D. Hennig, D. Robaschik, K. Scharnhorst: *Aspects of the renormalization of QED in a singular external gauge potential.* In: V.A. Matveev, V.A. Rubakov, A.N. Tavkhelidze, P.G. Tinyakov (Eds.): Proceedings of the International Seminar Quarks '90, Telavi, USSR, May 14-19, 1990. World Scientific, Singapore 1991, pp. 218-222.

ASPECTS OF THE RENORMALIZATION OF QED IN A SINGULAR EXTERNAL GAUGE POTENTIAL

D. HENNIG
Sektion Physik, Humboldt-Universität zu Berlin, Invalidenstr. 110, DDR — 1040 Berlin, GDR

D. ROBASCHIK
Sektion Physik and NTZ, Karl-Marx-Universität Leipzig, Karl-Marx-Platz, DDR — 7010 Leipzig, GDR

and

K. SCHARNHORST
INTSEM and NTZ, Karl-Marx-Universität Leipzig, Goethestr. 6, DDR — 7010 Leipzig, GDR

ABSTRACT

QED is considered in the presence of a delta function shaped external gauge potential with support on a plane. The divergent part of the 1-loop fermion self-energy taken in this background field configuration is calculated. On the basis of this example specific renormalization problems arising in QED in a singular background are commented.

QED in background fields has been studied widely for various reasons. Among them are the following. From one hand side it is attractive for its potential or real experimental relevance. On the other hand, from a quantum field theoretic point of view it draws attention due to the broad opportunities to obtain nonperturbative (in the background field) results which admit to explore the implications of the theory not easily be studied by other, mainly perturbative methods. Most investigations are using constant or/and periodic background fields which can be handled without serious technical difficulties. Interesting insights have been obtained among which the mechanism of pair production out of the vacuum in an external electric field discovered by Schwinger [1] perhaps is the best known.

However, localized and asymptotically decaying background fields as they are produced by a point charge or by a cosmic string are of major physical relevance too, but technical difficulties in treating these as background fields in the framework of QED are much more severe than in those cases mentioned above and have been less extensively studied therefore.

In the present paper we are dealing with QED in the presence of a delta function shaped external gauge potential with support on a plane. This background may be considered as a model for a strongly localized field configuration. As an attractive feature on the quantum mechanical level the Dirac equation taken in the presence of this external gauge potential exhibits bound states which also accounts for rich properties of the field theory. Relations of the corresponding model with two delta functions having its support on parallel planes to the Casimir effect problem should be mentioned too. The present investigation is a continuation of Refs. 2., 3..

We are considering the following external gauge potential.

$$eA_\mu = (eA_0, 0, 0, 0) \; ; \qquad eA_0 \;=\; a\, \delta(x_3) \tag{1}$$

For the discussion following below here one should take notice that the parameter a characterizing the external field is a dimensionless quantity. The Dirac propagator in the presence of the gauge potential (1) obeys the equation

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\; S(x,x') \;=\; -\; \delta^{(4)}(x-x') \tag{2}$$

and can be given as the sum of the free Dirac propagator and an additional term using the Ansatz

$$S(x,x') \;=\; S^c(x-x') \;+\; \bar{S}(x,x') \tag{3}$$

$$S^c(x) \;=\; -\int \frac{d^4p}{(2\pi)^4}\; e^{-ipx}\; \frac{\not p + m}{p^2 - m^2 + i\epsilon} \tag{4}$$

$$\bar{S}(x,x') \;=\; \int d^4y\; d^4y'\; S^c(x-y)\; C(y,y')\; S^c(y'-x)\;\; . \tag{5}$$

One finds for the form factor

$$C(y,y') = -\; a\; \delta(y_3)\; \delta(y'_3) \int \frac{d^3\tilde{p}}{(2\pi)^3}\; e^{-i\tilde{p}(\tilde{y}-\tilde{y}')}\; \frac{[\gamma^0\Gamma - (ia/2)(\not\tilde{p} - m)]}{\lambda_-\Gamma - iap_0} \tag{6}$$

$$\tilde{p} \;=\; (p_0, p_1, p_2, 0)\;,\;\; \tilde{p}^2 \;=\; p_0^2 - p_1^2 - p_2^2\;,\;\; \Gamma \;=\; (\tilde{p}^2 + i\epsilon)^{1/2}\;,\;\; \lambda_\pm \;=\; 1 \pm a^2/4$$

which leads to the additional propagator term [2]

