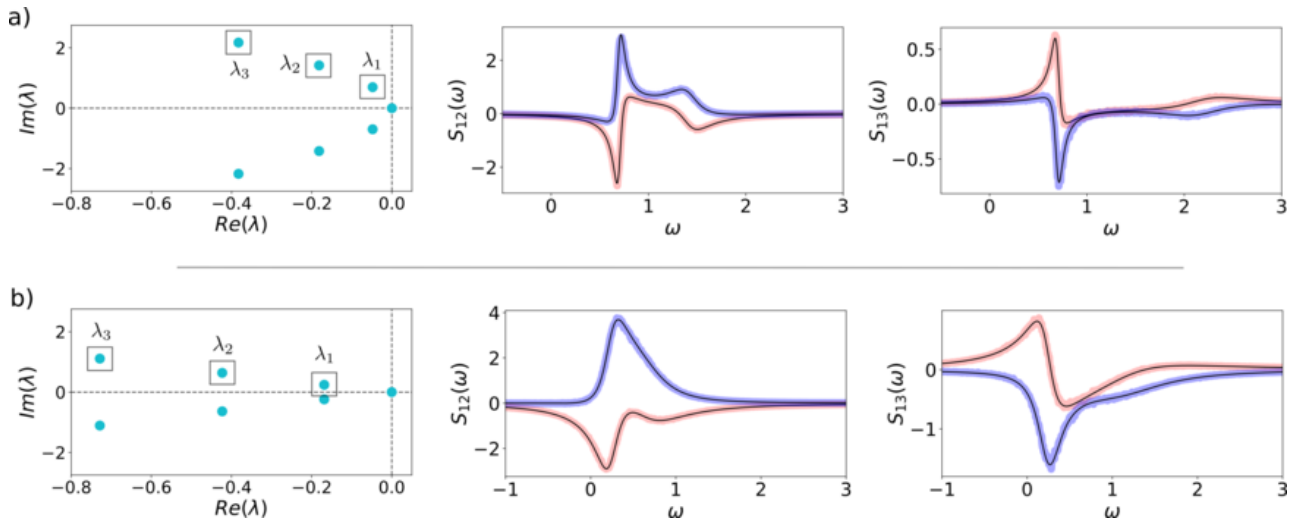


Fig. S1. Eigenvalue spectra (left panels) and cross-spectra (middle and right panels) of the backwards eigenfunctions $S_{\lambda,\lambda'}(\omega)$ for the noisy SNIC system (Eqs.[11] in the main text). **a:** Parameters $m = 1.216$, $n = 1.014$, $D_1 = D_2 = 0.0119$ (deterministic model would be in the oscillatory limit-cycle regime) leading to the eigenvalues (see also left panel) $\lambda_1 = -0.048 + 0.697i$, $\lambda_2 = -0.182 + 1.42i$, $\lambda_3 = -0.383 + 2.17i$ and the covariances $\langle Q_1^* Q_2\rangle = -0.106 + 0.11i$, $\langle Q_1^* Q_3\rangle = 0.022 - 0.03i$. Cross-spectrum between the first and the second slowest decaying eigenfunctions (middle panel) and between the first and the third slowest decaying eigenfunctions (right panel). **b:** Parameters $m = 0.99$, $n = 1$, $D_1 = D_2 = 0.01125$ (excitable regime with noise-induced oscillations) resulting in eigenvalues (see also left panel) $\lambda_1 = -0.168 + 0.241i$, $\lambda_2 = -0.423 + 0.638i$, $\lambda_3 = -0.728 + 1.109i$ and co-variances $\langle Q_1^* Q_2\rangle = -0.416 + 0.35i$, $\langle Q_1^* Q_3\rangle = 0.07 - 0.023i$. Cross-spectrum between the first and the second slowest decaying eigenfunctions (middle panel) and the cross-spectrum between the first and the third slowest decaying eigenfunctions (right panel). In all middle and right panels, stochastic simulation results in red (blue) correspond to the real (imaginary) part of $S_{\lambda,\lambda'}(\omega)$; theory is indicated by black lines.

sets (top and bottom rows). As can be expected, the exact result, Eq. (15), is in excellent agreement with the simulation results.

From Eq. (15) which we can straightforwardly obtain the power spectrum

$$S_\lambda(\omega) = \frac{2|\mu_\lambda|}{\mu_\lambda^2 + (\omega - \omega_\lambda)^2}. \qquad [16]$$

For the special case $\lambda = \lambda_1$ (with $\mu_\lambda = \mu_1$ and $\omega_\lambda = \omega_1$), we obtain the Lorentzian power spectrum (Eq.[14] in the main text) of the new variable $Q_1^*(\mathbf{x}(t))$.

Derivation of the linear response and susceptibility functions. Lets consider now the case in which an external time dependent input is applied to the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \varepsilon p(t)\mathbf{e} + \mathbf{g}(\mathbf{x})\xi(t) \qquad \mathbf{x}, \mathbf{e} \in \mathbb{R}^n \qquad [17]$$

because of the input, the Fokker-Planck equation is now modified and reads

$$\begin{aligned}\partial_t P &= -\nabla_\mathbf{x} \cdot ([\mathbf{f}(\mathbf{x}) + \varepsilon p(t)\mathbf{e}]P) + \partial^2_{x_i,x_j}\left(D_{ij}(\mathbf{x})P\right) = -\nabla_\mathbf{x} \cdot (\mathbf{f}(\mathbf{x})P) + \partial^2_{x_i,x_j}\left(D_{ij}(\mathbf{x})P\right) - \varepsilon p(t)\mathbf{e}\cdot\nabla_\mathbf{x} P \\ &= \mathcal{L}(\mathbf{x})[P] + \varepsilon p(t)\mathcal{L}_\mathbf{e}(\mathbf{x})[P]\end{aligned} \qquad [18]$$

where $\mathcal{L}_\mathbf{e}(\mathbf{x})$ is given by

$$\mathcal{L}_\mathbf{e}(\mathbf{x}) = -\mathbf{e}\cdot\nabla. \qquad [19]$$

Since the input is weak ($|\varepsilon| \ll 1$), we can make the usual linear-response ansatz (cf. (2)) and linearise the density

$$P(\mathbf{x}, t) = P_0(\mathbf{x}) + \varepsilon P_\mathbf{e}(\mathbf{x}, t) + \mathcal{O}(\varepsilon^2). \qquad [20]$$

Upon inserting this ansatz in the Fokker-Planck Eq. (18), we find that, to first order in ε, the density has to satisfy

$$\partial_t P_\mathbf{e}(\mathbf{x}, t) = \mathcal{L}(\mathbf{x})[P_\mathbf{e}(\mathbf{x}, t)] + p(t)\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})], \qquad [21]$$

the (formal) solution of which is given by

$$P_\mathbf{e}(\mathbf{x}, t) = \int_{-\infty}^{t} p(t') e^{\mathcal{L}(\mathbf{x})(t-t')}\Big[\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})]\Big]dt'. \qquad [22]$$

Here the operator $e^{\mathcal{L}(\mathbf{x})(t-t')}$ acts on the full product within the square brackets. We can now use this general result to calculate the time-dependent mean value of the eigenfunction $Q_\lambda^*(\mathbf{x}(t))$ in response to the perturbation and obtain for $\lambda \neq 0$

$$\langle Q_\lambda^*(\mathbf{x}(t))\rangle = \int P(\mathbf{x}, t) Q_\lambda^*(\mathbf{x}) d\mathbf{x} = \int \big(P_0(\mathbf{x}) + \varepsilon P_\mathbf{e}(\mathbf{x}, t)\big) Q_\lambda^*(\mathbf{x}) d\mathbf{x} = \langle Q_\lambda^*(\mathbf{x})\rangle_0 + \varepsilon \int P_\mathbf{e}(\mathbf{x}, t) Q_\lambda^*(\mathbf{x}) d\mathbf{x} = \varepsilon \int_{-\infty}^{t} dt' p(t') K_{\mathbf{e},\lambda}(t - t'), \qquad [23]$$

where in the last step we have used Eq. (5) and also introduced the linear response function $K_{\mathbf{e},\lambda}(t-t')$ as

$$K_{\mathbf{e},\lambda}(t-t') = \int Q^*_\lambda(\mathbf{x}) e^{\mathcal{L}(\mathbf{x})(t-t')} \Big[\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})]\Big] d\mathbf{x}. \quad [24]$$

