Neural noise and neural signals - spontaneous activity and signal transmission in models of single nerve cells

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Formal matter

Weekly tutorial on Mondays 13:15-15:45 starting on May 2nd

Problems are handed out two weeks before the tutorial; required for passing: 70% of the points

Oral examination before the new semester starts

ECTS: 6 (both Computational Neuroscience & Physics)

URL of the lecture: http://people.physik.hu-berlin.de/~neurophys/neusig/
Motivation
Sensory signals are represented by sequences of electrical discharges - neural spike trains.
Neurons fire randomly

Spontaneous firing
(Neocortex of rat)

Different responses to the same stimulus
(auditory neuron of locust)

Interspike interval (ISI) statistics?

Aim: reproduce and understand the spike statistics
Statistics shapes information transmission and processing!
Aim of the course

**Analytical calculation** of the statistics of **single-neuron** activity, modeled by **dynamical systems** in a **stochastic framework**

**Spontaneous activity**
- **neuron** → Spike train
- Statistics: Rate, Cv, ISI density & correlations; spike train spectrum & Fano factor

**Evoked activity**
- **Stimulus** → **neuron** → Spike train
- Statistics: Rate modulation, phase lag; signal-to-noise ratio, coherence, mutual information rate
What can be achieved by stochastic models of single-neuron activity?
Spiking statistics of a Poisson process ...

\[ p(I) = r_0 \exp[-r_0 I] \]

Spike train power spectrum

ISI probability density

Interval correlation coefficient
Spiking statistics of a Poisson process ...

... and of a sensory neuron

data: Neiman & Russell, Ohio University
Spiking statistics of a Poisson process ... ... and of a sensory neuron and the right theory!

data: Neiman & Russell, Ohio University
What can be achieved by stochastic models of single-neuron activity?

- understand neural spontaneous activity (single neurons are complex non-equilibrium systems)
Information theory of neural spiking

How much info does the spike train carry about the sensory signal?
Do neurons encode more information on slow or on fast components of a sensory stimulus?
Information filtering is ubiquitous

Visual System
...

Auditory System
...

Vestibular System
• Massot et al. *J Neurophysiol* (2011)
...

Electrosensory System
...
What can be achieved by stochastic models of single-neuron activity?

• understand neural spontaneous activity (single neurons are complex non-equilibrium systems)

• understand the neuron’s capability to transmit and filter information about time-dependent stimuli (equivalent to populations of uncoupled neurons)
Methods are useful for problem of recurrent neural nets
Methods are useful for problem of recurrent neural nets

Network of model neurons:

\[ \dot{v}_i = -v_i + \mu(t) + \sigma(t)\xi_i(t) \]

Equation for the probability density of the membrane voltage:

\[ \partial_t P(v, t) = \partial_v [v - \mu(t)] P(v, t) + \frac{\sigma^2(t)}{2} \partial_x^2 P(x, t) \]

Collective behavior of neurons in recurrent networks

- **Synchronous regular (SR) states**, where neurons are almost fully synchronized in a few clusters and behave as oscillators when excitation dominates inhibition and synaptic time distributions are sharply peaked;

- **Asynchronous regular (AR) states**, with stationary global activity and quasi-regular individual neuron firing when excitation dominates inhibition and synaptic time distributions are broadly peaked;

- **Asynchronous irregular (AI) states**, with stationary global activity but strongly irregular individual firing at low rates when inhibition dominates excitation in an intermediate range of external frequencies;

- **Synchronous irregular (SI) states**, with oscillatory global activity but strongly irregular individual firing at low (compared to the global oscillation frequency) firing rates, when inhibition dominates excitation and either low external frequencies (slow oscillations) or high external frequencies (fast oscillations). When the average synaptic time constant is high enough, these two regions merge together.

What can be achieved by stochastic models of single-neuron activity?

