Recent developments in AdS/CFT

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Overview

AdS/CFT correspondence provides a fascinating link between conformal quantum field theories without gravity and string theory with (both classical and quantized) gravity.

Major (recent) activities:

1. Integrability in AdS/CFT: Spectral problem solved (?)
2. Scattering amplitudes in maximally susy Yang-Mills, relation to light-like Wilson loops and dual superconformal symmetry
3. Novel well understood $AdS_4/CFT_3$ duality pair: IIA strings on $AdS_4 \times CP^3$ dual to max susy 3d Chern-Simons theory [Aharony,Bergmann,Jafferis,Maldacena '08]
5. Use AdS/CFT as tool to study quantum gravity
6. …
Overview

AdS/CFT correspondence provides a fascinating link between conformal quantum field theories without gravity and string theory with (both classical and quantized) gravity

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This talk!
\( \mathcal{N} = 4 \) super Yang Mills: The simplest interacting 4d QFT

- **Field content:** All fields in adjoint of \( SU(N) \), \( N \times N \) matrices
  - Gluons: \( A_\mu, \mu = 0, 1, 2, 3, \Delta = 1 \)
  - 6 real scalars: \( \Phi_I, I = 1, \ldots, 6, \Delta = 1 \)
  - 4 \times 4 \) real fermions: \( \Psi_{\alpha A}, \bar{\Psi}_{\dot{\alpha} A} \), \( \alpha, \dot{\alpha} = 1, 2 \). \( A = 1, 2, 3, 4, \Delta = 3/2 \)
  - Covariant derivative: \( D_\mu = \partial_\mu - i[A_\mu, \ast], \Delta = 1 \)

- **Action:** Unique model completely fixed by SUSY

\[
S = \frac{1}{g_{\text{YM}}^2} \int d^4 x \text{Tr} \left[ \frac{1}{4} F_{\mu \nu}^2 + \frac{1}{2} (D_\mu \Phi_i)^2 - \frac{1}{4} [\Phi_I, \Phi_J] [\Phi_I, \Phi_J] + \bar{\Psi}^A \sigma^{\dot{\alpha} \beta}_\mu D_\mu \Psi^A_B - \frac{i}{2} \Psi_{\alpha A} \sigma^I \epsilon^{\alpha \beta} [\Phi^I, \Psi^I_B] - \frac{i}{2} \bar{\Psi}^{\dot{\alpha}} \sigma^I \epsilon^{\dot{\alpha} \dot{\beta}} [\Phi^I, \bar{\Psi}^I_B] \right]
\]

- \( \beta g_{\text{YM}} = 0 \): Quantum Conformal Field Theory, 2 parameters: \( N \) \& \( \lambda = g_{\text{YM}}^2 N \)

- Shall consider 't Hooft planar limit: \( N \to \infty \) with \( \lambda \) fixed.
Most symmetric 4d gauge theory!

- Symmetry: $\mathfrak{so}(2, 4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2, 2|4)$

Poincaré: $p^{\alpha \dot{\alpha}} = p_\mu (\sigma^\mu)^{\dot{\alpha} \beta}, \ m_{\alpha \beta}, \ \bar{m}_{\dot{\alpha} \dot{\beta}}$

Conformal: $k^{\alpha \dot{\alpha}}, \ d \ (c : \text{central charge})$

R-symmetry: $r_{AB}$

Poncaré Susy: $q^{\alpha A}, \bar{q}_{\dot{\alpha}}^A$ \hspace{1cm} Conformal Susy: $s_{\alpha A}, \bar{s}_{\dot{\alpha}}^A$

- 4 + 4 Supermatrix notation $\bar{A} = (\alpha, \dot{\alpha}|A)$

$$J_{\bar{A} \bar{B}} = \begin{pmatrix}
m^{\alpha \beta} - \frac{1}{2} \delta^{\alpha \beta} (d + \frac{1}{2} c) & k^{\alpha \dot{\beta}} & s^{\alpha B} \\
p^{\dot{\alpha} \beta} & \bar{m}^{\dot{\alpha} \dot{\beta}} + \frac{1}{2} \delta^{\dot{\alpha} \dot{\beta}} (d - \frac{1}{2} c) & \bar{q}^{\dot{\alpha} B} \\
q^{A \beta} & \bar{s}^{A \dot{\beta}} & -r^{A \dot{B}} - \frac{1}{4} \delta^{A \dot{B}} c
\end{pmatrix}$$

- Algebra:

$$[J_i \bar{A}_{\bar{B}}, J_j \bar{C}_{\bar{D}}] = \delta_{ij} \left[ \delta_{\bar{B} \bar{D}}^{\bar{C} \bar{D}} J_i \bar{A}_{\bar{B}} - (-1)(|\bar{A}| + |\bar{B}|)(|\bar{C}| + |\bar{D}|) \delta_{\bar{D} \bar{B}}^{\bar{C} \bar{D}} J_i \bar{C}_{\bar{B}} \right]$$
Observables

- **Local operators:** \( \mathcal{O}_n(x) = \text{Tr}[\mathcal{W}_1 \mathcal{W}_2 \ldots \mathcal{W}_n] \) with \( \mathcal{W}_i \in \{ \mathcal{D}^k \Phi, \mathcal{D}^k \Psi, \mathcal{D}^k F \} \)

