P13.1 – WEIß MODELL

$$\mathcal{H} = -\frac{1}{2}\frac{J}{N}\sum_{\ell,\ell'=1}^{N}\sigma_\ell\sigma_{\ell'} - h\sum_{\ell=1}^{N}\sigma_\ell \qquad \sigma_\ell = \pm 1.$$

Mit $\sum_{\ell,\ell'=1}^{N}\sigma_\ell\sigma_{\ell'} = \left(\sum_{\ell=1}^{N}\sigma_\ell\right)^2$ ergibt sich

$$\mathcal{H} = -\frac{J}{2N}m^2 - hm \qquad m = \sum_{\ell=1}^{N}\sigma_\ell$$

□ Zustandssumme:

$$Z_N = \sum_{\sigma_\ell=\pm1} e^{-\beta\mathcal{H}} = \sum_{\sigma_\ell=\pm1}\delta\left(m-\sum_\ell\sigma_\ell\right) e^{\frac{\beta J}{2N}m^2+\beta hm}$$

□ Hubbard-Stratanovich transformation:

Wir schreiben

$$e^{\frac{\beta J}{2N}m^2} = \int_{-\infty}^{\infty}dy\sqrt{\frac{N\beta J}{2\pi}}\, e^{-N\frac{\beta J}{2}y^2+\beta J m y}$$

Dann folgt für die Zustandssumme:

$$Z_N = \sum_{\sigma_\ell=\pm1}\int_{-\infty}^{\infty}dy\sqrt{\frac{N\beta J}{2\pi}}\, e^{-N\frac{\beta J}{2}y^2+(\beta J y+\beta h)\sum_{\ell=1}^{N}\sigma_\ell}$$

$$= \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}}\, e^{-N\frac{\beta J}{2}y^2} \Big(\underbrace{e^{(\beta J y + \beta h)} + e^{-(\beta J y + \beta h)}}_{2\cosh[\beta J y + \beta h]} \Big)^N$$

$$= \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}}\, \exp\Big[-N\Big(\frac{\beta J}{2}y^2 - \log(2\cosh[\beta J y + h\beta])\Big)\Big]$$

$$= \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}}\, e^{-N g(y)}$$

mit $g(y) = \frac{\beta J}{2}y^2 - \log(2\cosh(\beta J y + \beta h))$

☐ $\underline{N \to \infty}$: Sattelpunktsmethode für Integral:

$$Z_N \sim e^{-N g(y_0)}$$

mit $g'(y_0) = 0$

$$\Rightarrow \boxed{y_0 = \tanh(\beta J y_0 + \beta h)}$$

Identische Gleichung zur Zustandsglg. der Mean Field Theorie für das ~~nächste Nachbar~~ Ising Modell aus der Vorlesung: $\boxed{m = \tanh[\beta(h + m J)]}$ dal

□ Magnetisierung:

$$M = \lim_{N\to\infty} \frac{1}{\beta N} \frac{\partial \ln Z_N}{\partial h} \qquad Z_N = e^{-N g(y_0)}$$

$$= \frac{1}{\beta N}(-N)\frac{\partial g(y_0)}{\partial h} = -\frac{1}{\beta}\frac{\partial g(y_0)}{\partial h}$$

$$\frac{\partial g(y_0)}{\partial h} = \underbrace{g'(y_0)}_{=0} \cdot \frac{\partial h}{\partial y_0} + (-\beta)\tanh(\beta J y_0 + \beta h)$$

$$= -\beta y_0$$

$$\Rightarrow \qquad \boxed{M = y_0}$$

[D.h. Ising Mean-Field-Theorie und das Weiß-Modell im thermodynamischen Limes sind identisch.]

H13.1 Suszeptibilität und Wärmekapazität des Isingmodells in MFT

a) Suszeptibilität. $\chi = \left(\frac{\partial m}{\partial h}\right)_T = \left(\left(\frac{\partial h}{\partial m}\right)_T\right)^{-1}$

$$\boxed{h(m,T) = -\tilde{J} m + \frac{zT}{2} \log \frac{1+m}{1-m}} \qquad \tilde{J} = z T_c$$

$$\frac{\partial h}{\partial m} = -zT_c + \frac{zT}{2}\left(\frac{1}{1+m} + \frac{1}{1-m}\right) = -zT_c + zT\,\frac{1}{1-m^2}$$

$$= \frac{z(T-T_c) + m^2 z T_c}{1-m^2}$$

$$\Rightarrow \boxed{\chi = \frac{1-m^2}{z(T-T_c) + m^2 z T_c}}$$

$$\lim_{h\to 0} \chi = \begin{cases} T>T_c,\ m=0: & \underline{\underline{\frac{1}{z(T-T_c)}}} \\ T<T_c,\ m=\pm m_0: & \frac{1-m_0^2}{z(T-T_c)+m_0^2 z T_c} = \frac{T_c - 3(T_c-T)}{z(T-T_c) - 3(T-T_c)z}\,\frac{1}{T_c} \end{cases}$$

