**E13: Gauge Freedom of Zero-Curvature and Lax Equation (3 Points)**

Consider the zero-curvature and Lax equations for matrices \((U, V)\) and \((L, M)\), respectively, as well as an invertible matrix \(g = g(t, x)\).

(a) Show that the zero curvature equation is invariant under the gauge transformation
\[
U \rightarrow g U g^{-1} + \frac{dg}{dx} g^{-1}, \quad V \rightarrow g V g^{-1} + \frac{dg}{dt} g^{-1}.
\]

(b) Show that the Lax equation is invariant under
\[
L \rightarrow g L g^{-1}, \quad M \rightarrow g M g^{-1} + \frac{dg}{dt} g^{-1}.
\]

**E14: Casimir Operators (4 Points)**

Consider a Lie algebra \(g\) with generators \(t_a\), which obey the defining commutation relations
\[
[t_a, t_b] = f_{abc} t_c,
\]
where \(f_{abc}\) denotes the totally antisymmetric structure constants. Let the quadratic Casimir be defined by \(C = t_a t_a\) and define the tensor Casimir by \(C_\otimes = t_a \otimes t_a\). Here the index \(a\) is summed over.

(a) Show that \(C\) is indeed a Casimir operator for the Lie algebra, i.e. that \([C, t_a] = 0\).

(b) Show that the tensor Casimir commutes with the tensor product representation of the Lie algebra generators, i.e. \([C_\otimes, \Delta(t_a)] = 0\), where \(\Delta(t_a) = t_a \otimes 1 + 1 \otimes t_a\).

(c) Consider now the specific Lie algebra \(g = \mathfrak{u}(2)\) consisting of skew-hermitian matrices with generators \(t_a \in \{i \mathbb{1}, i \sigma^1, i \sigma^2, i \sigma^3\}\). Here \(\sigma^a\) denotes the Pauli matrices. How does the following operator look as a matrix?:
\[
-\frac{1}{2} C_\otimes = \frac{1}{2} \left( \mathbb{1} \otimes \mathbb{1} + \sum_{a=1}^3 \sigma^a \otimes \sigma^a \right).
\]

How does it act on a tensor product state \(\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \otimes \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right)\)? How would you call this operator?

*Hint:* We have \([A_1 \otimes A_2, B_1 \otimes B_2] = A_1 B_1 \otimes A_2 B_2 - B_1 A_1 \otimes B_2 A_2\).
E15: Classical Yang–Baxter Equation (6 Points)

(a) Assume that the generators \( s \) and \( t \) obey the algebra \([s, t] = s\). Show that \( r = s \otimes t - t \otimes s \) obeys the classical spectral-parameter independent Yang–Baxter equation:

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{1}
\]

*Hint:* Write for instance \( r_{13} = s \otimes 1 \otimes t - t \otimes 1 \otimes s \).

(b) Demonstrate that Yang’s classical r-matrix 

\[
r(u) = \frac{C}{u},
\]

obeys the classical spectral-parameter dependent Yang–Baxter equation

\[
[r_{12}(u_{12}), r_{13}(u_{13})] + [r_{12}(u_{12}), r_{23}(u_{23})] + [r_{13}(u_{13}), r_{23}(u_{23})] = 0. \tag{2}
\]

(c) Assume that the generators \( t_{a,m} \) obey the loop algebra relations \([t_{a,m}, t_{b,n}] = f_{abc} t_{c,m+n}\). Show that these commutation relations imply that

\[
r = \sum_{n \geq 0} t_{a,n} \otimes t_{a,-n-1}
\]

obeys the classical Yang–Baxter equation (1).

E16: Compatibility Condition and Flat Currents (4 Points)

Consider a conserved and flat current \( j^{\mu} \) in two dimensions, i.e. with \( \mu = 0, 1 \).

(a) Make the ansatz

\[
D_\mu = \partial_\mu - L_\mu \quad \text{with} \quad L_\mu = a j_\mu + b \epsilon_{\mu\nu} j^\nu
\]

and arbitrary coefficients \( a, b \). Here indices are raised and lowered using the metric \((\eta_{\mu\nu}) = \text{diag}(+1, -1)\) and for the antisymmetric tensor we use the convention \( \epsilon_{01} = +1 \). Under which condition on \( a \) and \( b \) do we have

\[
[D_\mu, D_\nu] = 0, \tag{3}
\]

i.e. compatibility of the auxiliary linear problem \( D_\mu \Phi = 0 \)? Does the condition hold for the \( D_\mu(u) \) defined in the lecture?

(b) Show that the following nonlocal current is conserved if you assume that (3) holds:

\[
\hat{j}^{\mu}_u(t, x) = \epsilon^{\mu\nu} j_{\nu a}(t, x) - \frac{1}{2} f_{abc} j^{\mu}_b(t, x) \int_{-\infty}^{x} dy \; j^{0}_c(t, y).
\]

E17: Path-Ordered Exponential (3 Points)

Consider the path-ordered exponential

\[
T(x_0, x) = P \exp \int_{x_0}^{x} L(x') dx' = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_0}^{x} \int_{x_0}^{x} \cdots \int_{x_0}^{x} dx_1' \cdots dx_n' P[L(x'_1) \cdots L(x'_n)],
\]

where \( P \) denotes path-ordering with greater \( x \) to the left, e.g. \( P[L(1)L(2)] = L(2)L(1) \). Show that

\[
\frac{d}{dx} T(x_0, x) = L(x) T(x_0, x).
\]

*Hint:* Rewrite the n-fold integral over the path-ordered product as an ordered n-fold integral.