2 point fct: 
\[
\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^2 \Delta_a(\lambda)} \quad \Delta_a(\lambda) \quad \text{Scaling Dims}
\]

3 point fct: 
\[
\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \mathcal{O}_c(x_2) \rangle = \frac{c_{abc}(\lambda)}{x_{12}^{\Delta_a+\Delta_b-\Delta_c} x_{23}^{\Delta_b+\Delta_c-\Delta_a} x_{31}^{\Delta_c+\Delta_a-\Delta_b}}
\]

\( n \)-point functions follow from OPE

- **Wilson loops:**
\[
\mathcal{W}_C = \left\langle \text{Tr} P \exp i \oint_C ds \ (\dot{x}^\mu A_\mu + i|\dot{x}| \theta^I \Phi_I) \right\rangle
\]

- **Scattering amplitudes:**
\[
\mathcal{A}_n(\{p_i, h_i, a_i\}; \lambda) = \begin{cases} 
\text{UV-finite} \\
\text{IR-divergent}
\end{cases}
\]

helicities: \( h_i \in \{0, \pm \frac{1}{2}, \pm 1\} \)
Superstring in $AdS_5 \times S^5$

\[ I = \sqrt{\lambda} \int d\tau \, d\sigma \left[ G_{mn}^{(AdS_5)} \partial_a X^m \partial^a X^n + G_{mn}^{(S^5)} \partial_a Y^m \partial^a Y^n + \text{fermions} \right] \]

- $ds^2_{AdS} = R^2 \frac{dx^2_{3+1} + dz^2}{z^2}$ has boundary at $z = 0$
- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, classical limit: $\sqrt{\lambda} \to \infty$, quantum fluctuations: $\mathcal{O}(1/\sqrt{\lambda})$
- $AdS_5 \times S^5$ is max susy background (like $\mathbb{R}^{1,9}$ and plane wave)
- **Quantization unsolved!**
- String coupling constant $g_s = \frac{\lambda}{4\pi N} \to 0$ in 't Hooft limit
- **Isometries:** $so(2, 4) \times so(6) \subset psu(2, 2|4)$
- **Include fermions:** Formulate as $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ supercoset model

[Metsaev, Tseytlin]
The AdS/CFT landscape

(planar AdS/CFT)

(Picture by N. Beisert)
Gauge Theory - String Theory Dictionary of Observables

\[ \Delta_a(\lambda) \text{ spectrum of scaling dimensions} \quad \Leftrightarrow \quad E(\lambda) \text{ string excitation spectrum solved (?)} \]

\[ c_{abc}(\lambda) \text{ structure constants} \quad \Leftrightarrow \quad \text{Only SUGRA: } \mathcal{Z}_{\text{AdS}}[\phi|_\partial\text{AdS} = J] = \mathcal{Z}_{\text{CFT}}[J] \]

\[ A_n(\{p_i, h_i, a_i\}; \lambda) \quad \Leftrightarrow \quad \text{open string amplitude} \]

\[ \mathcal{W}_C \quad \Leftrightarrow \quad \text{minimal surface} \]
The spectral problem and integrability
The spectral problem of AdS/CFT

String states resp. gauge theory local operators classified by conserved Cartan charges \((E, S_1, S_2)\) of \(so(2,4)\) (energy and "spins") and \((J_1, J_2, J_3)\) of \(so(6)\) ("angular momenta")

Geometrical picture:

\[
\begin{align*}
AdS_5 : & \quad -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2 \\
S^5 : & \quad Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2
\end{align*}
\]

- \(Z_0 + iZ_5 = \rho_3 e^{it}\), \(Z_1 + iZ_2 = \rho_1 e^{i\alpha_1}\), \(Z_3 + iZ_4 = \rho_2 e^{i\alpha_2}\): 
  3 angles \(t, \alpha_1, \alpha_2 \rightarrow 3\) conserved quantities \(E, S_1, S_2\). \(E\) is the energy.

- \(Y_1 + iY_2 = r_1 e^{i\phi_1}\), \(Y_3 + iY_4 = r_2 e^{i\phi_2}\), \(Y_5 + iY_6 = r_3 e^{i\phi}\):
  3 angles \(\phi_1, \phi_2, \phi \rightarrow 3\) conserved angular momenta \(J_1, J_2, J_3\).

<table>
<thead>
<tr>
<th>string energy ↔ scaling dimension</th>
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<tbody>
<tr>
<td>(E(\lambda) = \Delta(\lambda))</td>
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- Solid Fact I: The \(AdS_5 \times S^5\) string \(\sigma\)-model is classically integrable.
  \(\text{[Bena, Polchinski, Roiban '03]}\)
  It has been completely solved in terms of an algebraic curve.
  \(\text{[Kazakov, Marshakov, Minahan, Zarembo '04, Beisert, Kazakov, Sakai, Zarembo '05]}\)

- Solid Fact II: The full one-loop dilatation operator of \(N=4\) SYM can be mapped to a quantum integrable spin chain. It has been completely diagonalized by means of the Bethe ansatz.
  \(\text{[Minahan, Zarembo '02, Beisert, MS '03]}\)

The Spectral Problem of AdS/CFT and Integrability

A key prediction of AdS/CFT:

\[string\ energy \leftrightarrow \text{scaling dimension}\]

\[E(\lambda) = \Delta(\lambda)\]
Example 1: Rotating point particle on $S^5$

$$t = \kappa \tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2} \quad \phi_1 = \kappa \tau \quad \phi_2 = \phi_3 = \psi = 0$$