Note that, because of causality, the linear response function satisfies $K_{\mathbf{e},\lambda}(t-t') = 0$ for $t' > t$. Often, the susceptibility, i.e. the Fourier transform of $K_{\mathbf{e},\lambda}(t)$, is considered:

$$\chi_{\mathbf{e},\lambda}(\omega) = \int_{-\infty}^{\infty} K_{\mathbf{e},\lambda}(t) e^{-i\omega t} dt. \quad [25]$$

Eq. (23) reads then in the Fourier domain by a simple multiplication instead of a convolution:

$$\langle \tilde{Q}^*_\lambda(\omega)\rangle = \chi_{\mathbf{e},\lambda}(\omega)\tilde{p}(\omega). \quad [26]$$

Finally, by expanding by $\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})]$ in the basis of forward eigenfunctions

$$\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})] = \sum_{\lambda'} \beta_{\mathbf{e},\lambda'} P_{\lambda'}(\mathbf{x}), \quad [27]$$

we can simplify both the response function $K_{\mathbf{e},\lambda}(t)$

$$K_{\mathbf{e},\lambda}(\tau) = \int Q^*_\lambda(\mathbf{x}) e^{\mathcal{L}(\mathbf{x})\tau} \Big[\mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})]\Big] d\mathbf{x} = \sum_{\lambda'} \beta_{\mathbf{e},\lambda'} \int Q^*_\lambda(\mathbf{x}) e^{\mathcal{L}(\mathbf{x})\tau} \Big[P_{\lambda'}(\mathbf{x})\Big] d\mathbf{x} = \sum_{\lambda'} \beta_{\mathbf{e},\lambda'} e^{\lambda'\tau} \int Q^*_\lambda(\mathbf{x}) P_{\lambda'}(\mathbf{x}) d\mathbf{x} = \beta_{\mathbf{e},\lambda} e^{\lambda\tau} \quad [28]$$

and the susceptibility

$$\chi_{\mathbf{e},\lambda}(\omega) = \int_{-\infty}^{t} K_{\mathbf{e},\lambda}(t-t') e^{-i\omega(t-t')} dt' = \beta_{\mathbf{e},\lambda} \int_0^\infty e^{(\lambda - i\omega)\tau} d\tau = \frac{\beta_{\mathbf{e},\lambda}}{\lambda - i\omega} e^{(\lambda-i\omega)\tau}\Big|_0^\infty = -\frac{\beta_{\mathbf{e},\lambda}}{\lambda - i\omega}, \quad [29]$$

where the coefficient $\beta_{\mathbf{e},\lambda}$ can be obtained by multiplying Eq. (27) with Q^*_λ, integrating over $\mathbf{x}$ and using the bi-orthogonality relation Eq. (4) to obtain

$$\beta_{\mathbf{e},\lambda} = \int Q^*_\lambda(\mathbf{x}) \mathcal{L}_\mathbf{e}(\mathbf{x})[P_0(\mathbf{x})] d\mathbf{x}. \quad [30]$$

A better interpretation for the role of $\beta_{\mathbf{e},\lambda}$ can be given for the cases in which the stationary density $P_0(\mathbf{x})$ and also its products with the backward eigenfunctions, $Q^*_\lambda(\mathbf{x})P_0(\mathbf{x})$, vanish at $\pm\infty$. In this case, because

$$\int \partial_i (Q^*_\lambda(\mathbf{x}) P_0(\mathbf{x})) d\mathbf{x} = \int \partial_i (Q^*_\lambda(\mathbf{x})) P_0(\mathbf{x}) d\mathbf{x} + \int Q^*_\lambda(\mathbf{x}) \partial_i (P_0(\mathbf{x})) d\mathbf{x} = 0,$$

we find

$$\beta_{\mathbf{e},\lambda} = -\int Q^*_\lambda(\mathbf{x}) [\mathbf{e}\cdot\nabla P_0(\mathbf{x})] d\mathbf{x} = \int P_0(\mathbf{x}) [\mathbf{e}\cdot\nabla Q^*_\lambda(\mathbf{x})] d\mathbf{x} = \mathbf{e}\cdot\langle \nabla Q^*_\lambda(\mathbf{x})\rangle \quad [31]$$

that is, the response of a given eigenfunction is proportional to the mean change of $Q^*_\lambda(\mathbf{x})$ in the direction of the perturbation. As a final remark, notice that since $Q^*_0(\mathbf{x}) = 1$, then, $\beta_{\mathbf{e},0} \equiv 0$.

Fluctuation-dissipation theorem for the backwards eigenfunctions $Q^*_\lambda(\mathbf{x})$. From the already found expressions for the response function $K_{\mathbf{e},\lambda}(\tau)$ in Eq. (28) and the autocorrelation function $C_\lambda(\tau)$ (Eq. [11] in the main manuscript) it is easy to derive the following fluctuation-dissipation relationships for the backwards eigenfunctions $Q^*_\lambda(\mathbf{x})$

$$K_{\mathbf{e},\lambda}(\tau) = \beta_{\mathbf{e},\lambda} C_\lambda(\tau), \qquad \tau > 0 \quad [32]$$

Moreover, using the expressions for the power spectrum $S_\lambda(\omega)$ in Eq. (16) and the susceptibility $\chi_{\mathbf{e},\lambda}(\omega)$ in Eq. (29) we can write this result in the frequency domain as follows:

$$S_1(\omega) = \frac{2|\mu_\lambda|}{\Im[\beta_{\mathbf{e},\lambda}]\mu_\lambda + \Re[\beta_{\mathbf{e},\lambda}](\omega - \omega_\lambda)} \Im[\chi_{\mathbf{e},\lambda}(\omega)]. \quad [33]$$

This result is not equal but resembles somewhat the classical result for a system in thermodynamic equilibrium (3)

$$S_{xx} = \frac{2k_B T}{\omega} \Im[\chi_{x,F}(\omega)], \quad [34]$$

where k_B is the Boltzmann constant, T is absolute temperature and $\chi_{x,F}$ is the susceptibility with respect to a mechanical perturbation by a weak force $F(t)$. Note that one of our example systems, the harmonic oscillator, can be regarded as a system at thermodynamic equilibrium. Still, even for this system, the two equations above are not equivalent, because the new variable $Q^*_1(\mathbf{x})$ constitutes a linear combination of position and velocity variables, whereas Eq. (34) applies to the position variable only.

We would like to emphasize that our general FDT does not require thermodynamic equilibrium nor Markovian dynamics for $Q^*_1(\mathbf{x}(t))$ [only the full system $\mathbf{x}(t)$ is supposed to follow a Markovian dynamics].