• understand neural spontaneous activity (single neurons are complex non-equilibrium systems)

• understand the neuron’s capability to transmit and filter information about time-dependent stimuli (equivalent to populations of uncoupled neurons)

• advanced theory of recurrent networks of spiking neurons rely on accurate theory of single-cell activity (methods are an extension of those for the single cell)
Schedule (approximate)

20.4. Motivation; spike-train statistics
27.4. Spike-train statistics II
  4.5. How much information does a spike train contain?
11.5. Mutual information - direct estimates and lower bound
18.5. Beyond statistical models: Sources of neural noise and their incorporation into dynamical neuron models
25.5. Effective time-constant approximation & IF models: Fokker-Planck analysis (intro)
  1.6. IF models: Fokker-Planck analysis of spontaneous activity
  8.6. IF models: Fokker-Planck analysis of driven activity I
15.6. IF models: Fokker-Planck analysis of driven activity II
22.6. Stochastic resonance in driven IF models
29.6. Multi-dimensional IF models: colored noise, spike-frequency adaptation, subthreshold oscillations
  6.7. Properties of spontaneous firing & signal transmission
13.7. Signal transmission in neural populations of uncoupled neurons
20.7. Outlook: recurrent networks
General references

• Stochastic Processes in general ...
  
  
  - Gardiner *Handbook of Stochastic Methods* Springer (1985)
  
  
  - Cox & Isham *Point Processes* Chapman and Hall (1980)

• ... and in neuroscience:
  
  
  - Tuckwell *Stochastic Processes in the Neuroscience* SIAM (1987)
  
  
  
Spike train statistics
From the membrane dynamics to the point process

Spike train: \( x(t) = \sum \delta(t - t_i) \)
A stationary process - time-independent firing rate

Spike trains

Estimate for the firing probability in $dt$

$$p(t, t + dt) = \frac{\text{# of neurons that fire in } (t, t + dt)}{\text{total # neurons}}$$

firing rate per unit time

$$r(t) = \frac{p(t, t + dt)}{dt}$$

$$r(t) = \left\langle \sum \delta(t - t_i) \right\rangle = \left\langle x(t) \right\rangle$$
A non-stationary process - time-dependent firing rate

Spike trains

Estimate for the firing probability in $dt$

$$p(t, t + dt) = \frac{\text{# of neurons that fire in } (t, t + dt)}{\text{total # neurons}}$$

firing rate per unit time

$$r(t) = \frac{p(t, t + dt)}{dt}$$

$$r(t) = \langle \sum \delta(t - t_i) \rangle = \langle x(t) \rangle$$

Simple example: Poissonian spike train (homogeneous)

\[ x(t) = \sum \delta(t - t_i) \]

Defining property: all \( t_i \) are independent!
Only one parameter: the firing rate \( r \)

Three ways to simulate a Poisson process

1. draw \( N \) points from a uniform density along the \( t \)-axis

If \( \frac{T_{\text{small}}}{T_{\text{large}}} \ll 1 \)

Poisson in \( T_{\text{small}} \)
with rate \( r = \frac{N}{T_{\text{large}}} \)
Simple example: Poissonian spike train (homogeneous)

\[ x(t) = \sum \delta(t - t_i) \]

Defining property: all \( t_i \) are independent!
Only one parameter: the firing rate \( \lambda \)

Three ways to simulate a Poisson process

2. discretize t-axis; draw independent random number for each bin

\[ \xi_8 < r \Delta t \Rightarrow \text{Spike in 8th bin} \]

\[ \xi_{13} > r \Delta t \Rightarrow \text{no Spike in 13th bin} \]
Simple example: Poissonian spike train (homogeneous)

\[ x(t) = \sum \delta(t - t_i) \]

Defining property: all \( t_i \) are independent!
Only one parameter: the firing rate \( \lambda \)

Three ways to simulate a Poisson process

3. draw time interval to the next spike (Gillespie method)

\[ T_1 \quad T_2 \quad T_3 \]

