Superprotected n-point functions of local operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills

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Motivation: $\mathcal{N} = 4$ super Yang Mills

- Gluons A_μ, 6 scalars Φ_I, 4 gluinos $(\psi^A_\alpha, \psi^A_{\dot\alpha})$ in adjoint rep. of $SU(N)$:

$$S = \frac{1}{{g_{\rm YM}}^2} \int d^4x \, \mathrm{Tr} \left[\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} (D_\mu \Phi_i)^2 - \frac{1}{4} [\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \text{ferm.} \right]$$

- Planar limit: $N \to \infty$ with $\lambda = {g_{\rm YM}}^2 N$ fixed.
- **Most symmetric 4d gauge theory!** Symmetry: $psu(2,2|4) \in so(2,4) \otimes so(6)$

Conformal: $P_\mu, K_\mu, J_{\mu\nu}, D$	R-symmetry: R_{IJ}	$I = 1, \ldots 6$
Susy: $Q^\alpha_A, \bar{Q}^{\dot\alpha}_A$	Superconformal: $S^A_\alpha, \bar{S}^A_{\dot\alpha}$	$A = 1, \ldots, 4$

$\Rightarrow \quad 16 + 16 = 32$ supersymmetries

- Dual to $AdS_5 \times S^5$ superstring theory.
- Tremendeous advances in our understanding of $\mathcal{N} = 4$ SYM due to AdS/CFT intergability

Motivation: Local operators

- Class of observables: Correlation functions of local gauge invariant operators, e.g scalars $\mathcal{O}_{I_1 I_2 \dots I_n}(x) = \mathrm{Tr}[\, \Phi_{I_1}\, \Phi_{I_2} \dots \Phi_{I_n}\,]$
- Form of two-point functions determined by conformal symmetry:

$$\langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2) \rangle = \frac{\delta_{\alpha\beta}}{(x_1 - x_2)^{2\, \Delta_\alpha(\lambda)}}$$

AdS/CFT Integrability: Close to **exact** knowledge of $\Delta_\alpha(\lambda)$ via Bethe Eqs.

- Question: What can be said for three and higher-point correlators?

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{C_{123}(\lambda)}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\; x_{23}^{\Delta_2+\Delta_3-\Delta_1}\; x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

- 4 and higher point function very complicated!

1/2 BPS "protected" chiral primary operators I

- Simplest' class of operators: $\mathcal{O}_{\text{CPO}} = \text{Tr}[\Phi_{\{I_1} \dots \Phi_{I_n\}}]$ with $_{\{\dots\}}$: sym, tracel.
- Introduce $\boxed{Z_u(x) = u^I \Phi^I(x)}$ and write $\mathcal{O}_{\text{CPO}} = \text{Tr}[Z(x)^J]$ with $u^I \in \mathbb{C}$ and null condition $u \cdot u = 0$. [Arutyunov,Dolan,Osborn,Sokatchev]

$$\text{Susy:} \qquad \delta\Phi^I = \bar{\psi}\,\gamma^5\,\rho^I(\epsilon_0 + \gamma_\mu x^\mu \epsilon_1)$$

$$\Rightarrow \quad \delta Z(x=0) \overset{!}{=} 0 \quad \Leftrightarrow \quad u^I \rho^I \epsilon_0 = 0 \quad \Leftrightarrow \quad u \cdot u = 0$$

Preserves **24** supercharges (8 super-Poincaré and 16 superconformal).

- Two-point functions receive no radiative corrections!

$$\left\langle \text{Tr} Z_1^J(x_1)\, \text{Tr} Z_2^J(x_2) \right\rangle = \left\langle \text{Tr} Z_1^J(x_1)\, \text{Tr} Z_2^J(x_2) \right\rangle_0$$

$$= \left\langle \text{Tr}[u_1 \cdot \Phi(x_1)]^J\, \text{Tr}[u_2 \cdot \Phi(x_2)]^J \right\rangle_0 = J \Bigg(\underbrace{\frac{u_1 \cdot u_2}{4\pi^2 (x_1 - x_2)^2}}_{=:[12]} \Bigg)^J$$

- $\text{Tr} Z_1^J(x_1)$ and $\text{Tr} Z_1^J(x_2)$ share **16** supercharges.

