E21: Spectrum from Bethe Equations (5 Points)

Mathematica: In Exercise 20 you found that the Bethe equations for the XXX spin chain take the form

\[
\left( \frac{u_j + i/2}{u_j - i/2} \right)^L = \prod_{k=1 \atop k \neq j}^M \frac{u_j - u_k + i}{u_j - u_k - i}.
\]

The corresponding energy for an \(M\)-magnon state can be written as

\[
E = \sum_{k=1}^M \frac{1}{u_k^2 + \frac{1}{4}}.
\]

Solve the Bethe equations for a spin chain of length \(L = 5\) and for \(M = 1\) and \(M = 2\) magnons using Mathematica and determine the corresponding spectrum. For \(M = 1\) you should find the distinct energies \(\{1.38197, 3.61803\}\) and for \(M = 2\) you should obtain \(\{1.76393, 4.623607\}\). How many physical solutions to the Bethe equations belong to each energy level?

Hint: To find all solutions, write the Bethe equations without denominators, i.e. multiply them by their denominators. Solve the resulting equations using the method \texttt{Solve} and make a list of the solutions. In the case \(M = 2\) you have to select the physical solutions, which are those having distinct \(u_1\) and \(u_2\) and whose rapidities have nonvanishing real parts. Use \texttt{N} to obtain numerical values for the solutions and \texttt{TableForm} to display lists in a nicer form. If you don’t want to select the solutions by hand, you may find the following methods useful: \texttt{SortBy}, \texttt{Round}, \texttt{Union}, \texttt{Select}.

E22: Quantum Yang–Baxter Equation (3 Points)

Consider the quantum Yang–Baxter equation

\[
R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2).
\] (1)

defined on a tensor product of three spaces \(V_1 \otimes V_2 \otimes V_3\).

(a) Assume that in addition to the spectral parameter, the R-matrix also depends on a parameter \(\hbar\) and that it has the quasi-classical expansion

\[
R_{jk}(u) = c \times \left( 1_{jk} + \hbar r_{jk}(u) + \mathcal{O}(\hbar^2) \right), \quad c = \text{const}.
\]

Expand the quantum Yang–Baxter equation in \(\hbar\) up to (including) order \(\hbar^2\) and derive a constraint equation for the matrix \(r\). Argue that Yang’s R-matrix

\[
R(u) = 1 + \frac{C_{\otimes}}{u}
\]

obeys the quantum Yang–Baxter equation, where \(C_{\otimes}\) denotes the tensor Casimir of a Lie algebra.

(b) Define an alternative version of the R-matrix as \(\tilde{R}_{ij}(u) = P_{ij} R_{ij}(u)\), with the permutation operator \(P_{ij}\). Obtain the analogue of (1) for the operator \(\tilde{R}\), i.e. an equation that relates two products of three matrices \(\tilde{R}\).
E23: RTT-Relations (4 Points)

Assume that the RLL-relations hold for the Lax operator \( L(u) \).

(a) Show explicitly the RTT-relations for \( L = 2 \) with the monodromy

\[
T_0(u) = L_{02}(u)L_{01}(u).
\]

(b) Translate the proof into a graphical argument and generalize it from \( L = 2 \) to generic \( L \).

Remark: The proof is also called train argument.

(c) Assume that the R-matrix is invertible and show that the transfer matrix \( t(u) \) obeys

\[
[t(u_1), t(u_2)] = 0.
\]

E24: Heisenberg Hamiltonian and Spin Chain Transfer Matrix (5 Points)

Consider the R-matrix \( R(u) = u \mathbb{1} + i \mathbb{P} \) and the periodic spin chain transfer matrix \( t(u) \) as defined in the lectures.

(a) Show that \( U = i^{-L}t(\frac{i}{2}) \) is the spin chain shift operator acting as

\[
U \ket{X_1, X_2, \ldots, X_L} = \ket{X_L, X_1, \ldots, X_{L-1}}.
\]

Hint: Use that \( \text{Tr}_0(A_0 \mathbb{P}_{0k}) = A_k \).

(b) Demonstrate that the Hamiltonian defined by

\[
H = \frac{J}{2} \left( i \frac{d}{du} \log t(u) \bigg|_{u = \frac{i}{2}} - L \mathbb{1} \otimes L \right) = \sum_{k=1}^{L} H_{k,k+1},
\]

is the Heisenberg (XXX) Hamiltonian

\[
H_{\text{XXX}} = \sum_{k=1}^{L} H_{k,k+1}^{\text{XXX}}, \quad H_{k,k+1}^{\text{XXX}} = \frac{J}{4} \left( \sigma_k^a \otimes \sigma_{k+1}^a - \mathbb{1}_k \otimes \mathbb{1}_{k+1} \right).
\]

E25: Nontrivial Coproduct (3 Points)

The Lie algebra generators \( J_a \) obey \( [J_a, J_b] = f_{abc} J_c \) with totally anti-symmetric structure constants \( f_{abc} \). The coproduct \( \Delta \) for these Lie algebra generators is defined as

\[
\Delta(J_a) = J_a \otimes \mathbb{1} + \mathbb{1} \otimes J_a.
\]

Now extend the algebra by a second type of generator \( \tilde{J}_a \) that obeys \( [J_a, \tilde{J}_b] = f_{abc} \tilde{J}_c \) and has the non-cocommutative coproduct

\[
\Delta(\tilde{J}_a) = \tilde{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{J}_a - \frac{1}{2} \hbar f_{abc} J_b \otimes J_c.
\]

Show that

\[
\Delta([J_a, J_b]) = [\Delta(J_a), \Delta(J_b)], \quad \Delta([J_a, \tilde{J}_b]) = [\Delta(J_a), \Delta(\tilde{J}_b)].
\]

Remark: For \( \Delta \) to be an algebra homomorphism we would like to have \( \Delta([A, B]) = [\Delta(A), \Delta(B)] \). However, \( \Delta([\tilde{J}_a, \tilde{J}_b]) = [\Delta(\tilde{J}_a), \Delta(\tilde{J}_b)] \) does not automatically follow from the above commutation relations and one has to postulate additional relations as will be discussed in the lectures.