Prob. density

\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]

\[ 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \]
Spike train correlation function

\[ C(\tau) = \langle x(t)x(t + \tau) \rangle - \langle x(t) \rangle^2 \]

For a stationary process

\[ C(\tau) = P(\text{fire at } t \text{ and } t + \tau) - r_0^2 \]
\[ = P(\text{fire at } t + \tau | \text{has fired at } t) \cdot P(\text{fire at } t) - r_0^2 \]
\[ = r_0 P(\text{fire at } t + \tau | \text{has fired at } t) - r_0^2 \]
\[ = r_0 [\delta(\tau) + P(\text{another spike at } t + \tau | \text{has fired at } t)] - r_0^2 \]
\[ = r_0 [\delta(\tau) + m(\tau)] - r_0^2 \]
Spike train correlation function

\[ C(\tau) = r_0[\delta(\tau) + m(\tau)] - r_0^2 \]

For a Poisson process

\[ C(\tau) = r_0 \delta(\tau) \]

corresponding to a flat spike train power spectrum

\[ S(f) = r_0 \]

Generally, for any stationary stochastic process

Wiener-Khinchin theorem: \[ S(f) = \int_{-\infty}^{\infty} d\tau \ e^{2\pi i f \tau} C(\tau) \]
Spectral measures

For a stationary process $x(t)$:

$$S(f) = \lim_{T \to \infty} \frac{\tilde{x}(f) \cdot \tilde{x}(-f)}{T}$$

and the power spectrum indicates possible stochastic periodicity by finite peaks.

- Afferent sensory neuron in paddlefish
- Hair bundle from a sensory hair cell of bullfrog

Spike train power spectra are not flat in general


Neiman & Russell *J. Neurophysiol.* 2004
Spike train -> spike count

\[ N(t) = \int_0^t dt' x(t') = \sum_{0 < t_i < t} \Theta(t - t_i) \]
Spike train -> spike count -> diffusion process with drift

\[ \langle N(t) \rangle = \int_0^t dt' \langle x(t') \rangle = \int_0^t dt' r(t') \]

For a stationary process

\[ \langle N(t) \rangle = r_0 T \]
Spike train -> spike count -> diffusion process with drift

Spike count variance

$$\langle \Delta N^2(t) \rangle = \int_0^t \int_0^t dt_1 dt_2 \langle x(t_1)x(t_2) \rangle - \langle x(t_1) \rangle \langle x(t_2) \rangle$$

Often, spike count shows normal diffusion

$$\langle \Delta N^2(t) \rangle \rightarrow 2D_{\text{eff}} t, \text{ for } t \rightarrow \infty$$

Temporal structure of variability is described in the Fano factor

$$F(t) = \frac{\langle \Delta N^2(t) \rangle}{\langle N(t) \rangle} \quad \quad \quad \quad F(t) \rightarrow \frac{2D_{\text{eff}}}{r_0} \text{ for } t \rightarrow \infty$$

For a Poisson process at all times

$$F(t) = 1$$
Fano factor does not always saturate ...

\[ F(t) = \frac{\langle \Delta N^2(t) \rangle}{\langle N(t) \rangle} \]
Interval statistics

Probability density $p(I)$ of the interspike interval (ISI)

Mean ISI $\langle I \rangle$

Variance of the ISI $\langle \Delta I^2 \rangle$

Coefficient of variation $C_V = \frac{\sqrt{\langle \Delta I^2 \rangle}}{\langle I \rangle}$

For a Poisson process $C_V = 1$
Examples for ISI densities

\[ p(I) = r_0 \exp[-r_0 I] \]

The moments do not determine the probability density

Example: \[ P(T) = \frac{1}{T \sqrt{2\pi}} \exp\left[ -\log^2(x)/2 \right] \left( 1 + \varepsilon \sin(2\pi \log(x)) \right) \]
N-th order intervals

interspike intervals (ISI)
with density $\rho(I)$

and

n-th order intervals
with densities $p_n(T_n)$
Interval correlations (in nonrenewal spike trains)

\[ \rho_k = \frac{\langle I_{i+k}I_i \rangle - \langle I_{i+k} \rangle \langle I_i \rangle}{\langle I_i^2 \rangle - \langle I_i \rangle^2} \]

\[ \rho_1 > 0 \] short short \[ \rho_1 < 0 \] short
Interval correlations (in nonrenewal spike trains)