Im Grenzfall $h \to 0$ lassen sich die Spontanen Magnetisierungen $m = \{0, +m_0, -m_0\}$ erreichen

$m_0^2 = 3(T_c - T)/T_c$

$$\underset{T\to T_c}{\approx} \underline{\underline{\frac{1}{2(T_c-T)\cdot z}}} \quad ✓$$

b) Wärmekapazität:

Aus Vorlesung:

$$f(T,h) = \frac{F_{MFT}(T,h)}{Nz} = \frac{T_c}{2} m^2 - T \ln\left[2\cosh\left(\beta(h + zT_c m)\right)\right]$$

Entwicklung für $T \sim T_c$ und $h \approx 0$: Impliziert

$|m| \lesssim |m_0| = \sqrt{3}\sqrt{\frac{T-T_c}{T_c}}$:

$$\log[2\cosh(\alpha)] = \log 2 + \frac{\alpha^2}{2} - \frac{\alpha^4}{12} + \mathcal{O}(\alpha^5)$$

$$\Rightarrow \quad f(T,h) \simeq \frac{T_c}{2}m^2 - T\log 2 - \frac{T}{2}\frac{1}{(kT)^2}(h+kT_c m)^2 + \frac{T}{12}\frac{1}{(kT)^4}(h+kT_c m)^4 + \dots$$

$$\underline{f(T,h=0)} = \frac{T_c}{2}m^2 - T\log 2 - \underbrace{\frac{T}{2}\left(\frac{T_c}{T}\right)^2 m^2}_{\frac{T_c}{2}+\frac{\Delta T}{2}-\Delta T = \frac{T_c}{2}-\frac{1}{2}\Delta T} + \underbrace{\frac{T}{12}\left(\frac{T_c}{T}\right)^4 m^4}_{=\frac{1}{12}(T_c-3\Delta T)m^4}$$

$$= \frac{1}{2}(T-T_c)m^2 + \frac{T_c}{12}m^4 - T\log 2 + \underbrace{\Delta T\, m^4}_{\mathcal{O}(\Delta T^3)}$$

$$= \underline{\frac{1}{2}(T-T_c)m^2 + \frac{T_c}{12}m^4 - T\log 2} + \mathcal{O}(\Delta T^3)$$

Dann:

$$C_{h=0} = -NkT\left.\frac{\partial^2 f}{\partial T^2}\right|_{h=0} = \begin{cases} T > T_c: \quad m=0 \Rightarrow f''=0. & (A) \\ T < T_c: \quad m^2 = m_0^2 = 3\frac{T_c-T}{T_c} & (B) \end{cases}$$

$$(B) - \frac{\partial^2}{\partial T^2}\left(\frac{1}{2}(T-T_c)\,3\,\frac{T_c-T}{T_c} + \frac{T_c}{12}\,9\,\frac{(T_c-T)^2}{T_c^2}\right)$$

$(B) = \frac{1}{T_c}\left[-3 + \frac{9\cdot 2}{12}\right] = -\frac{3}{2}\frac{1}{T_c}$

Somit:

$$C_{h=0} = \begin{cases} 0 & T>T_c, \quad m=0 \\ \frac{3}{2} Nk \frac{T}{T_c} & T<T_c, \quad m=\pm m_0 . \end{cases}$$

✓

H 13.2 – 1D Isingmodell

$$\boxed{\mathcal{H} = -\sum_{i=1}^{N-1} J_i \sigma_i \sigma_{i+1}} \qquad Z = Sp(e^{-\beta\mathcal{H}})$$

Definiere $K_i = \beta J_i$. Dann gilt

$$Z_{N+1} = \underbrace{\sum_{\sigma_1 \dots \sigma_N} \left(\prod_{i=1}^{N-1} e^{K_i \sigma_i \sigma_{i+1}}\right)}_{Z_N} \underbrace{\sum_{\sigma_{N+1}=(-1,1)} e^{K_N \sigma_N \sigma_{N+1}}}$$

$$2\cosh(K_N \sigma_N) \underset{\sigma_N = \pm 1}{=} 2\cosh(K_N)$$

$\Rightarrow \quad Z_{N+1} = Z_N \left(2\cosh(K_N)\right) = Z_{N-1} \left(2\cosh(K_N)\right)\left(2\cosh(K_{N-1})\right)$

$\dots = \prod_{i=1}^{N} \left[2\cosh(K_N)\right]$

Sei $J_i = J$:

$$\boxed{Z_N = \left[2\cosh(J/kT)\right]^{N-1}}$$