P7.1

Erinnerung Residuensatz: $f(\omega) = \frac{1}{2\pi i} \oint_{\omega} \frac{f(z)}{z-\omega} dz$ falls $f(z)$ keinen Pol bei ω hat.

$$\Rightarrow \oint \frac{dz}{2\pi i} \frac{\sum_n a_n z^n}{z^{N+1}} = a_N$$

Hier:

a) $$Z_{Groß} = \sum_{N=0}^{\infty} Z_N(T,V)\, e^{\beta\mu\cdot N} = \sum_N Z_N z^N$$

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{Z_{Groß}}{z^{N+1}} dz = \sum_{M=0}^{\infty} \oint \frac{dz}{2\pi i} Z_M \frac{z^M}{z^{N+1}} = Z_N \quad \square$$

b) Test:

$$Z_N = \oint \frac{dz}{2\pi i} \frac{e^{zV/\lambda^3}}{z^{N+1}} = \sum_{n=0}^{\infty} \oint \frac{dz}{2\pi i} \frac{1}{n!} \left(\frac{V}{\lambda^3}\right)^n \frac{z^n}{z^{N+1}}$$

$$= \frac{1}{N!} \frac{V^N}{\lambda^{3N}} \quad // \quad \square$$

②

c) Allgemeiner Fall

$$Z_{Groka} = e^{-\beta\Phi}$$

$$\oint \frac{dz}{2\pi i}\, e^{-\beta\Phi - (N+1)\log z} = \oint \frac{dz}{2\pi i}\, e^{g(z)} \approx e^{g(z_0)}$$

Sattelpunktbed.: $\frac{\partial}{\partial z}\left(-\beta\Phi - (N+1)\log z\right)\Big|_{z_0} = 0$

$$\Rightarrow \quad \beta \frac{\partial\Phi}{\partial z} + \frac{N+1}{z_0} = 0 \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\mu}{\partial z}\frac{\partial\Phi}{\partial\mu} =$$

$$= -\langle N\rangle \underbrace{\left(\frac{\partial z}{\partial\mu}\right)^{-1}}_{\frac{\partial}{\partial\mu}e^{\beta\mu} = \beta z}$$

$$\Rightarrow \quad \beta\left(-\langle N\rangle \frac{1}{\beta z_0}\right) + \frac{N+1}{z_0} = 0$$

$$\Rightarrow \quad \langle N\rangle = N+1 \quad \text{für } N\to\infty \quad \boxed{\langle N\rangle = N}$$

Einsetzen in Exponenten

$$g(z_0) = -\beta F = -\beta\Phi(z_0) - (N+1)\underbrace{\log z_0}_{\beta\mu}$$

$$\Rightarrow \quad \boxed{F = \Phi(z_0) + \mu\langle N\rangle}$$

①

P7.2

a) Die mittlere Besetzungszahl wurde in der Vorlesung hergeleitet und ist dimensionsunabhängig

$$n(\vec{p}, s) = \frac{1}{e^{\beta(\varepsilon_p - \mu)} \mp 1} \quad \binom{B}{F} \qquad (\text{IV}.5) \text{ im Skript}$$

$$\varepsilon_p = \frac{\vec{p}^2}{2m}$$

Da $s = 1/2$ haben wir es mit Fermionen zu tun.

Für die Gesamtzahl an Teilchen gilt dann:

$$\langle N \rangle = \sum_s \sum_{\vec{p}} n(\vec{p}, s) = 2 \sum_{\vec{p}} \frac{1}{e^{\beta \vec{p}^2/2m} \cdot z + 1}$$

$$\sum_{\vec{p}} \hat{=} \frac{L^2}{(2\pi\hbar)^2} \int d^2\vec{p} = \frac{L^2}{(2\pi\hbar)^2} \int_0^{2\pi} d\varphi \int_0^\infty dp\, p$$

$$\Rightarrow \langle N \rangle = \frac{L^2}{\pi\hbar^2} \int_0^\infty \frac{p\, dp}{z e^{\beta p^2/2m} + 1} = \frac{L^2 m k T}{\pi \hbar^2} \int_0^\infty \frac{dx}{\bar{z} e^x + 1}$$

$$x = \frac{\beta p^2}{2m}$$

$$dx = \frac{\beta}{m} p\, dp \qquad \bar{z} = e^{-\beta\mu}$$

Integral: Substitution $\bar{z} e^x + 1 = y \quad dy = (y-1)\, dx$

$$\int_0^\infty \frac{dx}{y} = \int_{\bar{z}+1}^\infty \frac{dy}{y(y-1)} = \int_{\bar{z}+1}^\infty \left(\frac{dy}{y-1} - \frac{dy}{y} \right)$$

$$= \ln(y-1) - \ln(y) \Big|_{z+1}^\infty = \ln \frac{y-1}{y} \Big|_{\bar{z}+1}^\infty = \ln \frac{\bar{z}+1}{\bar{z}}$$

$$= \ln(1 + 1/\bar{z}) = \ln(1 + e^{\beta\mu})$$

$$\Rightarrow \boxed{\langle u \rangle = \frac{L^2 m k T}{\pi \hbar^2} \ln\left(1 + e^{\beta\mu}\right)}$$