1/2 BPS "protected" chiral primary operators II

- Three-point functions: ($[ij] := \frac{u_i \cdot u_j}{4\pi^2 (x_i - x_j)^2}$)

$$\left\langle \mathrm{Tr} Z_1^{J_1}(x_1)\, \mathrm{Tr} Z_2^{J_2}(x_2)\, \mathrm{Tr} Z_3^{J_3}(x_3) \right\rangle = C_{123}\, [12]^{J_1+J_2-J_3} [23]^{J_2+J_3-J_1} [13]^{J_3+J_1-J_2}$$

  Also no radiative corrections! All three operators share **at least 8** supercharges.

- Four-point functions are notrivial:

$$\left\langle \mathrm{Tr} Z_1^{k}(x_1)\, \mathrm{Tr} Z_2^{k}(x_2)\, \mathrm{Tr} Z_3^{k}(x_3)\, \mathrm{Tr} Z_4^{k}(x_4) \right\rangle = \text{tree} + \mathcal{R} \sum_{m+n+l=k-2} \mathcal{F}^{(k)}_{mnl}(s,t,\lambda)\, \mathcal{X}^m\, \mathcal{Y}^n\, \mathcal{Z}^l$$

  Receive loop-correction. Generically share **no** supersymmetries.

- Would like to attribute the complexity of the 4-point function to this.

Structure of 4-point functions

- The 4-point function is very complicated [Arutyunov,Dolan,Nirschl,Osborn,Sokatchev]

$$\left\langle \mathrm{Tr} Z_1^k(x_1)\, \mathrm{Tr} Z_2^k(x_2)\, \mathrm{Tr} Z_3^k(x_3)\, \mathrm{Tr} Z_4^k(x_4) \right\rangle$$
$$= \text{tree} + \mathcal{R} \sum_{m+n+l=k-2} \mathcal{F}^{(k)}_{mnl}(s,t,\lambda)\, \mathcal{X}^m\, \mathcal{Y}^n\, \mathcal{Z}^l$$

where

$$\mathcal{X} = [12][34]\,, \qquad \mathcal{Y} = [13][24]\,, \qquad \mathcal{Z} = [14][23]\,.$$

- Conformal invariant cross ratios

$$s = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}\,, \qquad t = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}\,, \qquad x_{ij}^2 = (x_i - x_j)^2\,.$$

- Universal prefactor

$$\mathcal{R} = s\,\mathcal{X}^2 + \mathcal{Y}^2 + t\,\mathcal{Z}^2 + (s-t-1)\,\mathcal{Y}\,\mathcal{Z}$$
$$+ (1-s-t)\,\mathcal{X}\,\mathcal{Z} + (t-s-1)\,\mathcal{X}\,\mathcal{Y}\,.$$

$\mathcal{F}^{(k)}_{mnl}(s,t,\lambda)$ is known to two loop order. [Arutyunov, Penati,Santambrogio,Sokatchev]

Outline

We pose and answer two questions:

- Are there ways to choose four operators or more so that they share SUSY?
- Will the n-point function be protected?

Plan:

1. Trivial example
2. Example I
   - Four-point function
   - Symmetry
3. Example II
4. Side result: General one-loop insertion formula
5. Some five and six-point functions
6. Outlook

Trivial example

- Take $u^I = (1, i, 0, 0, 0, 0)$ so $Z_u = \Phi^1 + i\Phi^2$
- $\mathrm{Tr} Z^J(0)$ preserves 8 super-Poincaré charges (Qs) and all 16 superconformal ones (the Ss). At a different position $\mathrm{Tr} Z^J(x)$ will no longer preserve the Ss, but it will still preserve the same eight Q
- Indeed n-point function is protected

$$\left\langle \mathrm{Tr} Z^{J_1}(x_1)\, \mathrm{Tr} Z^{J_2}(x_2) \cdots \mathrm{Tr} Z^{J_n}(x_n) \right\rangle = 0\,.$$

This is a rather trivial example, as the R-symmetry charges are not balanced.