Derivation of the spectral density equations in the coupled case. As we state in the main document, we denote the two oscillator dynamics with $\mathbf{x}$ and $\mathbf{y}$, couple them with scalar functions of particular forms, $H_{\mathbf{x}}(\mathbf{x},\mathbf{y}) = H_{\mathbf{xx}}(\mathbf{x}) + H_{\mathbf{yx}}(\mathbf{y})$ and $H_{\mathbf{y}}(\mathbf{x},\mathbf{y}) = H_{\mathbf{xy}}(\mathbf{x}) + H_{\mathbf{yy}}(\mathbf{y})$ along the directions $\mathbf{e_x}$ and $\mathbf{e_y}$, and scale the coupling terms by a small parameter ε. The equations for the coupled system read

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f_x}(\mathbf{x}) + \varepsilon \mathbf{e_x}[H_{\mathbf{xx}}(\mathbf{x}) + H_{\mathbf{yx}}(\mathbf{y})] + \mathbf{g_x}(\mathbf{x})\xi_{\mathbf{x}}(t),\\ \dot{\mathbf{y}} &= \mathbf{f_y}(\mathbf{y}) + \varepsilon \mathbf{e_y}[H_{\mathbf{xy}}(\mathbf{x}) + H_{\mathbf{yy}}(\mathbf{y})] + \mathbf{g_y}(\mathbf{y})\xi_{\mathbf{y}}(t).\end{aligned} \tag{35}$$

Here, $\xi_{\mathbf{x}}(t)$ and $\xi_{\mathbf{y}}(t)$ are independent vectors of white Gaussian noise. We note that the particular shape of the coupling function includes a simple diffusive coupling between the oscillators, e.g. a weak spring coupling between two harmonic oscillators of the form $\varepsilon(y-x)$. Formally, it would be possible in Eq. (35) to lump the terms $\varepsilon \mathbf{e_x} H_{\mathbf{xx}}(\mathbf{x})$ and $\varepsilon \mathbf{e_y} H_{\mathbf{yy}}(\mathbf{y})$ into the respective drift terms $\mathbf{f_x}(\mathbf{x})$ and $\mathbf{f_y}(\mathbf{y})$, leaving only perturbations of the respective other variables in the two equations. This kind of ansatz will be presented below as an alternative perturbation calculation. The disadvantage of the procedure is that a change in the drift terms implies a (ε-dependent) change of the Q_1^* functions of the single systems. For the calculation in the following, we regard all terms proportional to ε as perturbations of the isolated oscillator dynamics.

The new system of coupled oscillators is much more complicated than the dynamics of the single systems, and it is not obvious how we can describe it in terms of the eigenfunctions of the isolated oscillators, i.e. by the functions that have been so helpful in simplifying the description of the single oscillator's spontaneous and externally perturbed activity. We can, however, achieve a likewise striking simplification of the correlation statistics of weakly coupled oscillators in terms of the Q_1^* functions by a particular ansatz for the coupled system. To this end, we use the response functions Eq. (24) in a realisation-wise version

$$\begin{aligned}Q^*_{\lambda_{\mathbf{x}}} &= Q^*_{\lambda_{\mathbf{x}},0} + \varepsilon \int_{-\infty}^{t} dt' K_{\mathbf{e_x},\lambda_{\mathbf{x}}}(t-t')[H_{\mathbf{xx}}(\mathbf{x}(t')) + H_{\mathbf{yx}}(\mathbf{y}(t'))],\\ Q^*_{\lambda_{\mathbf{y}}} &= Q^*_{\lambda_{\mathbf{y}},0} + \varepsilon \int_{-\infty}^{t} dt' K_{\mathbf{e_y},\lambda_{\mathbf{y}}}(t-t')[H_{\mathbf{xy}}(\mathbf{x}(t')) + H_{\mathbf{yy}}(\mathbf{y}(t'))],\end{aligned} \tag{36}$$

where we have introduced the notation $Q^*_{\lambda_{\mathbf{x}}} = Q^*_{\lambda}(\mathbf{x})$, identifying the eigenfunctions of the $\mathbf{x}$-unit in Eq. (35) (similarly, we introduce $Q^*_{\lambda_{\mathbf{y}}} = Q^*_{\lambda}(\mathbf{y})$). The functions $Q^*_{\lambda_{\mathbf{x}},0}$ and $Q^*_{\lambda_{\mathbf{y}},0}$ in Eq. (36) denote the spontaneous activity of the uncoupled oscillator, respectively. By making the above ansatz, we implicitly assume that the effect of the coupling and the intrinsic noise can be subdivided into independent parts, which does not appear very plausible when the dynamics of the system is strongly nonlinear (as for our example systems of the noisy Stuart-Landau model and SNIC model). Nevertheless, in all tested cases (linear and nonlinear systems, more coherent and less coherent cases, identical oscillators and detuned oscillators), the cross-spectra and cross-correlation functions that can be analytically calculated using the ansatz Eq. (36) agree excellently with numerical simulation results.

We expand the coupling functions into the backward eigenfunctions as follows

$$\begin{aligned}H_{\mathbf{xx}}(\mathbf{x}) + H_{\mathbf{yx}}(\mathbf{y}) &= \sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} Q^*_{\lambda'_{\mathbf{x}}} + \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} Q^*_{\lambda'_{\mathbf{y}}},\\ H_{\mathbf{xy}}(\mathbf{x}) + H_{\mathbf{yy}}(\mathbf{y}) &= \sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} Q^*_{\lambda'_{\mathbf{x}}} + \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} Q^*_{\lambda'_{\mathbf{y}}},\end{aligned} \tag{37}$$

which, upon insertion into Eq. (36), leads to

$$\begin{aligned}Q^*_{\lambda_{\mathbf{x}}} &= Q^*_{\lambda_{\mathbf{x}},0} + \varepsilon\left(\sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} \int_{-\infty}^{t} dt' K_{\mathbf{e_x},\lambda_{\mathbf{x}}}(t-t') Q^*_{\lambda'_{\mathbf{x}}}(\mathbf{x}(t')) + \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} \int_{-\infty}^{t} dt' K_{\mathbf{e_x},\lambda_{\mathbf{x}}}(t-t') Q^*_{\lambda'_{\mathbf{y}}}(\mathbf{y}(t'))\right),\\ Q^*_{\lambda_{\mathbf{y}}} &= Q^*_{\lambda_{\mathbf{y}},0} + \varepsilon\left(\sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} \int_{-\infty}^{t} dt' K_{\mathbf{e_y},\lambda_{\mathbf{y}}}(t-t') Q^*_{\lambda'_{\mathbf{x}}}(\mathbf{x}(t')) + \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} \int_{-\infty}^{t} dt' K_{\mathbf{e_y},\lambda_{\mathbf{y}}}(t-t') Q^*_{\lambda'_{\mathbf{y}}}(\mathbf{y}(t'))\right).\end{aligned} \tag{38}$$

The coefficients $\alpha_{\lambda'_{\mathbf{x}}}, \alpha_{\lambda'_{\mathbf{y}}}, \gamma_{\lambda'_{\mathbf{x}}}, \gamma_{\lambda'_{\mathbf{y}}}$ are given by

$$\alpha_{\lambda'_{\mathbf{x}}} = \int P_{\lambda'_{\mathbf{x}}}(\mathbf{x}) H_{\mathbf{xy}}(\mathbf{x}) d\mathbf{x}, \qquad \alpha_{\lambda'_{\mathbf{y}}} = \int P_{\lambda'_{\mathbf{y}}}(\mathbf{y}) H_{\mathbf{yx}}(\mathbf{y}) d\mathbf{y}, \qquad \gamma_{\lambda'_{\mathbf{x}}} = \int P_{\lambda'_{\mathbf{x}}}(\mathbf{x}) H_{\mathbf{xx}}(\mathbf{x}) d\mathbf{x}, \qquad \gamma_{\lambda'_{\mathbf{y}}} = \int P_{\lambda'_{\mathbf{y}}}(\mathbf{y}) H_{\mathbf{yy}}(\mathbf{y}) d\mathbf{y}.$$

Eq. (38) attains a simpler form in the Fourier domain, where convolutions turn into multiplications with the respective susceptibilities:

$$\begin{aligned}\tilde{Q}^*_{\lambda_{\mathbf{x}}}(\omega) &= \tilde{Q}^*_{\lambda_{\mathbf{x}},0}(\omega) + \varepsilon \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}}(\omega)\Big(\sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} \tilde{Q}^*_{\lambda'_{\mathbf{x}}}(\omega) + \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} \tilde{Q}^*_{\lambda'_{\mathbf{y}}}(\omega)\Big),\\ \tilde{Q}^*_{\lambda_{\mathbf{y}}}(\omega) &= \tilde{Q}^*_{\lambda_{\mathbf{y}},0}(\omega) + \varepsilon \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}}(\omega)\Big(\sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} \tilde{Q}^*_{\lambda'_{\mathbf{x}}}(\omega) + \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} \tilde{Q}^*_{\lambda'_{\mathbf{y}}}(\omega)\Big).\end{aligned} \tag{39}$$