Solves eqs. of motion & Virasoro constraint (here $S_1, S_2, J_2, J_3 = 0$)

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{X}_0 = \sqrt{\lambda} \kappa \quad \boxed{E = J}$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_1 \dot{Y}_2 - Y_2 \dot{Y}_1) = \sqrt{\lambda} \kappa =: J$$

Dual gauge theory operator: $Z = \Phi_1 + i\Phi_2$ \[\text{[Berenstein,Madacena,Nastase]}\]

$$\mathcal{O}_J = \text{Tr}[Z^J] \quad \text{with} \quad \Delta(\lambda) = \Delta(\lambda = 0) = J$$

Actually classical picture only good for $J \to \infty$
Example 2: Folded spinning string: $J_1 \neq J_2 
eq 0$

Ansatz:

$$t = \kappa \tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2}$$

$$\phi_1 = \omega_1 \tau \quad \phi_2 = \omega_2 \tau \quad \phi_3 = 0 \quad \psi = \psi(\sigma)$$

Solution yields Charges and Energy

$$J_1 = \sqrt{\lambda} \omega_1 \int_0^{2\pi} d\sigma \cos^2 \psi(\sigma) \quad J_2 = \sqrt{\lambda} \omega_2 \int_0^{2\pi} d\sigma \sin^2 \psi(\sigma).$$

$$E = J \left( 1 + \lambda \frac{J}{J^2} E_1 + \frac{\lambda^2}{J^4} E_2 + \ldots \right) \quad J = J_1 + J_2$$

where $E_1 = \frac{2}{\pi^2} K(q_0) \left( E(q_0) - (1 - q_0) K(q_0) \right)$ with $\frac{J_2}{J} = 1 - \frac{E(q_0)}{K(q_0)}$

Similarly $E_l$: $l$-loop gauge theory prediction.

Dual gauge theory operator: $Z = \Phi_1 + i\Phi_2 \quad W = \Phi_3 + i\Phi_4$

$$\mathcal{O}_J = \text{Tr}[Z^{J_1} W^{J_2}] + \ldots$$

with $\Delta(\lambda) = J_1 + J_2 + \lambda \Delta_1(J_1, J_2) + \ldots$

Indeed $\lim_{J \to \infty} \Delta_1(J_1, J_2) = \frac{\lambda}{J^2} E_1$!
Operator mixing and the dilatation operator

Composite operators are renormalized and operators with degenerate \((\Delta^0, S_1, S_2; J_1, J_2, J_3)\) charges mix:

\[
\mathcal{O}^A_{\text{ren}} = \mathcal{Z}^A_B \mathcal{O}^B_{\text{bare}}
\]

Mixing matrix (\textbf{dilatation operator} \(\hat{d} \in \mathfrak{psu}(2,2|4)\))

\[
(\mathcal{D})^A_B = (\mathcal{Z}^{-1})^A_C \frac{d}{\log \Lambda} \mathcal{Z}^C_B
\]

Acts on composite operators: \(\mathcal{O}(x) = \text{Tr}[\Phi_{i_1} \Phi_{i_2} \ldots \Phi_{i_n}]\)

Eigenvalues yield scaling dims. \(\mathcal{D} \circ \mathcal{O}(x) = \Delta \mathcal{O} \mathcal{O}(x)\) [Beisert,Kristjansen,Plefka,Staudacher]

\[
\mathcal{D} = \Delta^0 + \sum_{l=1}^{\infty} \lambda^l \mathcal{D}_{l+1}
\]

\[
\mathcal{D}_k = \sum_{p=1}^{L} \mathcal{D}_k
\]

\[
\mathcal{D}(x)
\]
The dilatation operator and spin chains

- For simplicity: Consider $\mathfrak{su}(2)$ subsector
  \[ Z = \Phi_1 + i \Phi_2 \quad \text{and} \quad W = \Phi_3 + i \Phi_4 \]
  & consider operators $\mathcal{O} = \text{Tr}(\text{word in } Z \& W)$

- **Spin chain picture:** Operator $\text{Tr}(ZZWZW) \overset{\hat{=}}{=} \text{State } |↓↓↑↓↑⟩ \overset{\hat{=}}{=}$

- **One-loop structure:** $\mathcal{D}_2$ is Hamiltonian of the Heisenberg spin chain, an integrable system! [Minahan,Zarembo]

\[
\mathcal{D}_2 = 2 \sum_{l=1}^{L} (1 - P_{l,l+1}) \quad P_{i,j} : \text{permutation operator}
\]

- **Ground state:** $|↓↓\ldots↓⟩ \overset{\hat{=}}{=} \text{Tr}(Z^J)$ with $\Delta = 0$
- **Excitations:** "Magnons": $|m⟩ = |\underbrace{↑↓\ldots↓↑}_m⟩ \overset{\hat{=}}{=} \text{Tr}(WZ^mWZ^{J-m})$
Integrability

- Heisenberg spin chain is **integrable**: Existence of $L$ commuting charges $Q_n$:
  $$[Q_m, Q_n] = 0 \quad \forall (m, n)!$$

- Spectrum determined by **Bethe equations**:

  $$e^{ip_k L} = \prod_{i=1, i \neq k}^{M} S(p_k, p_i) \quad k = 1, \ldots, M$$

  With S-Matrix:

  $$S(p_i, p_k) = \frac{x^+(p_i) - x^-(p_k)}{x^-(p_i) - x^+(p_k)} \quad \text{with} \quad x^\pm(p) = \frac{1}{2} (\cot\left(\frac{p}{2}\right) \pm i)$$

  Energy (one loop scaling dimensions) additive:

  $$\Delta = L + \lambda E_2 \quad \text{with} \quad E_2(p_1, \ldots, p_M) = \sum_{k=1}^{M} 4 \sin^2 \frac{p_k}{2}$$

  + Cyclicity of trace condition: $\sum_{k=1}^{M} p_k = 0$
The asymptotic Bethe Ansatz

What happens at higher loops?