Example I

- Central idea: Make $\boxed{u^I = u^I(x)}$ space-time-dependent!
- Choose following combination of the 6 real scalars: [de Medeiros, Hull, Spence, Figueroa-O'Farrill]

$$C(x) := 2i\, x_\mu\, \Phi^\mu(x) + i\,[1-(x_\mu)^2]\,\Phi^5(x) + [(1+(x_\mu)^2]\,\Phi^6(x)$$

I.e. $u^I = (2ix_\mu, i(1-x^2), (1+x^2))$. For example

$$C(0) \propto \Phi^6 + i\Phi^5\,, \quad C(\infty) \propto \Phi^6 - i\Phi^5 = C(0)^\dagger\,, \quad C(1,0,0,0) \propto \Phi^6 + i\Phi^1\,.$$

- Crucial property:

$$[12] = \langle C(x_1)\, C(x_2)\rangle_0 = \frac{-4x_1^\mu x_2^\mu - (1-x_1^2)(1-x_2^2) + (1+x_1^2)(1+x_2^2)}{4\,\pi^2(x_1-x_2)^2} = \frac{1}{2\pi^2}$$

Constant independent of x_1 and x_2! (similar to circular Wilson-loops)

4-point function of $C(x)^J$

- We have the general result

$$\left\langle \mathrm{Tr}C^k(x_1)\,\mathrm{Tr}C^k(x_2)\,\mathrm{Tr}C^k(x_3)\,\mathrm{Tr}C^k(x_4)\right\rangle$$
$$= \text{tree} + \mathcal{R} \sum_{m+n+l=k-2} \mathcal{F}^{(k)}_{mnl}(s,t)\,\mathcal{X}^m\,\mathcal{Y}^n\,\mathcal{Z}^l$$

- Recall $\quad \mathcal{X} = [12][34]\,, \qquad \mathcal{Y} = [13][24]\,, \qquad \mathcal{Z} = [14][23]\,.$

- But for our $C(x)$: $[ij] = \text{const!}$ Hence $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = 1/4\pi^4$, so

$$\mathcal{R} = s\,\mathcal{X}^2 + \mathcal{Y}^2 + t\,\mathcal{Z}^2 + (s-t-1)\,\mathcal{Y}\,\mathcal{Z}$$
$$+ (1-s-t)\,\mathcal{X}\,\mathcal{Z} + (t-s-1)\,\mathcal{X}\,\mathcal{Y} = 0$$

No one-loop corrections to the 4-point function!

Supersymmetry analysis

- Require: $\boxed{\delta C(x) = \bar{\psi}\,\gamma^5\, u^I_C(x)\rho^I(\epsilon_0 + \gamma_\mu x^\mu \epsilon_1) = 0}$ for ϵ_0 & ϵ_1 independent of x^μ:

$$\Rightarrow \qquad \boxed{(\gamma^{\mu\nu} + \rho^{\mu\nu})\epsilon_0 = 0 \qquad \epsilon_1 = i\gamma^1\,\rho^{16}\epsilon_0}$$

- $\epsilon_0 = (\epsilon_0^{+\alpha}{}_A, \epsilon_0^{-\dot{\alpha}}{}_A)$ in $(\mathbf{2},\mathbf{1},\mathbf{4}) \oplus (\mathbf{1},\mathbf{2},\bar{\mathbf{4}})$ rep of $su(2)_L \times su(2)_R \times su(4)$. Break R-symmetry to $su(2)_A \times su(2)_B$ and decompose $\mathbf{4} \longrightarrow (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$, i.e split $A = (a, \dot{a})$.