Multiplying both equations by $\tilde{Q}_{\lambda''_{\mathbf{x}}}$ and averaging yields

$$\begin{aligned}\langle \tilde{Q}^*_{\lambda_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle + \varepsilon \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle + \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle \Big), \\ \langle \tilde{Q}^*_{\lambda_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{y}},0} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle + \varepsilon \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle + \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle \Big),\end{aligned} \tag{40}$$

and, similarly, multiplying with $\tilde{Q}_{\lambda''_{\mathbf{y}}}$ and averaging gives

$$\begin{aligned}\langle \tilde{Q}^*_{\lambda_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle + \varepsilon \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle + \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle \Big), \\ \langle \tilde{Q}^*_{\lambda_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{y}},0} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle + \varepsilon \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle + \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle \Big).\end{aligned} \tag{41}$$

Next, we study the ε-dependence of the terms in Eq. (40). For the term $\langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle$, we can use $\tilde{Q}_{\lambda''_{\mathbf{x}}}$ in Eq. (39) to obtain

$$\begin{aligned}\langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \Big(\tilde{Q}_{\lambda''_{\mathbf{x}},0} + \varepsilon \chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \gamma^*_{\lambda'_{\mathbf{x}}} \tilde{Q}_{\lambda'_{\mathbf{x}}}(\omega) + \sum_{\lambda'_{\mathbf{y}}} \alpha^*_{\lambda'_{\mathbf{y}}} \tilde{Q}_{\lambda'_{\mathbf{y}}} \Big) \Big) \rangle \\ &= \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}},0} \rangle + \varepsilon \chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \Big(\sum_{\lambda'_{\mathbf{x}}} \gamma^*_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{x}},0} \rangle + \sum_{\lambda'_{\mathbf{y}}} \alpha^*_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{y}},0} \rangle \Big) + \mathcal{O}(\varepsilon^2) \\ &= \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}},0} \rangle + \varepsilon \chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{x}}} \gamma^*_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{x}},0} \rangle + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{42}$$

Here we have used in the last step, that the eigenfunctions of independent oscillators will be uncorrelated: $\langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{y}},0} \rangle = \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \rangle \langle \tilde{Q}_{\lambda'_{\mathbf{y}},0} \rangle = 0$. For the remaining terms in Eq. (40) and Eq. (41), we expand in a similar way to take only the leading linear order of ε into account, which results in the following expression:

$$\langle \tilde{Q}^*_{\lambda_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle = \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}},0} \rangle + \varepsilon \Big(\chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{x}}} \gamma^*_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{x}},0} \rangle + \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}},0} \rangle \Big) + \mathcal{O}(\varepsilon^2). \tag{43}$$

If we divide both sides by the time window T and take the limit $T \to \infty$, we arrive via Eq. (10) at spectral densities. Neglecting furthermore the higher-order terms in ε, we obtain for the cross-spectra of the eigenfunctions of the $\mathbf{x}$ oscillator in the presence of coupling (indicated here and below by a superscript c):

$$S^{\mathrm{c}}_{\lambda_{\mathbf{x}},\lambda''_{\mathbf{x}}} = S_{\lambda_{\mathbf{x}},\lambda''_{\mathbf{x}}} + \varepsilon \Big(\chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{x}}} \gamma^*_{\lambda'_{\mathbf{x}}} S_{\lambda_{\mathbf{x}},\lambda'_{\mathbf{x}}} + \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{x}}} \gamma_{\lambda'_{\mathbf{x}}} S_{\lambda'_{\mathbf{x}},\lambda''_{\mathbf{x}}} \Big). \tag{44}$$

These functions are to lowest (zeroth) order in ε simply given by the cross-spectra for the uncoupled oscillator, and, in first order of ε only affected by the self-coupling terms $H_{\mathbf{xx}}(\mathbf{x})$ and the related coefficient $\gamma^*_{\lambda'_{\mathbf{x}}}$.

More interesting than the statistics of the single oscillator are the cross-spectra between the two oscillators. From previous calculations, one obtains by the expansion in ε:

$$\langle \tilde{Q}^*_{\lambda_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{x}}} \rangle = \varepsilon \Big(\chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{y}}} \alpha^*_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{y}},0} \tilde{Q}_{\lambda'_{\mathbf{y}},0} \rangle + \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{x}},0} \tilde{Q}_{\lambda''_{\mathbf{x}},0} \rangle \Big) + \mathcal{O}(\varepsilon^2). \tag{45}$$

Again, by dividing by the time window T, letting $T \to \infty$, and neglecting higher-order terms in ε, we obtain expressions for spectral densities. Specifically, the cross-spectrum between eigenfunctions of the two coupled oscillators reads

$$S^{\mathrm{c}}_{\lambda_{\mathbf{y}},\lambda''_{\mathbf{x}}} = \varepsilon \Big(\chi^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{y}}} \alpha^*_{\lambda'_{\mathbf{y}}} S_{\lambda_{\mathbf{y}},\lambda'_{\mathbf{y}}} + \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{x}}} \alpha_{\lambda'_{\mathbf{x}}} S_{\lambda'_{\mathbf{x}},\lambda''_{\mathbf{x}}} \Big) \tag{46}$$

This result gives us the cross-spectrum of the two oscillators in terms of the susceptibilities of the two oscillators in the direction of the coupling force and in terms of the cross-spectra between the different modes in the single (uncoupled) stochastic oscillator. If we specifically choose the two eigenvalues $\lambda_{\mathbf{y}}$ and $\lambda''_{\mathbf{x}}$ to be the one with the least negative eigenvalue for the respective system (denoted by $\lambda_{1_{\mathbf{x}}}$ and $\lambda_{1_{\mathbf{y}}}$), we obtain the key result, Eq. [32] in the main text, for the cross-spectrum of the new observables $Q^*_{1_{\mathbf{x}}}(\mathbf{x}(t))$ and $Q^*_{1_{\mathbf{y}}}(\mathbf{y}(t))$.

Similar calculations for the second pair of equations in Eq. (41) yield

$$\begin{aligned}\langle \tilde{Q}^*_{\lambda_{\mathbf{x}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle &= \varepsilon \Big(\chi^*_{\mathbf{e_y},\lambda''_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{x}}} \alpha^*_{\lambda'_{\mathbf{x}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{x}},0} \tilde{Q}_{\lambda'_{\mathbf{x}},0} \rangle + \chi_{\mathbf{e_x},\lambda_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{y}}} \alpha_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}},0} \tilde{Q}_{\lambda''_{\mathbf{y}},0} \rangle \Big) + \mathcal{O}(\varepsilon^2) \\ \langle \tilde{Q}^*_{\lambda_{\mathbf{y}}} \tilde{Q}_{\lambda''_{\mathbf{y}}} \rangle &= \langle \tilde{Q}^*_{\lambda_{\mathbf{y}},0} \tilde{Q}_{\lambda''_{\mathbf{y}},0} \rangle + \varepsilon \Big(\chi^*_{\mathbf{e_y},\lambda''_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{y}}} \gamma^*_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda_{\mathbf{y}},0} \tilde{Q}_{\lambda'_{\mathbf{y}},0} \rangle + \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} \langle \tilde{Q}^*_{\lambda'_{\mathbf{y}},0} \tilde{Q}_{\lambda''_{\mathbf{y}},0} \rangle \Big) + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{47}$$

The first equation, upon renaming the eigenvalues appropriately, is just the complex conjugated of Eq. (45), while the second equation captures the linear corrections to the cross- and power spectra for the different eigenfunctions of the $\mathbf{y}$ oscillator

$$S^{\mathrm{c}}_{\lambda_{\mathbf{y}},\lambda''_{\mathbf{y}}} = S_{\lambda_{\mathbf{y}},\lambda''_{\mathbf{y}}} + \varepsilon \Big(\chi^*_{\mathbf{e_y},\lambda''_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{y}}} \gamma^*_{\lambda'_{\mathbf{y}}} S_{\lambda_{\mathbf{y}},\lambda'_{\mathbf{y}}} + \chi_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{y}}} \gamma_{\lambda'_{\mathbf{y}}} S_{\lambda'_{\mathbf{y}},\lambda''_{\mathbf{y}}} \Big) \tag{48}$$

which, we note, is the counterpart to Eq. (44).