$\lambda$ deformed variables:

$$x^{\pm}(p) = \frac{e^{\pm ip/2}}{4 \sin \frac{p}{2}} \left(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right) \quad \Leftrightarrow \quad e^{ip} = \frac{x^+(p)}{x^-(p)}$$

### Asymptotic all loop conjecture:

$x^\pm_k := x^\pm(p_k)$

$$\left(\frac{x^+_k}{x^-_k}\right)^L = \prod_{j=1, j \neq k}^{M} \frac{x^+_k - x^-_j}{x^-_k - x^+_j} \frac{1 - \lambda}{1 - \frac{16\pi^2 x^+_k x^-_j}{16\pi^2 x^-_k x^+_j}} \cdot S_0(\{p_k\}, \lambda)^2 \quad S_0: \text{dressing factor}$$

- Valid for $L >$ loop order, completely fixed by $\mathfrak{psu}(2,2|4)$ symmetry up to $S_0$.
- Conjectured all loop form of $S_0$ exists [Beisert, Hernandez, Lopez; Beisert, Eden, Staudacher]
- Perturbatively: $S_0 \sim \mathcal{O}(\lambda^4)$ [Bern, Czakon, Dixon, Kosower, Smirnov]

Scaling dimensions then

$$\Delta = \Delta_0 + \sum_{k=1}^{M} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} - 1.$$
Integrability in the AdS/CFT system

- $AdS_5 \times S^5$ string $\sigma$-model is classically integrable [Bena, Polchinski, Roiban]
  Can be solved completely in terms of algebraic curve
  [Kazakov, Marshakov, Minahan, Zarembo; Beisert, Kasazkov, Sakai, Zarembo]

- Full one-loop dilatation operator has been constructed in terms of an integrable super-spin chain and diagonalized by Bethe ansatz. [Minahan, Zarembo; Beisert, Staudacher]
  Super-magnon excitations scatter according to matrix Bethe equations:

\[
e^{ip_k L} |\Psi\rangle = \left( \prod_{j=1, j \neq i}^{M} S(p_k, p_j) \right) \cdot |\Psi\rangle, \quad E = \sum_{k=1}^{M} q_2(p_k).
\]

(Assumptotic) S-matrix is assumed to be factorized. So far only proven at one-loop for all and up to four-loop for some operators.

- Wrapping problem: For finite size chains and long-range interactions not allowed to assume exactness of S-matrix!
Full set of conjectured nested $\mathfrak{psu}(2, 2|4)$ Bethe equations

\[
1 = \prod_{j=1}^{K_4} \frac{x_{4,k}^+}{x_{4,k}^-} \quad \text{Spectral parameter}: \quad x_{4,k}^\pm = \frac{1}{4}(\cot p_k/2 \pm i) \left(1 + \sqrt{1 + 16g^2 \sin^2 p_k/2}\right)
\]

\[
1 = \prod_{j=1}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3+K_1} \frac{u_{2,k} - u_{3,j} + i\frac{\eta_1}{2}}{u_{2,k} - u_{3,j} - i\frac{\eta_1}{2}}
\]

\[
1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + i\frac{\eta_1}{2}}{u_{3,k} - u_{2,j} - i\frac{\eta_1}{2}} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_1} - x_{4,k}^{-\eta_1}}{x_{4,j}^{-\eta_1} - x_{4,k}^{+\eta_1}}
\]

\[
1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right) \prod_{j=1}^{K_3+K_1} \frac{x_{4,k}^- - x_{4,j}^-}{x_{4,k}^+ - x_{4,j}^+} \prod_{j=1}^{K_5+K_7} \frac{x_{4,j}^{+\eta_1} - x_{5,j}^-}{x_{4,j}^{-\eta_1} - x_{5,j}^+}
\]

\[
1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + i\frac{\eta_2}{2}}{u_{5,k} - u_{6,j} - i\frac{\eta_2}{2}} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_2} - x_{5,j}^-}{x_{4,j}^{-\eta_2} - x_{5,j}^+}
\]

\[
1 = \prod_{j=1}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5+K_7} \frac{u_{6,j} - u_{5,j} + i\frac{\eta_2}{2}}{u_{6,j} - u_{5,j} - i\frac{\eta_2}{2}}
\]

with $\eta_1, \eta_2$ related to four different choices of $\mathfrak{psu}(2, 2|4)$ Dynkin labels, e.g.

\[
\{\eta_1, \eta_2\} = \{+1, +1\}: \quad \begin{array}{cccccccc}
\times & - & \times & + & \times & - & \times & \\
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & \\
\end{array}
\]