- Above condition solved for $\epsilon_0^{+\alpha}{}_a = \delta^\alpha_a\,\epsilon_0^+$ and $\epsilon_0^{-\alpha}{}_a = \delta^{\dot{\alpha}}_{\dot{a}}\,\epsilon_0^-$

  $\Rightarrow$ **2** supercharges $\mathcal{Q}^\pm$ preserved:

$$\left\langle \mathcal{Q}^\pm\, \left(\mathrm{Tr}\,C^{J_1}(x_1)\,\mathrm{Tr}\,C^{J_2}(x_2)\,\ldots\,\mathrm{Tr}\,C^{J_n}(x_n)\right)\right\rangle = 0$$

$\Rightarrow$ n-point function of $\mathrm{Tr}\,C^J(x)$ is **1/16 BPS**.

Twisted conformal symmetry

- Bosonic symmetries act on field Φ^i of scaling dimensions Δ:

$$P_\mu \Phi^i = -i\partial_\mu \Phi^i , \qquad K_\mu \Phi^i = -i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu)\Phi^i ,$$
$$D\,\Phi^i = -i(x^\mu\partial_\mu + \Delta)\Phi^i , \qquad M_{\mu\nu}\,\Phi^i = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi^i ,$$
$$R_{ij}\,\Phi^k = -i(\delta_i^k \Phi_j - \delta_j^k \Phi_i) .$$

- Our operator $C(x)$ transforms covariantly under the twisted generators

$$\begin{aligned}
\tilde{P}_\mu &= P_\mu + R_{5\mu} + iR_{6\mu} , \\
\tilde{M}_{\mu\nu} &= M_{\mu\nu} + R_{\mu\nu} , \\
\tilde{D} &= D + iR_{56} , \\
\tilde{K}_\mu &= K_\mu - R_{5\mu} + iR_{6\mu} .
\end{aligned}$$

  with twisted scaling dimensions $\tilde{\Delta} = 0$.

- Explains two-point function $\langle C(x)\, C(y)\rangle_0 =$ const

Proof of protectedness of $\mathrm{Tr}\, C^J$ n-point functions

- Topological field theory description: Action was shown to be BRS-exact w.r.t. $\mathcal{Q}^{\pm}$ [de Medeiros, Hull, Spence, Figueroa-O'Farrill,01]

$$S = \{\, \mathcal{Q}^{\pm} \,,\, \Psi^{\pm} \,\} \qquad \text{with suitable fermion field } \Psi^{\pm}$$

- Then radiative corrections to n-point correlator vanish:

$$\begin{aligned} &\frac{\partial}{\partial\, g_{YM}^2} \big\langle \mathrm{Tr}\, C^{J_1}(x_1)\ \mathrm{Tr}\, C^{J_2}(x_2) \cdots\ \mathrm{Tr}\, C^{J_n}(x_n) \big\rangle \\ &\quad = \frac{i}{{g_{\mathrm{YM}}}^4} \big\langle \{\, \mathcal{Q}^{+} \,,\, \Psi^{+} \,\}\, \mathrm{Tr}\, C^{J_1}(x_1)\ \mathrm{Tr}\ \cdots\ \mathrm{Tr}\, C^{J_n}(x_n) \big\rangle \\ &\quad \propto \mathcal{Q}^{+} \big\langle \Psi^{\pm}\ \mathrm{Tr}\, C^{J_1}(x_1)\ \cdots\ \mathrm{Tr}\, C^{J_n}(x_n) \big\rangle = 0 \end{aligned}$$

- Hence $\big\langle \mathrm{Tr}\, C^{J_1}(x_1) \ldots\ \mathrm{Tr}\, C^{J_n}(x_n) \big\rangle = \big\langle \mathrm{Tr}\, C^{J_1}(x_1) \ldots\ \mathrm{Tr}\, C^{J_n}(x_n) \big\rangle_0$