Alternative treatment of the coupling equations. As we have stated in the beginning of the section, there is an alternative treatment for this problem consisting in lumping the diagonal terms $H_{\mathbf{xx}}(\mathbf{x})$ and $H_{\mathbf{yy}}(\mathbf{y})$ into the drift terms $\mathbf{f_x}(\mathbf{x})$ and $\mathbf{f_y}(\mathbf{y})$, respectively. More precisely, one can compute the backwards eigenfunctions $\hat{Q}^*_{\lambda'_{\mathbf{x}}}(\mathbf{x})$ of the following system

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}_{\mathbf{x}}(\mathbf{x}) + \mathbf{g_x}(\mathbf{x})\xi_{\mathbf{x}}(t), \quad \hat{\mathbf{f}}_{\mathbf{x}}(\mathbf{x}) = \mathbf{f_x}(\mathbf{x}) + \varepsilon \mathbf{e_x} H_{\mathbf{xx}}(\mathbf{x}) \tag{49}$$

and similarly obtain and $\hat{Q}^*_{\lambda'_{\mathbf{y}}}(\mathbf{y})$ from

$$\dot{\mathbf{y}} = \hat{\mathbf{f}}_{\mathbf{y}}(\mathbf{y}) + \mathbf{g_y}(\mathbf{y})\xi_{\mathbf{y}}(t), \quad \hat{\mathbf{f}}_{\mathbf{y}}(\mathbf{x}) = \mathbf{f_y}(\mathbf{y}) + \varepsilon \mathbf{e_y} H_{\mathbf{yy}}(\mathbf{y}). \tag{50}$$

In this situation, the ansatz in Eq. (36) becomes

$$\begin{aligned}\hat{Q}^*_{\lambda_{\mathbf{x}}} &= \hat{Q}^*_{\lambda_{\mathbf{x}},0} + \varepsilon \int_{-\infty}^{t} dt' \hat{K}_{\mathbf{e_x},\lambda_{\mathbf{x}}}(t-t') H_{\mathbf{yx}}(\mathbf{y}(t')), \\ \hat{Q}^*_{\lambda_{\mathbf{y}}} &= \hat{Q}^*_{\lambda_{\mathbf{y}},0} + \varepsilon \int_{-\infty}^{t} dt' \hat{K}_{\mathbf{e_y},\lambda_{\mathbf{y}}}(t-t') H_{\mathbf{xy}}(\mathbf{x}(t')).\end{aligned} \tag{51}$$

Put differently, the perturbation ansatz is now done only with respect to the true interaction between the oscillators. We note that, of course, also the response functions will now be modified (because they are calculated for the modified eigenfunctions $\hat{Q}^*_{\lambda_{\mathbf{y}},0}$) and are thus endowed with a hat.

We can now repeat the entire calculations as above, leading to cross-spectra between the eigenfunctions of one oscillator that are no longer affected by ε (because we have taken the effect of the *self-coupling terms* already into account). More importantly, similarly to Eq. (45) we find,

$$\langle \tilde{\hat{Q}}^*_{\lambda_{\mathbf{y}}} \tilde{\hat{Q}}_{\lambda''_{\mathbf{x}}} \rangle = \varepsilon \Big(\hat{\chi}^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{y}}} \hat{\alpha}^*_{\lambda'_{\mathbf{y}}} \langle \tilde{\hat{Q}}^*_{\lambda_{\mathbf{y}},0} \tilde{\hat{Q}}_{\lambda'_{\mathbf{y}},0} \rangle + \hat{\chi}_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{x}}} \hat{\alpha}_{\lambda'_{\mathbf{x}}} \langle \tilde{\hat{Q}}^*_{\lambda'_{\mathbf{x}},0} \tilde{\hat{Q}}_{\lambda''_{\mathbf{x}},0} \rangle \Big). \tag{52}$$

and hence, the cross-spectra of the eigenfunctions of the two oscillators are now given by

$$\hat{S}^{\mathrm{c}}_{\lambda_{\mathbf{y}},\lambda''_{\mathbf{x}}} = \varepsilon \Big(\hat{\chi}^*_{\mathbf{e_x},\lambda''_{\mathbf{x}}} \sum_{\lambda'_{\mathbf{y}}} \hat{\alpha}^*_{\lambda'_{\mathbf{y}}} \hat{S}_{\lambda_{\mathbf{y}},\lambda'_{\mathbf{y}}} + \hat{\chi}_{\mathbf{e_y},\lambda_{\mathbf{y}}} \sum_{\lambda'_{\mathbf{x}}} \hat{\alpha}_{\lambda'_{\mathbf{x}}} \hat{S}_{\lambda'_{\mathbf{x}},\lambda''_{\mathbf{x}}} \Big) \tag{53}$$

The hat on top of all parameters and functions is a reminder that these quantities have to be calculated for the modified drift terms $\hat{\mathbf{f}}_{\mathbf{x}}$ and $\hat{\mathbf{f}}_{\mathbf{y}}$. A comparison between the original and the alternative theories for the spectral measures of the coupled oscillators will be done somewhere else.

2. Computations for the damped stochastic oscillator

Eigenvalue, eigenfunction, and susceptibility of the single oscillator. A one-dimensional harmonic oscillator with unit mass[†] which is subject to Stokes friction and white Gaussian noise obeys the following equations

$$\dot{x} = v, \qquad \dot{v} = -\gamma v - \omega_0^2 x + \sqrt{2D}\xi(t). \tag{54}$$

For an oscillator in thermodynamic equilibrium, the noise intensity would be related to the dissipation strength via $D = \gamma k_B T$ with k_B being the Boltzmann constant and T absolute temperature; we will, however, use only the noise intensity in the following.

[†] The dependence on the mass could be easily restored by using parameters $\hat{\gamma} = \gamma/M$ and $\hat{D} = D/M^2$ in place of γ and D, respectively.

The associated Fokker-Planck equation

$$\partial_t P(x,v,t) = -v\partial_x P(x,v,t) + \partial_v \left(\gamma v + \omega_0^2 x + D\partial_v\right) P(x,v,t) \quad [55]$$

has the well-known Maxwell-Boltzmann distribution as its stationary solution

$$P_0(x,v) = \frac{\gamma}{2\pi\omega_0 D} \exp\left[-\frac{\omega_0^2 x^2 + v^2}{2D/\gamma}\right]. \quad [56]$$

We will assume an oscillator in the underdamped regime ($\gamma < 2\omega_0$); in the absence of noise, the time evolution is then described by two complex conjugated eigenvalues:

$$\lambda_1 = \mu_1 + i\omega_1 = -\frac{\gamma}{2} + i\frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}, \quad \lambda_1^* = \mu_1 - i\omega_1. \quad [57]$$

As our notation suggests, λ_1 happens to be the eigenvalue with the least negative real part for the Fokker-Planck equation (2).

The linear change of variables

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -\gamma & -\sqrt{4\omega_0^2 - \gamma^2} \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} \quad [58]$$

transforms Eq. (54) to another two-dimensional Ornstein-Uhlenbeck process

$$\begin{pmatrix} \dot{z} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \mu_1 & -\omega_1 \\ \omega_1 & \mu_1 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} - \frac{1}{2\omega_1} \begin{pmatrix} 0 \\ \varepsilon p(t) + \sqrt{2D}\xi(t). \end{pmatrix} \quad [59]$$

that has the same eigenvalues in the deterministic case and the same eigenvalue spectrum in the stochastic case. According to (4), the slowest decaying complex eigenfunction attains the simple form:

$$Q_1^*(z,u) = \sqrt{\frac{4|\mu_1|\omega_1^2}{D}}(z + iu) \quad [60]$$

(where the prefactor ensures the condition $\left\langle |Q_1^*(\mathbf{x}(t))|^2 \right\rangle = 1$ is met). Inverting the change of variables in Eq. (58), we recover an expression for Q_1^* in the x, v variables (see also chap. 10.2.2 in the textbook by Risken (2))

$$Q_1^*(x,v) = \sqrt{\frac{|\mu_1|}{D}} \left(\omega_1 x + i\left(\mu_1 x - v\right)\right). \quad [61]$$

The complex argument of this function is easily computed

$$\psi(x,v) = \arctan\left(\frac{\mu_1 x - v}{\omega_1 x}\right) \quad [62]$$

and has been plotted in Fig. 1a and Fig. 2a of the main text.