$g := \sqrt{\lambda}/4\pi$
The AdS/CFT (internal) S-matrix

- Describes scattering of two super-magnons, should be unitary and satisfy Yang-Baxter equation:

\[ S_{12} S_{21} = 1, \quad S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \]

- Was (ad hoc) conjectured to possess crossing symmetry:

\[ S_{12} S_{\bar{1}2} = f_{12}^2 \]

\[ \Rightarrow \text{can be used to fix dressing factor } S_0. \]

- AdS/CFT S-matrix has the structure

\[ S_{12} = \left( S_{12}^{\text{psu}(2|2)_L} \otimes S_{12}^{\text{psu}(2|2)_R} \right) S_0^2 \]

- First motivated from gauge theory spin chain, subsequently found in light-cone quantized string theory

\[ \text{[Arutyunov, Frolov, Plefka, Zamaklar '06]} \]

\[ \text{[Arutyunov, Frolov, Staudacher '04; Beisert, Staudacher '05 + '06; Beisert, Hernandez, Lopez '06, Beisert, Eden, Staudacher '06]} \]
Large Spin Limit of Twist Operators

- Consider twist operators: \( : \text{Spin} \ J_3 : \) “twist”

\[
\mathcal{O}_{S_1,J_3} = \text{Tr}(D^{S_1} Z^{J_3}) + \ldots
\]

with \( D = D_+ \) covariant derivative in light-cone direction.

- General spin chain state of length \( J_3 \) is \( \text{Tr}(D^{s_1} Z D^{s_2} Z \ldots D^{s_3} Z) \) where \( S_1 = s_1 + s_2 + \ldots s_{J_3} =: M = \text{Magnon number} \).

- Scaling dims in \( S_1 \to \infty \) limit:

\[
\Delta \mathcal{O}_{S_1,J_3} - S_1 - J_3 = \gamma(\lambda) \log S_1 + \mathcal{O}(S_1^0)
\]

\( \gamma(\lambda) \): Universal scaling function, aka cusp anomalous dimension.

- \( \gamma(\lambda) \) also appears in 4 gluon MHV amplitudes \( \mathcal{A}_{4,\text{MHV}} \) and in light-cone segmented Wilson loops \( \mathcal{W} \):

\[
\mathcal{A}^{\text{all-loop}}_{4,\text{MHV}} \sim \exp \left[ \gamma(\lambda) \mathcal{A}^{\text{one-loop}}_{4,\text{MHV}} \right], \quad \mathcal{A}^{\text{all-loop}}_{4,\text{MHV}} \sim \langle \mathcal{W} \rangle
\]

[Bern, Dixon, Smirnov]
Asymptotic Bethe equations reduce in $S_1 \to \infty$, $L = J_3 \to \infty$ with $L \ll \log S_1$ to integral equation for density $\hat{\sigma}$ of Bethe roots: ($g = \sqrt{\lambda/4\pi}$)

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[ \hat{K}(2gt,0) - 4g^2 \int_0^\infty dt' \hat{K}(2gt,2gt') \hat{\sigma}(t') \right].$$

Cusp anomalous dimensions: $\gamma(g) = 16g^2 \hat{\sigma}(0)$ **All loop prediction!**

Solution yields weak and strong coupling predictions: [BES, Basso, Korchemsky, Kotanski '07]

$$\gamma(g) = \begin{cases} 8g^2 - \frac{8}{3} \pi^2 g^4 + \frac{88}{45} \pi^4 g^6 - 16(\frac{73}{630} \pi^6 + 4 \zeta(3)^2) g^8 + \ldots & g \ll 1 \\ 4g - \frac{3 \log 2}{\pi} - \frac{K}{4\pi^2} \frac{1}{g-3 \log 2/4\pi} - \frac{27 \zeta(3)}{2^9 \pi^3} \frac{1}{g^2} - \ldots & g \gg 1 \end{cases}$$

Agrees with: 1) Four loop gauge theory calculation [Bern,Czakon,Dixon,Kosower,Smirnov '06] 
2) 2 loop superstring calculation [Roiban,Tseytlin '07]
Cusp anomalous dimension of $\mathcal{N} = 4$ SYM:

(Plot by N. Beisert)
Wrapping interactions

Asymptotic Bethe equations yield ‘half’ of the perturbative spectrum of $\mathcal{N} = 4$ SYM:

- Wrapping graphs contribute generically at order $g^{2L}$.
- Asymptotic Bethe eqs. describes $L \to \infty$ spin chain or string with worldsheet geometry $\mathbb{R}^2$ $\Rightarrow$ Existence of S-Matrix and asymptotic states.
**Thermodynamic Bethe Ansatz**

- Magnitude of finite size corrections: $\sim e^{-E_{TBA}(p_{TBA}) L}$ with $E_{TBA} = -ip$ and $p_{TBA} = -iE$ in ‘mirror’ theory, i.e. original theory with space and time interchanged.

- Approach was successfully implemented by generalization of Lüscher’s formulas for 2d Lorentz invariant FT: Computation of four loop scaling dimension of Konishi operator $\text{Tr}([Z, W][Z, W])$ from asymptotic S-matrix. [Bajnok, Janik '08]

- Agrees with perturbative four loop supergraph calculation! [Fiamberti, Santambrogio, Sieg, Zanon '08]

$$\Delta = \Delta_{aBE} + \Delta_{\text{wrapping}} \hspace{1cm} \Delta_{\text{wrapping}} = (324 + 864\zeta(3)1440\zeta(5))g^8$$

- Highly nontrivial test of AdS/CFT!!
**Recent conjecture:** Implementation of TBA through a “Y-system” to describe planar AdS/CFT at finite size. Passes all known tests!