Example II

- Restrict local operators to plane in $\mathbb{R}^4$.
- Introduce complex coordinates $w = x_1 + ix_2$ and $\bar{w} = x_1 - ix_2$ in $\mathbb{R}^2$:

$$Z = i(1-\bar{w}^2)\,\Phi^1 + (1+\bar{w}^2)\,\Phi^2 - 2i\bar{w}\,\Phi^3$$

  Not the same as C restricted to $\mathbb{R}^2$!
- Susy analysis: Break $su(4) \to su(2)_{A'} \times su(2)_{B'}$ and now decompose $\mathbf{4} \to (\mathbf{2},\mathbf{2})$. Generically four supercharges are preserved $\Rightarrow$ 1/8 BPS correlation functions.
- Now the two-point function is

$$[12] = \langle Z(w_1)\,Z(w_2)\rangle_0 \propto \frac{(\bar{w}_1 - \bar{w}_2)^2}{|w_1 - w_2|^2} = \frac{\bar{w}_1 - \bar{w}_2}{w_1 - w_2}$$

  Same as a ($N \times N$ matrix valued) 2d conformal field of weight $(\frac{1}{2}, -\frac{1}{2})$.

Four-Point function

- Parametrize the complex cross ratio differently (recall $x_{12}^2 = w_{12}\,\bar{w_{12}}$)

$$s = \mu\bar{\mu}\,, \qquad t = (1-\mu)(1-\bar{\mu})\,, \qquad \mu = \frac{w_{12}w_{34}}{w_{13}w_{24}} \in \mathbb{C}.$$

- Universal prefactor then

$$\mathcal{R} = \Big(\mu\,(\mathcal{X}-\mathcal{Z}) + \mathcal{Z} - \mathcal{Y}\Big)\,\Big(\bar{\mu}\,(\mathcal{X}-\mathcal{Z}) + \mathcal{Z} - \mathcal{Y}\Big)\,.$$

- Now

$$\frac{\mathcal{X}}{\mathcal{Y}} = \frac{\bar{\mu}}{\mu}\,, \qquad \frac{\mathcal{Z}}{\mathcal{Y}} = \frac{1-\bar{\mu}}{1-\mu} \qquad \Rightarrow \quad \mu\,(\mathcal{X}-\mathcal{Z}) + \mathcal{Z} - \mathcal{Y} = 0\,,$$

- Again there are no quantum corrections to the four-point functions! Topological theory?
- First step ⇒ Test protectedness for higher point functions at one-loop!

General structure of one loop n-point functions of chiral primaries

- Study the general n-point problem @ 1-loop: $\mathcal{O}_k^u(x) := \mathrm{Tr}[(u\cdot\Phi(x))^k]$

$$\boxed{\Big\langle \mathcal{O}_{k_1}^{u_1}(x_1)\,\mathcal{O}_{k_2}^{u_2}(x_2)\,\mathcal{O}_{k_3}^{u_3}(x_3)\ldots\mathcal{O}_{k_n}^{u_n}(x_n)\Big\rangle_{\text{1-loop}} = ?}$$

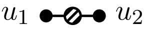
- Relevant integrals

$$\begin{aligned}
I_{12} &= \frac{1}{(2\pi)^2(x_1-x_2)^2}, \\
Y_{123} &= \int d^4w\, I_{1w}I_{2w}I_{3w}, \\
X_{1234} &= \int d^4w\, I_{1w}I_{2w}I_{3w}I_{4w}, \\
H_{12,34} &= \int d^4u\, d^4v\, I_{1u}I_{2u}I_{uv}I_{3v}I_{4v}.
\end{aligned}$$

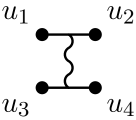
with $X_{1234} = \dfrac{\pi^2\Phi(s,t)}{(2\pi)^8(x_1-x_3)^2(x_2-x_4)^2}$, $\qquad Y_{123} = \lim_{x_4\to\infty}(2\pi)^2\, x_4^2\, X_{1234}$

$\Phi(s,t)$ is explicitly known scalar box integral.

