E10: Two-Soliton Solution of the KdV Equation (10 Points)

(a) Consider the two-soliton solution of the KdV equation as a function of the parameters $\chi_1, \chi_2, \beta_1(0), \beta_2(0)$ as well as of $t, x$. Assume that the scattering data consists of two energy levels $E_1 = -\chi_1^2$ and $E_2 = -\chi_2^2$ with $\chi_1 > \chi_2$ and that the reflection coefficient vanishes. Solve the GLM equation.

*Hint:* Follow the one-soliton solution determined in the lectures. Look for solutions to the GLM equation of the form $K(x, y) = K_1(x)e^{-\chi_1 y} + K_2(x)e^{-\chi_2 y}$ and note that the functions $e^{ax}$ and $e^{bx}$ are linearly independent for different constants $a$ and $b$.

(b) Show that the two-soliton solution can be written in the form

$$
\phi(t, x) = -2\frac{\partial^2}{\partial x^2} \log[\det A(x)], \quad A_{mn}(x) = \delta_{mn} + \frac{\beta_m e^{-(\chi_m + \chi_n)x}}{\chi_m + \chi_n},
$$

where $A$ denotes the $2 \times 2$ matrix defined by the elements $A_{mn}$.

*Remark:* This is the generic form of the $N$-soliton solution for an $N \times N$ matrix $A$.

(c) **Mathematica:** Use Mathematica to check that the obtained solution indeed satisfies the KdV equation (print and hand in the implementation).

*Hint:* Define a function \( \Phi[t, x, \chi_1, \chi_2, \beta_{10}, \beta_{20}] := \ldots \) in Mathematica, with the \( \ldots \) replaced by your two-soliton solution. Implement the derivatives in the KdV equation using for instance

\[
D[\Phi[t, x, \chi_1, \chi_2, \beta_{10}, \beta_{20}], \{x, 2}\]
\]

for the second derivative in $x$. Use `Simplify[...]` to simplify your expressions. Mathematica knows a function `KroneckerDelta[m, n]`. `Log[...]` is the natural logarithm, `Det[...]` the determinant. The $2 \times 2$ matrix $A$ can be defined by \( A = \{\ldots, \ldots, \ldots\} \) with the dots replaced by the matrix elements. It is often helpful to use replacement rules such as \( \ldots \beta[1] \rightarrow \beta[10]*\text{Exp}[8 \chi_1^3 t] \), which replaces every term $\beta[1]$ in \( \ldots \) by the right hand side of $->$.

(d) **Mathematica:** Plot the two-soliton solution $\Phi[t, x, 1, 2, 3, 4]$ obtained in (a) for $\chi_1=1$, $\chi_2=2$, $\beta_{10}=3$, $\beta_{20}=4$ as a function of $x$ and for different times $t$ using Mathematica and the command:

\[
\begin{align*}
f[0][x_] &= -\Phi[0, x, 1, 2, 3, 4]; \\
f[1][x_] &= -\Phi[0.5, x, 1, 2, 3, 4]; \\
f[2][x_] &= -\Phi[1.0, x, 1, 2, 3, 4]; \\
Plot[{f[0][x], f[1][x], f[2][x]}, \{x, -10, 10\}, \text{PlotRange} \rightarrow \{-1, 10\}]
\end{align*}
\]

To enjoy the result of your hard work try the following and play with the time slider:

\[
\begin{align*}
f[t, x_] &= -\Phi[t, x, 1, 2, 3, 4]; \\
\text{Manipulate}[\text{Plot}[f[t, x], \{x, -10, 10\}, \text{PlotRange} \rightarrow \{-1, 10\}], \{t, -2, 2\}]
\end{align*}
\]

*Hint:* In order to understand the above Mathematica commands in more detail, you can find useful explanations in the Mathematica-Menu under Help $\rightarrow$ Wolfram Documentation.
E11: Lax Formulation of KdV Equation (4 Points)

Consider the operators
\[ L = -\frac{d^2}{dx^2} + \phi(t, x), \quad M = 4\frac{d^3}{dx^3} - 3\left[ \phi(t, x)\frac{d}{dx} + \frac{d}{dx}\phi(t, x) \right]. \]

Show that the Lax equation
\[ L_t = [L, M] \]

is equivalent to the KdV equation by acting on a test function.

E12: Lax and Zero-Curvature Representation (6 Points)

Let \( L = -\partial_x^2 + \phi(t, x) \) be a Schrödinger operator with a real potential \( \phi \) and let another operator \( M \) have the form
\[ M = a_n\partial_x^n + \cdots + a_1\partial_x + a_0, \]
with \( a_k = a_k(t, x) \). Assume that \( L_t = [L, M] \).

(a) Show that the eigenvalues of \( L \) are independent of \( t \).

(b) Let \( f \) be an eigenfunction of \( L \) corresponding to an eigenvalue \( u \) which is non-degenerate. Show that there exists a function \( \hat{f} = \hat{f}(t, x, u) \) such that
\[ L\hat{f} = u\hat{f}, \quad \hat{f}_t + M\hat{f} = 0. \]

\textit{Hint:} It may be useful to define an integrating factor \( A(t) = \exp\left[-\int_{t_0}^{t} c(t')dt'\right] \) and to use its properties under the time derivative.

(c) Now assume that \( n = 3 \) and \( a_3 = 1, a_2 = 0 \). Show that the Lax representation (1) yields a zero-curvature representation with
\[ \partial_t U_L - \partial_x V_M + [U_L, V_M] = 0, \]
where \( U_L \) and \( V_M \) are some \( 2 \times 2 \) matrices which should be determined.

\textit{Hint:} Consider (2) as a system of first-order differential equations on a pair of functions \((\hat{f}, \partial_x \hat{f})\).