**Result:**

\[
\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a-1,s}} = \frac{1 + Y_{a,s+1}}{1 + Y_{a+1,s}} \frac{1 + Y_{a,s-1}}{1 + Y_{a+1,s}}
\]

\[ Y_{a,s}(u) \]

**Asymptotics**

\[ Y_{a,s} \neq 0 (u \to \infty) \to \text{const}_{a,s} \]

\[ Y_{a,0} (u \to \infty) \to \left( \frac{x[-a]}{x[a]} \right)^L \times \text{const}_a \]

(from talk of V. Kazakov at KITP 02/09)
Scattering Amplitudes
Scattering amplitudes in \( \mathcal{N} = 4 \) SYM I

- \( N \)-particle scattering amplitude

\[
A_n(p_i, h_i, a_i) = (2\pi)^4 \delta^{(4)}(\sum_{i=1}^n p_i) \sum_{\sigma \in S_n/Z_n} g^{n-2} \text{tr}[t^{a_1} \ldots t^{a_n}]
\times A_n(p_{\sigma_1}, h_{\sigma_1}, \ldots, p_{\sigma_1}, h_{\sigma_1}; \lambda = g^2 N)
\]

\( A_n \): Color ordered, planar amplitude
Helicities: \( h = 0 \) scalar, \( h = \pm 1 \) gluon, \( h = \pm \frac{1}{2} \) gluino

- Commuting spinor helicity formalism:

\[
p^{\alpha \dot{\alpha}} = (\sigma^\mu)^{\alpha \dot{\alpha}} p_\mu = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \iff p_\mu p^{\mu} = \det p^{\alpha \dot{\alpha}} = 0
\]

2 spinors + choice of helicity determines polarization vector \( \varepsilon^\mu \) of gluon

\[
h = +1 \quad \varepsilon^{\alpha \dot{\alpha}} = \frac{\lambda^\alpha \tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda} \tilde{\mu}]} \quad [\tilde{\lambda} \tilde{\mu}] := \epsilon_{\dot{\alpha} \dot{\beta}} \tilde{\lambda}^{\dot{\alpha}} \tilde{\mu}^{\dot{\beta}}
\]

\[
h = -1 \quad \tilde{\varepsilon}^{\alpha \dot{\alpha}} = \frac{\mu^{\alpha \dot{\alpha}}}{\langle \lambda \mu \rangle} \quad \langle \lambda \mu \rangle := \epsilon_{\alpha \beta} \lambda^\alpha \mu^\beta \quad \mu, \bar{\mu} \text{ arbitrary}
\]
Scattering amplitudes in $\mathcal{N} = 4$ SYM II

- **Gluon amplitudes:** $A_n(1^+, 2^+, \ldots, n^+) = 0 = A_n(1^-, 2^+, \ldots, n^+)$

- **Maximally helicity violating (MHV) amplitudes**

  $A_n(1^- 2^+, \ldots j^- \ldots n^+) = A^{\text{MHV}}_{n;0} + \lambda \cdot A^{\text{MHV}}_{n;1} + \ldots = A^{\text{MHV}}_{n;0} \cdot M^{\text{MHV}}_n (\{p_i \cdot p_j\}; \lambda)$

  Park-Taylor formula: $A^{\text{MHV}}_{n;0} = i \frac{\langle 1, j \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}$  
  [Park, Taylor]

- **BDS conjecture**

  $\log M^{\text{MHV}}_n = \gamma(\lambda) \cdot M^{\text{MHV}}_{n,1\text{-loop}} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon}$  
  (true for $n = 4, 5$ known to fail for $n \geq 6$)

- **N$^k$MHV amplitudes** have rather complicated structure!  
  $\Rightarrow$ Could there be a better formulation?
On-shell superspace

- Introduce Grassmann variables $\eta_i^A$ $A = 1, 2, 3, 4$ $i = 1, \ldots, n$
- Superwavefunction:

$$
\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)
$$

- Express amplitudes compactly in on-shell superspace $(\lambda^\alpha_i, \tilde{\lambda}^{\dot{\alpha}}_i, \eta^A_i)$

$$
\mathbb{A}^{\text{MHV}}_{n;0}(\lambda_1, \tilde{\lambda}_1, \eta_1; \ldots; \lambda_n, \tilde{\lambda}_n, \eta_n) = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda^\alpha_i \tilde{\lambda}^{\dot{\alpha}}_i) \delta^{(8)}(\sum_i \lambda^\alpha_i \eta^A_i)}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}
$$

- MHV-superamplitude: General gluon$^\pm$-gluino$^{\pm1/2}$-scalar amplitude

Factor $\delta^{(8)}(\sum_i \lambda^\alpha \eta^A_i) = \left(\sum_i \lambda^\alpha \eta^A_i\right)^8$

$\eta$-expansion associates $(\eta_i)^n := \prod_{k=1}^n \eta_i^{A_k}$ with $i$th particle of helicity $1 - h/2$

$$
\Rightarrow \mathbb{A}^{\text{MHV}}_n = i(2\pi)^4 \delta^{(4)}(\sum_i p_i) \sum_{j \neq k} (\eta_j)^4 (\eta_k)^4 \mathbb{A}^{\text{MHV}}_n (1^+ \ldots j^- \ldots k^- \ldots n^+)
$$
Superamplitudes