As we now know $Q_1^*(x,v)$ and λ_1, it is simple to find the expression for the susceptibility in the v-direction with $\mathbf{e} = (0,1)^T$ using Eq. (29) and Eq. (31)

$$\chi(\omega) = \frac{i\sqrt{|\mu_1|/D}}{\mu_1 + i(\omega_1 - \omega)}. \quad [63]$$

Two weakly coupled damped stochastic oscillators. We now consider the case of two identical harmonic oscillators that are weakly coupled ($|\varepsilon| \ll 1$) by a spring

$$\begin{aligned} \ddot{x} + \gamma\dot{x} + \omega_0^2 x &= \varepsilon(y - x) + \sqrt{2D}\xi_x(t), \\ \ddot{y} + \gamma\dot{y} + \omega_0^2 y &= \varepsilon(x - y) + \sqrt{2D}\xi_y(t). \end{aligned} \quad [64]$$

First, we compute for this linear system the cross-spectrum between the position variables x and y. To this end, if we consider a single unit ($\varepsilon = 0$) and take the Fourier transform; we obtain

$$\tilde{x} = \frac{\sqrt{2D}}{\omega_0^2 - \omega^2 - i\gamma\omega}\tilde{\xi}_x = \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}\sqrt{2D}\tilde{\xi}_1 = \chi(\omega)\sqrt{2D}\tilde{\xi}_x. \quad [65]$$

and similarly for $\tilde{y}$. For this linear system, the response to an external perturbation is similar to the response to the noise and we can identify the susceptibility $\chi(\omega)$ from the above relation (in nonlinear stochastic systems, the susceptibility will also depend on the level of background noise and has to be calculated by a perturbation calculation of the Fokker-Planck equation).

Considering now the coupled system, we may write

$$\begin{aligned} \tilde{x} &= \chi(\omega)\left(\varepsilon(\tilde{y} - \tilde{x}) + \sqrt{2D}\tilde{\xi}_x\right), \\ \tilde{y} &= \chi(\omega)\left(\varepsilon(\tilde{x} - \tilde{y}) + \sqrt{2D}\tilde{\xi}_y\right). \end{aligned} \quad [66]$$

This linear system can be easily solved by Rice method for the matrix of cross-spectra (see (2)); however, because for weak coupling we are only interested in the lowest order term in ε, we use the above equations recursively and omit high order terms. By doing so, we obtain

$$\begin{aligned}\tilde{x} &= \sqrt{2D}\left[\chi(\omega)\tilde{\xi}_x + \varepsilon\chi^2(\omega)\tilde{\xi}_y - \varepsilon\chi^2(\omega)\tilde{\xi}_x\right] + \mathcal{O}(\varepsilon^2),\\ \tilde{y} &= \sqrt{2D}\left[\chi(\omega)\tilde{\xi}_y + \varepsilon\chi^2(\omega)\tilde{\xi}_x - \varepsilon\chi^2(\omega)\tilde{\xi}_y\right] + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{67}$$

from which we can calculate the cross-spectrum (using that the noise sources in $\mathbf{x}$ and $\mathbf{y}$ are uncorrelated, $\langle\tilde{\xi}_x\tilde{\xi}_y^*\rangle/T_w = \delta_{x,y}$):

$$S_{\mathbf{xy}}(\omega) = \frac{\langle\tilde{x}\tilde{y}^*\rangle}{T_w} \approx 2D\varepsilon|\chi|^2\left(\chi+\chi^*\right) = 4D\varepsilon\Re\left(|\chi|^2\chi^*\right). \tag{68}$$

It is instructive to compare this cross-spectrum of the original variables to the one for the transformed variables that we are going to calculate now.

In order to obtain the cross spectrum $S^{\mathrm{c}}_{1,\mathbf{xy}}$ of system Eq. (64) for the eigenfunctions $Q^*_{1_\mathbf{x}}$, $Q^*_{1_\mathbf{y}}$, we first notice that we can take advantage of the analytical expression for $Q^*_1(\mathbf{x})$ (see Eq. (61)), to find that the coupling function is exactly given by a linear combination of $Q^*_1(\mathbf{x})$ and $Q_1(\mathbf{x})$. Indeed,

$$H_{\mathbf{xy}}(\mathbf{x}) = x = \sqrt{\frac{D}{4|\mu_1|\omega_1^2}}\left(Q^*_{1_\mathbf{x}} + Q_{1_\mathbf{x}}\right), \qquad H_{\mathbf{yx}}(\mathbf{y}) = y = \sqrt{\frac{D}{4|\mu_1|\omega_1^2}}\left(Q^*_{1_\mathbf{y}} + Q_{1_\mathbf{y}}\right) \tag{69}$$

and, in consequence the α coefficients of the expansion Eq. (37) (those which are relevant for the cross-spectrum between the oscillators, Eq. (46)) are known:

$$\alpha_1 = \alpha_{1^*} = \sqrt{\frac{D}{4|\mu_1|\omega_1^2}}, \qquad \alpha_{\lambda'} = 0 \quad \text{for} \quad \lambda' \neq \{\lambda_1, \lambda_1^*\} \tag{70}$$

and in consequence, from Eq. (46) we obtain that the cross-spectrum can be expressed as

$$\begin{aligned}S^{\mathrm{c}}_{1_\mathbf{y},1_\mathbf{x}} &= \varepsilon\left(\chi^*_{\mathbf{e_x}}\sum_{\lambda'_\mathbf{y}}\alpha^*_{\lambda'_\mathbf{y}}S_{1_\mathbf{y},\lambda'_\mathbf{y}} + \chi_{\mathbf{e_y}}\sum_{\lambda'_\mathbf{x}}\alpha_{\lambda'_\mathbf{x}}S_{\lambda'_\mathbf{x},1_\mathbf{x}}\right) = \varepsilon\sqrt{\frac{D}{|\mu_1|\omega_1^2}}\Re\left(\chi^*_\mathbf{e}[S_1 + S_{1,1^*}]\right)\\ &= \frac{2\varepsilon}{\omega_1}\Re\left(\frac{i}{|\mu_1| + i(\omega_1-\omega)}\left[\frac{|\mu_1|}{\mu_1^2+(\omega_1-\omega)^2} - \langle Q_1^*Q_1^*\rangle\frac{\lambda_1}{(\mu_1+i(\omega_1+\omega))(\mu_1+i(\omega_1-\omega))}\right]\right)\end{aligned} \tag{71}$$

Finally we state how to obtain the covariance $\langle Q_1^*Q_1^*\rangle$ of a single unit (which is needed to compute the cross-spectrum $S_{1,1^*}$ in the above formula):

$$\langle Q_1^*Q_1^*\rangle = \omega_1^2\frac{|\mu_1|}{D}\left\langle\left(x + i\frac{(\mu_1 x - v)}{\omega_1}\right)\left(x + i\frac{(\mu_1 x - v)}{\omega_1}\right)\right\rangle = \omega_1^2\frac{|\mu_1|}{D}\left(\left\langle x^2\right\rangle\left(1 + i\frac{\mu_1}{\omega_1}\right)^2 - \frac{\left\langle v^2\right\rangle}{\omega_1^2}\right) = \frac{1}{2}\left(\frac{\omega_1^2}{\omega_0^2}\left(1+i\frac{\mu_1}{\omega_1}\right)^2 - 1\right)$$

where we have used $\langle xv\rangle = 0$ and read off the variances $\left\langle x^2\right\rangle = D/(\gamma\omega_0^2)$ and $\left\langle v^2\right\rangle = D/\gamma$ from the stationary distribution Eq. (56).