H7.1

a)

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty du \, \frac{u^{x-1}}{e^u - 1} = \frac{1}{\Gamma(x)} \int_0^\infty du \sum_{l=0}^\infty u^{x-1} e^{-u} e^{-lu}$$

$$\underset{\text{Integral existiert}}{=} \frac{1}{\Gamma(x)} \sum_{l=0}^\infty \int_0^\infty du \, u^{x-1} e^{-u(l+1)} \underset{v = u(l+1)}{=} \frac{1}{\Gamma(x)} \sum_{l=0}^\infty \frac{1}{(l+1)^x} \int_0^\infty dv \, v^{x-1} e^{-v}$$

$$= \frac{\Gamma(x)}{\Gamma(x)} \sum_{l=0}^\infty \frac{1}{(l+1)^x} = \sum_{l=1}^\infty \frac{1}{l^x} \qquad \square$$

$$\sum_{l=1}^\infty l = \zeta(-1) = -\frac{1}{12}$$

c)

$$\frac{g_\nu(z)}{f_\nu(z)} = \frac{1}{\Gamma(\nu)} \int_0^\infty du \, \frac{u^{\nu-1}}{e^u z^{-1} \mp 1} = \frac{1}{\Gamma(\nu)} \int_0^\infty du \, \frac{u^{\nu-1}}{1 \mp z e^{-\mu}} \, z e^{-u}$$

$$= \frac{z}{\Gamma(\nu)} \int_0^\infty du \, u^{\nu-1} e^{-\mu} \sum_{\ell=0}^\infty (\pm z)^\ell e^{-\mu \cdot \ell} \qquad (\text{if } |z|<1)$$

$$= \pm \sum_{\ell=0}^\infty (\pm z)^{\ell+1} \frac{1}{\Gamma(\nu)} \int_0^\infty du \, u^{\nu-1} e^{-\mu(\ell+1)}$$

$$\underset{u \cdot (\ell+1) = x}{=} \frac{1}{\Gamma(\nu)} (\pm) \sum_{\ell=0}^\infty \frac{(\pm z)^{\ell+1}}{(\ell+1)^\nu} \underbrace{\int_0^\infty dx \, x^{\nu-1} e^{-x}}_{\Gamma(\nu)}$$

$$\Rightarrow \quad = \pm \sum_{\ell=0}^\infty \frac{(\pm z)^{\ell+1}}{(\ell+1)^\nu} = \pm \, Li_\nu(\pm z) \qquad \square$$

H7.2 2-D Elektronengas

a) Großkanon. Potential aus (IV.c):

$$\Phi = \pm \frac{1}{\beta} \sum_p \log\left(1 \mp e^{-\beta(\varepsilon_p - \mu)}\right) \qquad \binom{B}{F}$$

Nun ist (F) Fall und $\sum_p = 2 \cdot \frac{L^2}{(2\pi\hbar)^2} \int d^2p$

mit $\varepsilon_p = \frac{\vec{p}^2}{2m}$ zu nehmen:

$$\Phi = -\frac{1}{\beta} 2 \frac{L^2}{(2\pi\hbar)^2} 2\pi \int_0^\infty dp \cdot p \, \log\left(1 + e^{-\beta p^2/2m} \cdot z\right)$$

Integral: $\varepsilon = p^2/2m \quad d\varepsilon \cdot 2m = dp \cdot p$

$$I = \int_0^\infty dp \cdot p \, \log\left(1 + e^{-\beta\varepsilon} \cdot z\right) = 2m \int_0^\infty d\varepsilon \, \log\left(1 + e^{-\beta\varepsilon} \cdot z\right)$$

$$= 2m \int_0^\infty d\varepsilon \left(\frac{d}{d\varepsilon}\varepsilon\right) \log\left(1 + e^{-\beta\varepsilon} \cdot z\right)$$

P.I.

$$= -2m \int_0^\infty d\varepsilon \frac{\varepsilon e^{-\beta\varepsilon} z (-\beta)}{1 + e^{-\beta\varepsilon} \cdot z} + 2m \, \varepsilon \log\left(1 + e^{-\beta\varepsilon} z\right)\Big|_0^\infty$$

$$= 2m\beta z \int_0^\infty d\varepsilon \frac{\varepsilon}{e^{\beta\varepsilon} + z} \qquad - 0 \cdot \log(1+z) + \lim_{\varepsilon \to \infty} \varepsilon \log\left(1 + e^{-\beta\varepsilon}\right) \to 0$$