- General form of superamplitudes:
  \[
  A_n = i(2\pi)^4 \delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_i \lambda_i \eta_i) \frac{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle}{p_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})}
  \]

- \( A_n \) is invariant under full superconformal group \( \mathfrak{psu}(2, 2|4) \):
  \( p, m, \bar{m}, k, d \oplus r \oplus q, \bar{q}, s, \bar{s} \oplus (c) \)

- Realization of \( \mathfrak{psu}(2, 2|4) \) generators in on-shell superspace, e.g. [Witten]

\[
\begin{align*}
p^{\alpha \dot{\alpha}} &= \sum_i \lambda^{\alpha}_i \tilde{\lambda}^{\dot{\alpha}}_i \\
q^{\alpha} A &= \sum_i \lambda^{\alpha}_i \eta^A_i \\
k_{\alpha \dot{\alpha}} &= \sum \frac{\partial}{\partial \lambda^{\alpha}_i} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}_i} \\
s_A &= \sum \frac{\partial}{\partial \lambda^{\alpha}_i} \frac{\partial}{\partial \eta^A_i}
\end{align*}
\]

- Less obvious sym:

- We have

\[
 p^{\alpha \dot{\alpha}} A_n = q^{\alpha} A A_n = k_{\alpha \dot{\alpha}} A_n = s_A A A_n = 0
\]

- Also: Local relation
  \[
  h_i A_n = 1 \cdot A_n
  \]
  Helicity operator:
  \[
  h_i = -\frac{1}{2} \lambda^{\alpha}_i \partial_{\alpha} + \frac{1}{2} \tilde{\lambda}^{\dot{\alpha}}_i \partial_{\dot{\alpha}} + \frac{1}{2} \eta^A_i \partial_A = 1 - c_i
  \]
  \( \Rightarrow \) (Tree) amplitudes are \( \mathfrak{su}(2, 2|4) \) invariant
On shell recursion techniques

- Efficient way of computing tree level gluon amplitudes: BCFW On shell recursion techniques
  - Closed formula for ‘split helicity’ gluon amplitudes (+ ... + − ... −)
  - Reformulation of recursion relations in on-shell superspace through shift in \((\lambda_i, \tilde{\lambda})\) and \(\eta_i\)
    - Recursion much simpler and can be solved!

\[
\mathcal{A}_n = i(2\pi)^4 \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1, 2 \rangle \langle 2, 3 \rangle \ldots \langle n, 1 \rangle} \mathcal{P}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})
\]

⇒ \(\mathcal{P}_n^{\text{tree}}\) now known analytically (implies in particular pure Yang-Mills result).

[Britto,Cachazo,Feng+Witten '04,05]
[Roiban,Spradlin,Volovich,Britto,Feng]
[Elvang et al 08, Arkani-Hamed et al 08, Brandhuber et al 08]
[Drummond,Henn]
MHV Scattering amplitudes in AdS/CFT

- Dual string description of scattering amplitudes

  ![Diagram of T-dual ↔ z=0]

  Open string amplitude on IR-brane $\leftrightarrow$ Wilson loop with light-like segments

- Cusp points determined by gluon momenta via key relation

  $$p_i^\mu = x_{i+1}^\mu - x_i^\mu$$

- Yields strong coupling prediction for four-gluon MHV amplitude via classical string theory!

- Indeed BDS conjecture for $n = 4$ gluons tested:

  $$\lim_{g \to \infty} \log M_4^{\text{MHV}} = 4g \cdot M_{n,1\text{-loop}}^{\text{MHV}} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon}$$

  $\gamma(\lambda \to \infty)$
Scattering amplitude ⇔ Wilson loop duality at perturbative level

\[ x_{i+1}^\mu - x_i^\mu = p_i^\mu \]

Planar relation:

\[
\ln \mathcal{M}_{\text{MHV}}^n = \ln \mathcal{W}_n + \text{div} + \mathcal{O}(\epsilon)
\]

\[
\mathcal{W}_n = \frac{1}{N} \left\langle \text{Tr} P \exp[ig \oint_{C_n} dx^\mu A_\mu] \right\rangle
\]

Checked up to two loops and \( n \leq 6 \) points

String interpretation: Combination of bosonic and ‘fermionic’ T-duality transformation for \( AdS_5 \times S^5 \) superstring.

\[ \Rightarrow \] Conformal invariance in dual space

\[ \Rightarrow \] Dual conformal covariance of scattering amplitudes!
Dual Superconformal symmetry

- Dual superspace

\[(x_i - x_{i+1})^{\alpha \dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \quad (\theta_i - \theta_{i+1})^{\alpha A} = \lambda_i^\alpha \eta_i^A\]

Then \(x_i^{\alpha \dot{\alpha}}\) and \(\theta_i^{\alpha A}\) have standard transformation law under (dual) conformal transformations.