**long-range positive correlations for auditory neurons**


**P-units in weakly electric fish**


**Electroreceptors in paddle fish**


ISI correlations reviewed in
Akerberg & Chacron 2011
Renewal spike trains

All intervals are mutually independent!

Example: Poisson process

Consequence for the Fourier transforms of the n-th order interval density

\[ \tilde{p}_n(\omega) = \tilde{p}^n(\omega) \]
Relation between different spike train statistics

Time-averaged firing rate \(=\) inverse mean ISI

(stationary processes)

\[
r = \lim_{N \to \infty} \frac{N}{T_{\text{window}}} = \lim_{N \to \infty} \left( \frac{T_0}{N} + \frac{1}{N} \sum_{i=1}^{N} I_i + \frac{T_{N+1}}{N} \right)^{-1} = \frac{1}{\langle I \rangle}
\]
Relation between different spike train statistics

Power spectrum - Fourier transforms of n-th order interval densities

\[ S(\omega) = \int_{-\infty}^{\infty} d\tau C(\tau) e^{i\omega \tau} = r_0 \int_{-\infty}^{\infty} d\tau [\delta(\tau) + m(\tau) - r_0] e^{i\omega \tau} \]

\[ = r_0 - r_0^2 \delta(\omega) + r_0 \int_{-\infty}^{\infty} d\tau m(\tau) e^{i\omega \tau} \]

With \( m(-\tau) = m(\tau) \) and \( m(\tau) = \sum_{n=1}^{\infty} p_n(\tau) \) one obtains for \( \omega > 0 \)

\[ S(\omega) = r_0 + r_0 \int_{0}^{\infty} d\tau m(\tau) (e^{i\omega \tau} + e^{-i\omega \tau}) = r_0 \left[ 1 + \sum_{n=1}^{\infty} \tilde{p}_n(\omega) + \tilde{p}_n^*(\omega) \right] \]

For a renewal spike train \( (\tilde{p}_n(\omega) = \tilde{p}^n(\omega)) \)

\[ S(\omega) = r_0 \left[ 1 + \frac{\tilde{p}}{1 - \tilde{p}} + \frac{\tilde{p}^*}{1 - \tilde{p}^*} \right] \]

\[ S(\omega) = r_0 \frac{1 - |\tilde{p}|^2}{|1 - \tilde{p}|^2} \]
Relation between different spike train statistics

Given the $n$-th order interval densities $\rightarrow$ calculation of variances $\langle \Delta T^2_n \rangle$. Variances and SCC obey (Cox and Lewis, 1966)

$$\langle \Delta T^2_n \rangle / \langle \Delta T^2_1 \rangle = n + 2 \sum_{k=1}^{n-1} (n - k) \rho_k$$

For a renewal spike train

\[ \rho_k = 0, \quad k > 0 \]

\[ \langle \Delta T^2_n \rangle = n \langle \Delta T^2_1 \rangle \]

For a nonrenewal spike train

\[ \langle \Delta T^2_n \rangle \neq n \langle \Delta T^2_1 \rangle \]
Relation between different spike train statistics

Given the $n$-th order interval densities → calculation of variances $\langle \Delta T_n^2 \rangle$. Variances and SCC obey (Cox and Lewis, 1966)

$$\frac{\langle \Delta T_n^2 \rangle}{\langle \Delta T_1^2 \rangle} = n + 2 \sum_{k=1}^{n-1} (n-k) \rho_k$$

from which we find

$$\rho_k = \frac{\langle \Delta T_{k+1}^2 \rangle + \langle \Delta T_{k-1}^2 \rangle}{2\langle \Delta T_1^2 \rangle} - \frac{\langle \Delta T_k^2 \rangle}{\langle \Delta T_1^2 \rangle}$$