$\varepsilon \cdot \beta = x$

$$\Rightarrow \quad I = \frac{2m}{z\beta} \int_0^\infty dx \, \frac{\varepsilon}{e^x z^{-1} + 1} = \frac{2m}{z\beta} f_2(z)$$

$$\Rightarrow \quad \underline{\underline{\Phi}} = - \frac{L^2}{\pi\hbar^2} \frac{1}{\beta} \frac{2m}{z\beta} f_2(z) = \underline{\underline{\frac{L^2 \, 2m}{\beta^2 z \pi\hbar^2} Li_2(-z)}}$$

da $\quad f_\nu(z) = - Li_\nu(z)$

b) Innere Energie:

$$E = \sum_p \varepsilon_p \, n(\varepsilon_p) = \frac{2L^2}{(2\pi\hbar)^2} \int d^2p \left(\frac{p^2}{2m}\right) \frac{1}{e^{\beta p^2/2m} z^{-1} + 1}$$

$$= \frac{L^2}{\pi\hbar^2} \int_0^\infty dp \, p \, \frac{p^2/2m}{e^{\beta p^2/2m} z^{-1} + 1} \qquad dp\, p = d\Big(\underbrace{\tfrac{p^2}{2m}\beta}_{x}\Big) \frac{m}{\beta}$$

$$= \frac{L^2}{\pi\hbar^2} \frac{m}{\beta^2} \int_0^\infty dx \, \frac{x}{e^x z^{-1} + 1} = \frac{L^2 m}{\pi\hbar^2 \beta^2} f_2(z)$$

$$\Rightarrow \quad \boxed{E = - \Phi}$$

Dies folgt auch aus der Relation

$$E = \left(\frac{\partial(\phi \cdot \beta)}{\partial \beta}\right)_{\beta\cdot\mu = \text{const}} = \beta \frac{\partial}{\partial\beta} \phi + \phi = (-2+1)\phi$$

c) Zustandsgleichung: Gibbs-Duhem: $pV = -\phi$

$$\Rightarrow \quad \boxed{pV = \frac{L^2 m}{\pi \hbar^2 \beta^2 z} f_2(z)} = E$$

H7.3 Schwankung der Besetzungszahlen:

$$\langle n_p \rangle = \frac{1}{e^{\beta \varepsilon_p} z^{-1} \mp 1}$$

$$\langle n_p^2 \rangle = \frac{\sum_{n_q} e^{-\beta(\varepsilon_q - \mu) n_q} n_q^2}{\sum_{n_q} e^{-\beta(\varepsilon_q - \mu) n_q}} = \left. \frac{\frac{\partial}{\partial x^2} \sum_n e^{-xn}}{\sum_n e^{-xn}} \right|_{x = \beta(\varepsilon_q - \mu)}$$

Nun ist $\displaystyle\sum_{n=0}^{\infty/1} e^{-xn} = \begin{pmatrix} \frac{1}{1-e^{-x}} \\ 1 + e^{-x} \end{pmatrix} \begin{matrix} (B) \\ (F) \end{matrix}$

$$\frac{\partial}{\partial x^2} \begin{pmatrix} \frac{1}{1-e^{-x}} \\ 1+e^{-x} \end{pmatrix} = \frac{\partial}{\partial x} \begin{pmatrix} \frac{-e^{-x}}{(1-e^{-x})^2} \\ -e^{-x} \end{pmatrix} = \begin{pmatrix} \frac{e^{-x}}{(1-e^{-x})^2} + 2 \frac{e^{-x2}}{(1-e^{-x})^3} \\ e^{-x} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{-x}(1+e^{-x})}{(1-e^{-x})^3} \\ e^{-x} \end{pmatrix}$$

Dann ist

$$\langle n_p^2 \rangle = \begin{pmatrix} \dfrac{e^{-x}(1+e^{-x})}{(1-e^{-x})^2} = \dfrac{(e^x+1)}{(e^x-1)^2} \\ \dfrac{e^{-x}}{1+e^{-x}} = \dfrac{1}{e^x+1} \end{pmatrix}$$

$$\Rightarrow \quad \langle n_p^2 \rangle = \frac{e^{\beta(\varepsilon_p-\mu)}+1}{\left(e^{\beta(\varepsilon_p-\mu)}-1\right)^2} \qquad \text{(BOSON)}$$

$$\langle n_p^2 \rangle = \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} = \langle n_p \rangle \qquad \text{(FERMION)}$$

Schwankungsquadrat:

$$\langle n_p^2 \rangle - \langle n_p \rangle^2 = \frac{e^{\beta(\varepsilon_p-\mu)}}{\left(e^{\beta(\varepsilon_p-\mu)}-1\right)^2} \qquad \text{(B)}$$

$$\langle n_p^2 \rangle - \langle n_p \rangle^2 = \frac{1}{e^x+1} - \frac{1}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$$

$$= \frac{e^{\beta(\varepsilon_p-\mu)}}{\left(e^{\beta(\varepsilon_p-\mu)}+1\right)^2} \qquad \text{(F)}$$

Bzw: $\boxed{\dfrac{(\Delta n_p)^2}{\langle n_p \rangle^2} = e^{\beta(\varepsilon_p-\mu)}}$ für B & F