$$\bar{S}(x,x') \;=\; -\,a \int \frac{d^3\tilde{p}\; dq_3\; dk_3}{(2\pi)^5}\; e^{-i\tilde{p}(\tilde{x}-\tilde{x}') + ik_3x_3 - iq_3x'_3}\; \frac{\not\tilde{p} - \not k_3 + m}{\tilde{p}^2 - k_3^2 - m^2 + i\epsilon}\; \cdot$$
$$\cdot\; \frac{[\gamma^0\Gamma - (ia/2)(\not\tilde{p} - m)]}{\lambda_-\Gamma - iap_0}\, \frac{\not\tilde{p} - \not q_3 + m}{\tilde{p}^2 - q_3^2 - m^2 + i\epsilon} \qquad , \tag{7}$$
$$= \frac{a}{4} \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\Gamma^2}\; e^{-i\tilde{p}(\tilde{x}-\tilde{x}') + i\Gamma(|\, x_3 \,| + |\, x'_3 \,|)}\; \cdot$$
$$\cdot\; (\not\tilde{p} + m - \epsilon(x_3)\; \not\Gamma_3)\frac{[\gamma^0\Gamma - (ia/2)(\not\tilde{p} - m)]}{\lambda_-\Gamma - iap_0}(\not\tilde{p} + m + \epsilon(x'_3)\; \not\Gamma_3)\;,$$
$$\not\Gamma_3 \;=\; \gamma^3\, \Gamma \qquad . \tag{8}$$

For illustration, let us also display the x-space expression for the form factor (6) in the $m = 0$ case.

$$
\begin{aligned}
C(y, y') \;=\; & - a\,\delta(y_3)\,\delta(y'_3) \left\{ \frac{1}{\lambda_-}\,\gamma^0\,\delta^{(3)}(\tilde{Y}) - \right. \\
& - \frac{a}{4\pi^2}\,(\lambda_+\,\not{\partial}_0 - \lambda_-\,\tilde{\tilde{\not{\partial}}})\,\frac{1}{\lambda_-^2 Y_0^2 + \lambda_+^2 \tilde{\tilde{Y}}^2 - i\epsilon}\left[1 + \right. \\
& \left.\left. + \frac{\epsilon(\lambda_-) a Y_0}{(\lambda_-^2 Y_0^2 + \lambda_+^2 \tilde{\tilde{Y}}^2 - i\epsilon)^{1/2}}\,\mathrm{arcctg}\left(\frac{-\epsilon(\lambda_-) a Y_0}{(\lambda_-^2 Y_0^2 + \lambda_+^2 \tilde{\tilde{Y}}^2 - i\epsilon)^{1/2}} \right) \right] \right\}, \\
& \tilde{Y} = \tilde{y} - \tilde{y}' \quad , \quad \tilde{\tilde{Y}} = (0, Y_1, Y_2, 0) \qquad (9)
\end{aligned}
$$

Note, that the Dirac propagator (3) in the strongly localized background (1) has been obtained in a closed form well suited for loop calculations later on. Using the propagator (3) one may proceed to study fully interacting QED in the background (1).

Now, first thought has to be devoted to the renormalization of QED considered in the presence of this external gauge potential. Generally accepted folklore is that after having renormalized ordinary QED without external fields the corresponding theory in external fields automatically is finite inasmuch as these external fields always show up in the renormalization group invariant combination of coupling constant times external field only. But this is only true as long as the external gauge potential does not have any singularities itself (cf. the argumentation in Ref. 4.). Definitely this is the case for constant and free wave external fields mentioned above but certainly not applicable in the situation we are dealing with. Therefore, special attention has to be paid to the renormalization problem of quantum field theory in a singular background.

As one piece of the problem here we are focusing on the study of the divergent part of the 1-loop self-energy diagram where the external gauge potential (1) has been taken into account exactly.

The 1-loop self-energy is given by the following expression

$$
\begin{aligned}
\Sigma(x, x') &= \Sigma^c(x - x') + \bar{\Sigma}(x, x') \\
&= - i\,e^2\,\gamma^\mu\,S^c(x - x')\,\gamma^{\mu'}\,D^c_{\mu\mu'}(x - x') \\
&\quad - i\,e^2\,\gamma^\mu\,\bar{S}(x, x')\,\gamma^{\mu'}\,D^c_{\mu\mu'}(x - x') \qquad (10)
\end{aligned}
$$

where the photon propagator

$$
D^c_{\mu\mu'}(x) = - \int \frac{d^4k}{(2\pi)^4}\,e^{-ikx}\,\frac{1}{k^2 + i\epsilon}\left[g_{\mu\mu'} - (1 - \alpha)\frac{k_\mu k_{\mu'}}{k^2 + i\epsilon} \right] \qquad (11)
$$

is taken in the covariant gauge with gauge parameter α. The first term in Eq. (10) is standard, therefore we have to calculate the second term only. Explicitly it reads