**Last time ...**  For a stationary process:

1. Firing rate \( r_0 = 1/\langle I \rangle \)

2. n-th order interval variance
   \[
   \langle \Delta T_n^2 \rangle = n \langle \Delta I^2 \rangle \left[ 1 + 2 \sum_{k=1}^{n-1} \left( 1 - \frac{k}{n} \right) \rho_k \right]
   \]

3. Interval correlation coefficient
   \[
   \rho_k = \frac{\langle \Delta T_{k+1}^2 \rangle + \langle \Delta T_{k-1}^2 \rangle - 2 \langle \Delta T_k^2 \rangle}{2 \langle \Delta I^2 \rangle}
   \]

4. Power spectrum
   \[
   S(f) = r_0 \left( 1 + \sum_{n=1}^{\infty} \tilde{p}_n(f) + \tilde{p}_n^*(f) \right)
   \]
   \[\text{--> renewal process: } S(f) = r_0 \frac{1 - \left| \tilde{p}(f) \right|^2}{\left| 1 - \tilde{p}(f) \right|^2}\]

5. Fano factor
   \[
   S(0) = r_0 \lim_{t \to \infty} F(t)
   \]

\[S_{\text{renewal}}(0) = r_0 C_v^2\]
\[
\lim_{t \to \infty} F(t) = C_v^2
\]
For a stationary process:

1. Firing rate \( r_0 = 1/\langle I \rangle \)

2. n-th order interval variance
\[
\langle \Delta T_n^2 \rangle = n \langle \Delta I^2 \rangle \left[ 1 + 2 \sum_{k=1}^{n-1} \left( 1 - \frac{k}{n} \right) \rho_k \right]
\]

3. Interval correlation coefficient
\[
\rho_k = \frac{\langle \Delta T_{k+1}^2 \rangle + \langle \Delta T_{k-1}^2 \rangle - 2 \langle \Delta T_k^2 \rangle}{2 \langle \Delta I^2 \rangle}
\]

4. Power spectrum
\[
S(f) = r_0 \left( 1 + \sum_{n=1}^{\infty} \tilde{p}_n(f) + \tilde{p}_n^*(f) \right)
\]

--- renewal process: \( S(f) = r_0 \frac{1 - |\tilde{p}(f)|^2}{|1 - \tilde{p}(f)|^2} \)

5. Fano factor \( S(0) = r_0 \lim_{t \to \infty} F(t) \)

\( \text{nonrenewal: } S(0) = r_0 C_v^2 \left[ 1 + 2 \sum_{k=1}^{\infty} \rho_k \right] \)

\( \text{renewal: } S_{\text{renewal}}(0) = r_0 C_v^2 \quad \lim_{t \to \infty} F(t) = C_v^2 \)
Negative ISI correlations cause a drop in spectral power at low frequency

electroreceptor (P-units) in the weakly electric fish

Chacron et al. *Proc. SPIE* 2005
Negative ISI correlations cause a drop in spectral power at low frequency.

electroreceptor (P-units) in the weakly electric fish

Chacron et al. Proc. SPIE 2005
Gamma process - ISI density and spike train power spectrum

\[ r_0 C_v^2 = 1 \]
\[ r_0 C_v^2 = \frac{1}{2} \]
\[ r_0 C_v^2 = \frac{1}{25} \]
**Summary: spike-train statistics**

- Point processes can be characterized
  - by spike train (rate) and count statistics (Fano factor)
  - by interval statistics (ISI and nth-order intervals, interval correlations)

- ISI densities often unimodal; spectra non-flat --- not well described by Poisson statistics!

- Spike train and interval statistics are related in a nontrivial manner

**References**

- Cox & Isham *Point Processes* Chapman and Hall (1980)