- Dual superconformal algebra, with generators \(P, M, \bar{M}, K, D \oplus R \oplus Q, \bar{Q}, S, \bar{S}\), e.g.

\[K^{\alpha \dot{\alpha}} = \sum_{i=1}^{n} x_i^{\alpha \dot{\beta}} x_i^{\dot{\alpha} \beta} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_i^{\dot{\alpha} \beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta \dot{B}}}\]

Structure:

\[
\begin{array}{ccc}
p & K & \\
q & \bar{q} = \bar{S} & S \\
s & \bar{s} = \bar{Q} & Q \\
k & P & \\
\end{array}
\]

Also observed in dual string theory

[Drummond, Henn, Korchemsky, Sokatchev '08]

Also observed in the AdS dual

[Maldacena, Berkovits'08], [Beisert, Ricci, Tseytlin, Wolf '08]

[Beisert, Ricci, Tseytlin, Wolf; Berkovits, Maldacena '08]
Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Indeed $K^{\alpha\dot{\alpha}} A_n = - \sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}} A_n \Rightarrow K' = K + \sum_i x_i$ annihilates the amplitude.

- Beyond tree-level: Dual superconformal symmetry broken by IR divergences. However, breaking is under control and proportional to $\gamma(g)$ for MHV amplitudes. Conjecture: Dual superconformal 'anomaly' is the same for MHV and non-MHV amplitudes [Drummond,Henn,Korchemsky,Sokatchev '08]

- **Question:** What algebraic structure emerges when one commutes conformal with dual conformal generators? [Drummond,Henn,Plefka]

- **Task:** Transform dual superconformal generators expressed in $(x_i, \theta_i)$ into original on-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$. 

Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Open chain by dropping $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$ conditions, implemented via $\delta$-fcts: $\delta^{(4)}(p) \delta^{(8)}(q) = \delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})$

- “Non-local’ relations:

  $$x_i^{\alpha \dot{\alpha}} = x_1^{\alpha \dot{\alpha}} + \sum_{j<i} \lambda^\alpha_j \tilde{\lambda}^\dot{\alpha}_j \quad \theta_i^{\alpha A} = \theta_1^{\alpha A} + \sum_{j<i} \lambda^\alpha_j \eta^A_j$$

Set $x_1 = \theta_1 = 0$ by dual translation and susy.

- Can show that dual superconformal generator may be lifted to level 1 generators of a Yangian algebra $Y[\mathfrak{psu}(2, 2|4)]$:

  $$[J_a^{(0)}, J_b^{(0)}] = f_{ab}^\ c \ J_c^{(0)} \quad \text{conventional superconformal symmetry}$$
  $$[J_a^{(1)}, J_b^{(0)}] = f_{ab}^\ c \ J_c^{(1)} \quad \text{from dual conformal symmetry}$$

with nonlocal generators

$$J_a^{(1)} = f^c_{ab} \sum_{1<j<i<n} J_{i,b}^{(0)} J_{j,c}^{(0)}$$

and super Serre relations (representation dependent).
Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- E.g. $p^{(1)}_{\alpha\dot{\alpha}} A_n = 0$ with

  $$p^{(1)}_{\alpha\dot{\alpha}} = K'_{\alpha\dot{\alpha}} + \Delta K_{\alpha\dot{\alpha}} = \frac{1}{2} \sum_{i<j} (m_{i,\alpha} \gamma \delta_{\dot{\alpha}} + \bar{m}_{i,\dot{\alpha}} \hat{\gamma} \delta_{\alpha} - d_i \delta_{\alpha} \delta_{\dot{\alpha}}) p_{j,\gamma} + \bar{q}_i, \dot{\alpha} C q_{j,\alpha} - (i \leftrightarrow j)$$

- In supermatrix notation: $\bar{A} = (\alpha, \dot{\alpha}|A)$

  $$J^{\bar{A}}_{\bar{B}} = \begin{pmatrix}
  m^{\alpha}_{\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} (d + \frac{1}{2} c) & k^{\alpha}_{\dot{\beta}} & s^{\alpha}_{\dot{B}} \\
  p^{\alpha}_{\beta} & m^{\dot{\alpha}}_{\dot{\beta}} + \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} (d - \frac{1}{2} c) & q^{\dot{\alpha}}_{\dot{B}} \\
  q^{A}_{\beta} & \bar{s}^{A}_{\dot{\beta}} & -r^{A}_{B} - \frac{1}{4} \delta^{A}_{B} c
  \end{pmatrix}$$

  and

  $$J^{(1)}^{\bar{A}}_{\bar{B}} := - \sum_{i>j} (-1)^{|C|} (J_{i}^{\bar{A}} C J_{j}^{\bar{C}} \bar{B} - J_{j}^{\bar{A}} C J_{i}^{\bar{C}} \bar{B})$$

- Integrable spin chain picture also for colour ordered scattering amplitudes!

- Implies an infinite-dimensional symmetry algebra for $\mathcal{N} = 4$ SYM scattering amplitudes!
Same Yangian symmetry appears in the spectral problem of AdS/CFT!

[Dolan,Nappi,Witten; Beisert, Zwiebel, Torrielli, de Leeuw, ...]

Strong hint for integrability in scattering amplitudes!

Some Questions:
Does this generalize to higher loops? Most certainly yes, from string picture
Can it constrain the form of the higher loop amplitudes? In particular the ‘remainder’ function for MHV amplitudes ...
Great progress in our understanding of the maximally supersymmetric $AdS_4/CFT_3$ system

- Spectral problem (close) to exact solution!
- Integrability in scattering amplitudes at higher loops?
- What can be said about gauge theory three-point functions?
We regret that we do not have information about this document.