3. Computations for the coupled Stuart-Landau model

Next, we consider two different noisy Stuart-Landau oscillators diffusively coupled by their first coordinates x_1 and y_1

$$\begin{aligned}\dot{x}_1 &= a_x x_1 - x_2 - a_x(x_1^2+x_2^2)(x_1 + b_x x_2) + \sqrt{2D}\xi_{x_1}(t) + \varepsilon(y_1 - x_1),\\ \dot{x}_2 &= a_x x_2 + x_1 - a_x(x_1^2+x_2^2)(x_2 - b_x x_1) + \sqrt{2D}\xi_{x_2}(t),\\ \dot{y}_1 &= a_y y_1 - y_2 - a_y(y_1^2+y_2^2)(y_1 + b_y y_2) + \sqrt{2D}\xi_{y_1}(t) + \varepsilon(x_1 - y_1),\\ \dot{y}_2 &= a_y y_2 + y_1 - a_y(y_1^2+y_2^2)(y_2 - b_y y_1) + \sqrt{2D}\xi_{y_2}(t),\end{aligned} \tag{72}$$

As we state in the main text, we study two cases to inspect how inhomogeneities of the oscillators affect the cross-spectrum: we set parameters in case (i) $a_x = 1, b_x = -0.3, a_y = 1, b_y = -0.25, D = 0.04$, so both oscillators are slightly detuned ($\lambda_{1_\mathbf{x}} = -0.048 + 0.698i$, $\lambda_{1_\mathbf{y}} = -0.047 + 0.748i$) and in case (ii) $a_x = 1, b_x = -0.3, a_y = 1, b_y = -0.1, D = 0.04$, so the two oscillators are more strongly detuned ($\lambda_{1_\mathbf{x}} = -0.048 + 0.698i$, $\lambda_{1_\mathbf{y}} = -0.047 + 0.9i$, note that the frequencies $\omega_{1_\mathbf{x}} = 0.698$ and $\omega_{1_\mathbf{y}} = 0.9$ differ more than in case (i)).

As all quality factors in this example are small and the system is rotationally symmetric, we expect very few modes to contribute. We check this hypothesis comparing the output of numerical simulations with the reduction of $S^{\mathrm{c}}_{1,\mathbf{yx}}$ to the mere power spectrum contribution

$$S^{\mathrm{c}}_{1,\mathbf{yx}} = \varepsilon\left(\chi^*_{\mathbf{e_x}}\alpha^*_{1_\mathbf{y}}S_{1_\mathbf{y}} + \chi_{\mathbf{e_y}}\alpha_{1_\mathbf{x}}S_{1_\mathbf{x}}\right). \tag{73}$$

To test this prediction, we first obtain numerically the necessary forward and backward eigenfunctions (see Section 4B for full details of the procedure). We then use the forward eigenfunctions $P_{1_{\mathbf{x}}}(\mathbf{x}), P_{1_{\mathbf{y}}}(\mathbf{y})$ to compute the expansion coefficients

$$\alpha_{1_{\mathbf{x}}} = \int P_{1_{\mathbf{x}}}(\mathbf{x}) x_1 d\mathbf{x} \qquad \alpha_{1_{\mathbf{y}}} = \int P_{1_{\mathbf{y}}}(\mathbf{y}) y_1 d\mathbf{y}; \tag{74}$$

for the given parameters, this leads to the values $\alpha_{1_{\mathbf{x}}} \approx 0.497, \alpha_{1_{\mathbf{y}}} \approx 0.4965$ for the slightly detuned case (i) and $\alpha_{1_{\mathbf{x}}} \approx 0.497, \alpha_{1_{\mathbf{y}}} \approx 0.4956$ for the more strongly detuned case (ii).

Next, we use the numerically obtained backwards eigenfunctons $Q^*_{1_{\mathbf{x}}}(\mathbf{x}), Q^*_{1_{\mathbf{y}}}(\mathbf{y})$ to compute the susceptibility coefficients

$$\beta_{1_{\mathbf{x}}} = -\int Q^*_{1_{\mathbf{x}}}(\mathbf{x}) \partial_{x_1} P_0(\mathbf{x}) d\mathbf{x} \qquad \beta_{1_{\mathbf{y}}} = -\int Q^*_{1_{\mathbf{y}}}(\mathbf{y}) \partial_{y_1} P_0(\mathbf{y}) d\mathbf{y}. \tag{75}$$

finding the values $\beta_{1_{\mathbf{x}}} \approx 0.55 - 0.153i, \beta_{1_{\mathbf{y}}} \approx 0.55 - 0.128i$ for the slightly detuned case (i) and $\beta_{1_{\mathbf{x}}} \approx 0.55 - 0.153i, \beta_{1_{\mathbf{y}}} \approx 0.55 - 0.05i$ for the more strongly detuned case (ii).

Once we have the necessary coefficients, we can compare Eq. (73) with numerical simulations. As we showed in the main text, Eq. (73) agrees very well with numerical simulations for both cases.

4. Computations for the coupled SNIC system

We consider two identical SNIC systems diffusively coupled by their first coordinates x_1 and y_1

$$\begin{aligned}
\dot{x}_1 &= nx_1 - mx_2 - x_1(x_1^2 + x_2^2) + \frac{x_2^2}{\sqrt{x_1^2 + x_2^2}} + \sqrt{2D}\xi_{x_1}(t) + \varepsilon(y_1 - x_1), \\
\dot{x}_2 &= mx_1 + nx_2 - x_2(x_1^2 + x_2^2) - \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} + \sqrt{2D}\xi_{x_2}(t), \\
\dot{y}_1 &= ny_1 - my_2 - y_1(y_1^2 + y_2^2) + \frac{y_2^2}{\sqrt{y_1^2 + y_2^2}} + \sqrt{2D}\xi_{y_1}(t) + \varepsilon(x_1 - y_1), \\
\dot{y}_2 &= my_1 + ny_2 - y_2(y_1^2 + y_2^2) - \frac{y_1 y_2}{\sqrt{y_1^2 + y_2^2}} + \sqrt{2D}\xi_{y_2}(t)
\end{aligned} \tag{76}$$

We test our theory here for the cases of two rather coherent oscillators with high quality factor (referred to as the coherent case) and for two less coherent oscillators with a smaller quality factor, set in the excitable regime (the less coherent case). In contrast to the previous subsection, here, the two coupled oscillators have each the same parameters.

A. The coherent case. First, we set parameters as follows: $m = 1.216, n = 1.014, D = 0.01125$ so the system is in the oscillatory regime and the quality factor is high. In this case, the slowest decaying eigenvalue is given by $\lambda_1 = -0.048 + 0.698i$. Similarly to the previously studied Stuart-Landau case, we also expect very few modes to contribute and hence we start by comparing the output of numerical simulations with the reduction of $S^{\mathrm{c}}_{1,\mathbf{yx}}$ to the power spectrum contribution

$$S^{\mathrm{c}}_{1,\mathbf{yx}} = 2\varepsilon \Re\Big(\chi_{\mathbf{e}} \alpha_1 S_1\Big), \tag{77}$$

where for symmetry reasons, we drop the $\mathbf{x}, \mathbf{y}$ indices and obtain a purely real-valued cross-spectrum.

In line with the procedure in Section 3, we just need to compute numerically the following integrals

$$\alpha_1 = \int P_1(\mathbf{x}) x_1 d\mathbf{x} \qquad \beta_1 = -\int Q^*_1(\mathbf{x}) \partial_{x_1} P_0(\mathbf{x}) d\mathbf{x} \tag{78}$$

finding the values $\alpha_1 \approx 0.24 - 0.284i, \beta_1 \approx 0.562 + 0.68i$, to compare Eq. (77) with numerical simulations. As we showed in the main manuscript, Eq. (77) agrees very well with numerical simulations.