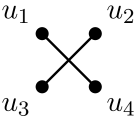
- Established plug-in formulas to ‘dress’ tree-graphs:



$$= -\lambda\,(u_1\cdot u_2)\,(Y_{112}+Y_{122})$$



$$= \frac{\lambda}{2}\,(u_1\cdot u_2)\,(u_3\cdot u_4)\,I_{12}\,I_{34}\,F_{12,34}$$



$$= \frac{\lambda}{2}\Big[\,2\,(u_2\cdot u_3)\,(u_1\cdot u_4) - (u_2\cdot u_4)\,(u_1\cdot u_3) - (u_1\cdot u_2)\,(u_3\cdot u_4)\Big]\,X_{1234}$$

- Important relation [Beisert,Krstjansen,Plefka,Staudacher,Semenoff]

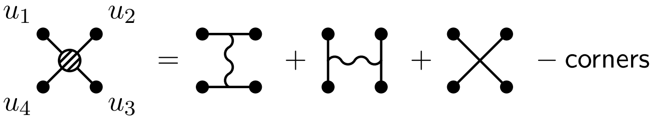
$$F_{12,34} = \frac{(\partial_1-\partial_2)\cdot(\partial_3-\partial_4)H_{12,34}}{I_{12}I_{34}} = \frac{X_{1234}}{I_{13}I_{24}} - \frac{X_{1234}}{I_{14}I_{23}} + G_{1,34} - G_{2,34} + G_{3,12} - G_{4,12}$$

$$G_{1,34} = \frac{Y_{134}}{I_{14}} - \frac{Y_{134}}{I_{13}}\,.$$

Approach: Dress tree level graphs with insertion vertices

- Dressed two-gons: Cancel!
- Dressed three-gons: Cancel!
- Dressed $(n > 3)$-gons: Cancelation of two and three-point insertions. Only effective four-point interaction survives:

$D_{1234} \equiv$



$$= \frac{\lambda}{2} \frac{X_{1234}}{I_{13} I_{24}} \big(2\,[13][24] + (s-1-t)[14][23] + (t-1-s)[12][34] \big)$$

Y_{ijk} functions cancel in effective vertex.

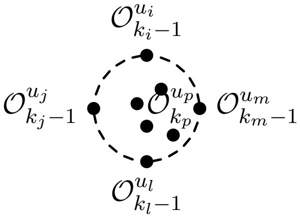
General one-loop insertion formula

- Reduction of general one-loop n-point function to tree-level disc amplitudes:

$$\left\langle \mathcal{O}^{u_1}_{J_1} \cdots \mathcal{O}^{u_n}_{J_n} \right\rangle_{\text{1-loop}} = \sum_{i,j,k,l} J_i J_j J_k J_l \, D_{ijkl} \left\langle \mathcal{O}^{u_i}_{J_i-1} \mathcal{O}^{u_j}_{J_j-1} \mathcal{O}^{u_k}_{J_k-1} \mathcal{O}^{u_l}_{J_l-1} \,\middle|\, \mathcal{O}^{u_1}_{J_1} \cdots \mathcal{O}^{u_n}_{J_n} \right\rangle_{\text{tree, disc}}$$

with

$$\left\langle \mathcal{O}^{u_i}_{k_i-1} \mathcal{O}^{u_j}_{k_j-1} \mathcal{O}^{u_l}_{k_l-1} \mathcal{O}^{u_m}_{k_m-1} \,\middle|\, \prod_{p\neq i,j,l,m} \mathcal{O}^{u_p}_{k_p} \right\rangle_{\text{tree, disc}} =$$



- Cancelation may only occur within set $ijlm$ of 4points points via identity $D_{ijlm} + D_{iljm} + D_{ijml} = 0$

