H1

$$\langle N_1 \rangle = \sum_{N_1=0}^{N} N_1 \, W_N(N_1) = \sum_{N_1=0}^{N} N_1 \binom{N}{N_1} p_1^{N_1} p_2^{N-N_1}$$

$$= p_1 \frac{\partial}{\partial p_1} \sum_{N_1=0}^{N} \binom{N}{N_1} p_1^{N_1} p_2^{N-N_1} = p_1 \cdot \frac{\partial}{\partial p_1} (p_1 + p_2)^N$$

$$= N p_1 (p_1 + p_2)^{N-1} = N p_1$$

$\square$

(wichtig zunächst p_2 als unabhängig von p_1 zu sehen)

Ebenso

$$\langle N_1^2 \rangle = \left(p_1 \frac{\partial}{\partial p_1}\right)^2 \sum_{N_1=0}^{N} \binom{N}{N_1} p_1^{N_1} p_2^{N-N_1} = (p_1 \partial_{p_1})^2 (p_1 + p_2)^N$$

$$= p_1 \frac{\partial}{\partial p_1} \left(N p_1 (p_1 + p_2)^{N-1} \right) = N p_1 (p_1 + p_2)^{N-1} + N(N-1) p_1^2 (p_1 + p_2)^{N-2}$$

$$= N p_1 + N(N-1) p_1^2$$

$$\Rightarrow \quad \langle N_1^2 \rangle - \langle N_1 \rangle^2 = N p_1 + N(N-1) p_1^2 - N^2 p_1^2$$

$$= N(p_1 - p_1^2) = N p_1 (1 - p_1)$$

$\square$

ÜBUNG 1

H2

a) $\Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t}$

$$\Gamma(n+1) = \int_0^\infty dt\, t^n \left(-\frac{d}{dt}\right) e^{-t} = \int_0^\infty dt\, n t^{n-1} e^{-t} + \underbrace{t^n e^{-t}\Big|_{t=0}^{t=\infty}}_{0}$$

$$= n \int_0^\infty dt\, t^{n-1} e^{-t} = n\,\Gamma(n)$$

$$\Gamma(0) = \int_0^\infty dt\, e^{-t} = 1$$

$$\Rightarrow \quad \Gamma(n+1) = n\cdot(n-1)\cdot\ldots\cdot 1\cdot\Gamma(0) = n!$$

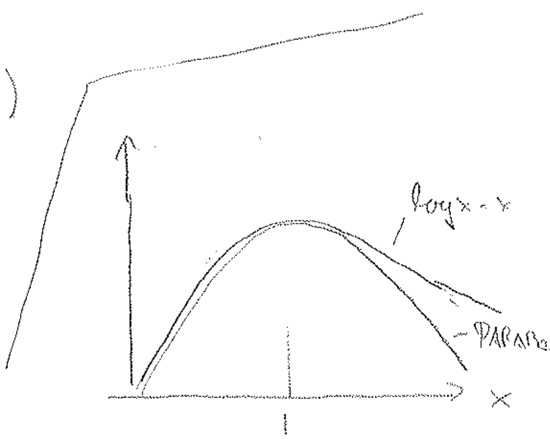
$\square$

b)

$$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t+n\log t} = \int_0^\infty dt\, e^{n\left(\log t - \frac{t}{n}\right)}$$

$$\underset{\tilde{t}=\frac{t}{n}}{=} n \int_0^\infty d\tilde{t}\, e^{n(\log\tilde{t}-\tilde{t}) + n\log n}$$

$$= n^{n+1} \int_0^\infty d\tilde{t}\, e^{n(\log\tilde{t}-\tilde{t})}$$



Sattelpunktsmethode für $n \to \infty$:

$$e^{-nf(x)} \approx e^{n\left[f(x_0) + \frac{(x-x_0)^2}{2} f''(x_0)\right]}$$

mit $f'(x_0) = 0$

$\Rightarrow$ Maximum

$$f(x) = \log x - x \qquad f'(x) = 0 = \frac{1}{x} - 1 \quad \Rightarrow \quad x = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow \quad f''(x=1) = -1 < 0 \quad \Rightarrow \quad \text{Maximum}$$

$$f(x) = -1 + \frac{1}{2}(x-1)^2(-1) + O(x^3) = -1 - \frac{1}{2}(x-1)^2$$

$$\Rightarrow \int_0^\infty dx\, e^{n(-1-\frac{1}{2}(x-1)^2)} = e^{-n} \int_0^\infty dx\, e^{-\frac{n}{2}(x-1)^2}$$

$$\approx e^{-n} \int_{-\infty}^\infty dx\, e^{-\frac{n}{2}(x-1)^2} = e^{-n} \int_{-\infty}^\infty dx\, e^{-\frac{n}{2}x^2} = \sqrt{n}^{-1} e^{-n} \underbrace{\int_{-\infty}^\infty dy\, e^{-\frac{1}{2}y^2}}_{\sqrt{2\pi}}$$

$$= \sqrt{\frac{2\pi}{n}}\, e^{-n}$$

Und somit

$$n! = \Gamma(n+1) \underset{n \gg 1}{=} \sqrt{2\pi n}\, n^n e^{-n} \qquad \square$$

H3

HÜ 1

1. a) Der letzte gespülte Teller (der n-te) ist immer ein „Zerbrochener". Es gibt daher $\binom{n-1}{k-1}$ verschiedene Möglichkeiten, die übrigen k−1 Teller beim Spülen von insgesamt n−1 ununterscheidbaren Tellern zu zerbrechen (für $k \leq n$). $\rightarrow P_n = \binom{n-1}{k-1} p^k (1-p)^{n-k}$

(b) $$F(x) = \sum_{n=0}^{\infty} P_n x^n = \sum_{n=k}^{\infty} P_n x^n = p^k x^k \sum_{n=k}^{\infty} \binom{k-1+n-k}{k-1} [(1-p)x]^{n-k}$$
$$= \underline{\underline{\left(\frac{px}{1-(1-p)x} \right)^k}} = \langle x^n \rangle$$

$$\partial_x F(x)|_{x=1} = \langle n \rangle = \underline{\underline{\frac{k}{p}}} \quad ; \quad \partial_x^2 F(x)|_{x=1} = \langle n^2 \rangle - \langle n \rangle = \frac{k(k+1)}{p^2} - 2\frac{k}{p}$$

$$\Rightarrow \Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \sqrt{\partial_x^2 F + \partial_x F - (\partial_x F)^2}\Big|_{x=1} = \underline{\underline{\frac{\sqrt{k(1-p)}}{p}}}$$

$$\Delta n / \langle n \rangle = \sqrt{(1-p)/k} \rightarrow \underline{\underline{0}} \text{ für } k \rightarrow \infty \quad \checkmark$$