B. The less coherent (excitable) case. If we set the parameters as follows $m = 0.99, n = 1, D = 0.01125$, the system is in the excitable regime and produces less coherent oscillation with a low quality factor. In this case, the slowest decaying eigenvalue is given by $\lambda_1 = -0.168 + 0.241i$. As we discuss in the main text, if we were using just the power spectrum approximation in Eq. (77) with the coefficients from Eq. (78) thus obtaining $\alpha_1 \approx 0.158 - 0.21i, \beta_1 \approx 1.38 + 1.3i$, this provides us with a quantitatively inaccurate approximation of the simulations output. It turns out that in this less coherent case we need to add more modes to the approximation

$$S^{\mathrm{c}}_{1,\mathbf{yx}} = 2\varepsilon \Re\Big(\chi^*_{\mathbf{e}} \alpha^*_1 S_1 + \chi^*_{\mathbf{e}} \sum_{\lambda' \in \Lambda} \alpha^*_\lambda S_{1,\lambda'}\Big) \qquad \text{for } \Lambda = \{\lambda^*_1, \lambda_2, \lambda_3, \lambda_4\}. \tag{79}$$

Further inspection reveals that we need to take into account the following set of eigenvalues $\Lambda = \{\lambda^*_1 = -0.168 - 0.241i, \lambda_2 = -0.423 + 0.638i, \lambda_3 = -0.728 + 1.108i, \lambda_4 = -1.074 + 1.63i\}$ and thus compute for each of them the following integrals

$$\alpha_\lambda = \int P_\lambda(\mathbf{x}) x_1 d\mathbf{x} \qquad \langle Q^*_1 Q_\lambda \rangle = \int P_0(\mathbf{x}) Q_\lambda(\mathbf{x}) Q^*_1(\mathbf{x}) d\mathbf{x} \tag{80}$$

in order to achieve a good agreement between simulations and theory. The numerical values that enter the theory are as follows: $\alpha_{1^*} = 0.158 + 0.21i, \alpha_2 = 0.218 - 0.214i, \alpha_3 = 0.285 - 0.193i, \alpha_4 = 0.343 - 0.1417i$ and $\langle Q_1^* Q_1^* \rangle = -0.08 + 0.068i, \langle Q_1^* Q_2 \rangle = -0.416 + 0.355i, \langle Q_1^* Q_3 \rangle = 0.07 - 0.23i, \langle Q_1^* Q_4 \rangle = 0.023 + 0.11i$.

5. Numerical computations

A. Stochastic simulations. The stochastic simulations yielding the results in this manuscript were performed using the stochastic Heun method (5) with a time step h such that $10^{-2} < h < 10^{-3}$.

B. Determination and normalization of the forward and backward eigenfunctions. To generate the numerical results in the main text, we followed the procedure in (6) (see also (7) and (8)). All our examples are two-dimensional $n = 2$. Given the eigenvalue equations in Eq. (4) for a function $T(x_1, x_2)$ (which can be $Q_\lambda^*(x_1, x_2)$ or $P_\lambda(x_1, x_2)$) with the operators as implicitly defined in Eq. (2) and Eq. (3), we first chose a (finite) rectangular domain

$$\mathcal{X} = [x_1^-, x_1^+] \times [x_2^-, x_2^+]. \tag{81}$$

Since the phase space for all the systems that we consider in this manuscript is unbounded, we consider a truncated domain $\mathcal{X}$ whose size is chosen large enough so that the probability for individual trajectories $\mathbf{x}(t)$ to reach the boundaries is very low[‡]. Then, we just need to discretise the domain $\mathcal{X}$ in N and M points such that $\Delta x_1 = (x_1^+ - x_1^-)/N$ and $\Delta x_2 = (x_2^+ - x_2^-)/M$, to build $\mathcal{L}^\dagger$ (and/or $\mathcal{L}$) by using a standard finite-difference scheme. In general, we used centered finite differences, for instance

$$(\partial_{x_1} T)_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x_1}, \qquad (\partial_{x_1 x_1} T)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x_1)^2}, \tag{82}$$

and, as for boundary conditions, we used adjoint reflecting boundary conditions

$$\sum_{j=1,2} n_j \sum_{k=1,2} D_{jk} \partial_{x_k} T(x_1, x_2) = 0 \tag{83}$$

where n is the local unit normal vector at $\mathcal{X}$ boundaries and $D = \frac{1}{2} g g^\intercal$ (see Eq. (2)).

After diagonalizing the resulting $(N \cdot M, N \cdot M)$ matrix, we obtain the eigenvalues and the associated eigenfunctions of $\mathcal{L}^\dagger$ ($\mathcal{L}$). We recall that we are not interested in the complete spectrum of $\mathcal{L}^\dagger$ ($\mathcal{L}$). For $\mathcal{L}^\dagger$ we just consider (and hence present in main text) the part of the spectrum which is relevant for our analysis. That is, we consider mainly the eigenvalue associated with the slowest decaying complex eigenfunction $Q_1^*(\mathbf{x})$ and a few higher backward modes (and its corresponding forward eigenmodes) to study the coupled case. For $\mathcal{L}$ we mainly consider the eigenmode associated with the eigenvalue $\lambda = 0$ which gives the stationary probability distribution P_0 and a few forward eigenmodes for the coupled case.

We finally state that we normalised each pair of eigenfunctions $Q_\lambda^*(\mathbf{x})$ and $P_\lambda(\mathbf{x})$ such that they satisfy the previously mentioned conditions Eq. (4) and Eq. (7))

$$\int d\mathbf{x} |Q_\lambda^*(\mathbf{x})|^2 P_0(\mathbf{x}) = 1, \qquad \int Q_{\lambda'}^*(\mathbf{x}) P_\lambda(\mathbf{x}) d\mathbf{x} = \delta_{\lambda' \lambda}, \tag{84}$$

so, while the left integral fixes the normalisation of $Q_\lambda^*(\mathbf{x})$ up to an arbitrary complex factor, the second integral fixes the norm and phase of $P_\lambda(\mathbf{x})$.

C. Computing the susceptibility functions. If we consider the effect of a weak external perturbations $\varepsilon p(t)\mathbf{e}$ (with $|\varepsilon| \ll 1$) on the SDE Eq. (1),

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \varepsilon p(t)\mathbf{e} + \mathbf{g}(\mathbf{x})\xi(t) \qquad \mathbf{x}, \mathbf{e} \in \mathbb{R}^n \tag{85}$$

we can follow the well established linear response theory (see Eq. (17) to Eq. (26) and Risken Ch. 7 (2)). Specifically, the evolution of the time-dependent mean value of a given observable $Z(\mathbf{x}(t))$ is given by a convolution of the linear response function with the perturbation (see e.g. eq.(18) in the main text). This convolution relation turns into a simple multiplication in the Fourier domain

$$\langle \tilde{z} \rangle = \chi_z(\omega) \tilde{p}, \tag{86}$$

where $\tilde{p}$ is the Fourier transform of the perturbation and $\chi_z(\omega)$ is the susceptibility, i.e. the Fourier transform of the linear response function. Considering a stochastic perturbation (that is unrelated to the intrinsic noise in the driven system), Eq. (86) leads after mulitiplication with $\tilde{p}$ and averaging over both ensembles of intrinsic noise and stochastic perturbation to the well-known relation between susceptibility and cross- and power spectra:

$$\chi_z(\omega) = \frac{S_{zp}(\omega)}{S_{pp}(\omega)}. \tag{87}$$

In order to estimate the susceptibility numerically, we generated many trials of a bandpass-limited white Gaussian noise process $p(t)$ (9) (also explained in detail in (10)), apply it to the Langevin dynamics that is in each trial run with independent

[‡] See (6–8) for examples of application of this methodology in systems, as the noisy heteroclinic oscillator, in which the phase is not bounded.

realizations of intrinsic white noise, and measure the cross-spectra of the Q_1^* function and the weak perturbation. From Eq. (87) we can then obtain the complex-valued susceptibility function. By repeating the whole procedure for different values of the perturbation-amplitude, ε, we ensure that we are indeed in the linear-response regime, i.e. for sufficiently small values of the amplitude there is no systematic dependence of the resulting susceptibility on ε.

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