Sample calculation

Computed a selection of four, five and six point functions at one loop:

- Simplest example $\langle 2|2|2|2\rangle$:

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\rangle_{\text{1-loop}} = 16\big(D_{1234}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}\rangle_{\text{tree, disc}} + D_{1324}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\rangle_{\text{tree, disc}} + D_{1243}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\mathcal{O}_1^{u_3}\rangle_{\text{tree, disc}}\big).$$

- For given ordering there are two planar tree diagrams with a pair of contractions: $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\rangle_{\text{1-loop}} &= 16\big(D_{1234}(\mathcal{X}+\mathcal{Z}) + D_{1243}(\mathcal{Y}+\mathcal{X}) + D_{1324}(\mathcal{Z}+\mathcal{Y})\big)\\ &= -16\big(D_{1234}\mathcal{Y} + D_{1243}\mathcal{Z} + D_{1324}\mathcal{X}\big) = -\frac{\lambda}{\pi^2}\,\Phi(s,t)\,\mathcal{R}.\end{aligned}$$

Used $(D_{1234}+D_{1243}+D_{1324}) = 0$.

Examples: 5-point functions

- $\langle 2|2|2|3|3\rangle$:

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_3^{u_5}\rangle_{\text{1-loop}} = \frac{9}{4}[45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} + \frac{9}{2}[41][15]\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} + \frac{9}{2}[42][25]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} + \frac{9}{2}[43][35]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}}\,.$$

- $\langle 2|2|2|5|5\rangle$:

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_5^{u_4}\mathcal{O}_5^{u_5}\rangle = \frac{25}{4}[45]^3\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + \frac{75}{2}[35][35][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + \frac{75}{2}[25][25][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + \frac{75}{2}[15][15][45]^2\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle$$

- In all the 12 examples we checked these can be written as the basic four- $\langle 2|2|2|2\rangle$ and five-point functions $\langle 2|2|2|2|2\rangle$ times free contractions.
  $\Rightarrow$ No general proof yet.

Back to the "superprotected" operators

- Important relation:

$$D_{1234} = \frac{\lambda}{2}\,\frac{X_{1234}}{I_{13}I_{24}}\big(2\mathcal{Y} + (s-1-t)\mathcal{Z} + (t-1-s)\mathcal{X}\big) \propto \frac{\partial\mathcal{R}}{\partial\mathcal{Y}}$$

- $\frac{X_{1234}}{I_{13}I_{24}}$ is a transcendental function of the cross-ratios s and t, so cancelations should happen for each choice of four points independently.

- Indeed for example I: $\mathcal{X} = \mathcal{Y} = \mathcal{Z} =$ const, so $D_{ijkl} = 0$!
$\Rightarrow$ Perurbative proof of vanishing of all n-point one-loop functions

- For example II this does not happen. Here have to use the modular property

$$D_{1234} = -\frac{1}{\mu}\,D_{1324} = -\frac{1}{1-\mu}\,D_{1243}.$$

We checked five and six-point functions with total weight ≤ 16 and found that the sum of these triplets always vanished for example II!

But: No general proof yet.

Conclusions and Outlook

- The answer is **yes we can**:
  - One can choose more than three operators that share SUSYs.
  - They seem to receive no perturbative corrections ("superprotected").
- Possible to combine with BPS Wilson loop operators, surface operators, etc.
- Many open questions:
  - Prove that there are no quantum corrections.
  - Do the strong-coupling supergravity AdS calculation.
  - Understand the twisted symmetry and its multiplets.
  - Understand the topological theories.
  - Calculate instanton corrections.
- Other examples: Any three $1/2$ BPS operators share eight supercharges. Should be possible to add more operators on the line. Use $\mathcal{R} = 0$?
- Grand goal: Use these operators as the starting point to calculate n-point functions of general operators $\Leftrightarrow$ Good ground states in a spin-chain picture?