$$
\begin{aligned}
\bar{\Sigma}(x,x') \;&=\; -\,i\;a\;e^2 \int \frac{d^4k}{(2\pi)^4}\; e^{-ik(x-x')} \int \frac{d^3\tilde{p}\;dq_3\;dp_3}{(2\pi)^5}\; e^{ip_3x_3 - iq_3x_3'}\;\cdot \\
&\cdot\;\gamma^{\mu}\frac{\tilde{\not p} + \not p_3 + m}{\tilde{p}^2 - p_3^2 - m^2 + i\epsilon}\,\frac{[\gamma^0\Gamma - (ia/2)(\tilde{\not p} - m)]}{\lambda_-\Gamma - iap_0}\;\cdot \\
&\cdot\;\frac{\tilde{\not p} + \not q_3 + m}{\tilde{p}^2 - q_3^2 - m^2 + i\epsilon}\gamma^{\mu'}\frac{1}{K^2 + i\epsilon}\left[g_{\mu\mu'} - (1-\alpha)\frac{K_\mu K_{\mu'}}{K^2 + i\epsilon}\right]\;, \\
K \;&=\; (\tilde{p} - \tilde{k}, k_3)\;.
\end{aligned} \tag{12}
$$

Inasmuch as the further calculation is not completely standard we are going to describe its main steps. After successive shifts of the integration variables $p_3 \to p_3 + q_3$, $k_3 \to k_3 - q_3$, $q_3 \to q_3 + k_3$ we reach at

$$
\begin{aligned}
\bar{\Sigma}(x,x') \;&=\; -\,i\;a\;e^2 \int \frac{d^4k}{(2\pi)^4}\; e^{-ik(x-x')} \int \frac{d^3\tilde{p}\;dq_3\;dp_3}{(2\pi)^5}\; e^{ip_3x_3}\;\cdot \\
&\cdot\;\gamma^{\mu}\frac{\tilde{\not p} + \not p_3 + \not q_3 + \not k_3 + m}{\tilde{p}^2 - (p_3 + q_3 + k_3)^2 - m^2 + i\epsilon}\,\frac{[\gamma^0\Gamma - (ia/2)(\tilde{\not p} - m)]}{\lambda_-\Gamma - iap_0}\;\cdot \\
&\cdot\;\frac{\tilde{\not p} + \not q_3 + \not k_3 + m}{\tilde{p}^2 - (q_3 + k_3)^2 - m^2 + i\epsilon}\gamma^{\mu'}\frac{1}{K^2 + i\epsilon}\left[g_{\mu\mu'} - (1-\alpha)\frac{K_\mu K_{\mu'}}{K^2 + i\epsilon}\right]\;, \\
K \;&=\; (\tilde{p} - \tilde{k}, q_3)\;.
\end{aligned} \tag{13}
$$

Here, by superficial power counting we already recognize that a potential logarithmic divergency is present. This divergency is independent of the mass and in the further calculation the mass may be set to zero. Now, the q_3-integration can be performed explicitly by introducing auxiliary integrations in order to merge the first, third and fourth denominator in Eq. (13) into one. Then, using spherical coordinates for the remaining 3-dimensional loop integration (after having applied a Wick rotation) and for simplicity considering a UV cut-off Λ in the radial integral we obtain the desired divergent part of Eq. (12). It reads as follows

$$
\bar{\Sigma}(x,x')\big|_{div.} \;=\; -i\;\frac{e^2}{8\pi^2}\;\gamma^0\;f(a)\;\delta(x_3)\;\delta^{(4)}(x-x')\;\ln\Lambda^2\;, \tag{14}
$$

$$
f(a) \;=\; \frac{1}{a}\left[3\left(\frac{\lambda_+}{a}\arctan\frac{a}{\lambda_-} - 1\right) \;+\; \alpha\;\frac{a^2}{4}\left(\frac{\lambda_+}{a}\arctan\frac{a}{\lambda_-} + 1\right)\right]\;. \tag{15}
$$

The same result has also been obtained by using dimensional regularization starting from the propagator representation (8).

Now, what result we would have expected in the case of a non-singular background A_μ? Clearly,

$$
\bar{\Sigma}(x,x')\big|_{div.} \sim\; i\;e^3\;\gamma^{\mu}\;A_\mu(x)\;\delta^{(4)}(x-x')\;\ln\Lambda^2 \tag{16}
$$

which comes from the primitively divergent triangle diagram. But, instead being linear in the external field parameter a the divergent term (14) is a nontrivial function f of a, which is possible because a has no dimension. All other features are indeed well in accordance with the expectation (16). Of course, the first term in the expansion of f in powers of a is in complete agreement with the known coefficient of the triangle diagram divergency. But an infinite number of higher 1-loop diagrams with external fields attached are contributing too. For $\alpha > 0$, $f(a)$ is a monotonous function having no nontrivial zeros. However, this is not true if one formally considers the domain $\alpha < 0$. Then $f(a)$ has a nontrivial zero depending on α.

In addition, one may ask oneself whether the inclusion of the tadpole term

$$e^2 \, \delta^{(4)}(x - x') \, \gamma^\mu \int d^4y \, D^c_{\mu\mu'}(x - y) \, tr[\, \gamma^{\mu'} \, \bar{S}(y, y) \,] \tag{17}$$

alters the behaviour of Eq. (14). Explicit calculation using dimensional regularization shows, that on the qualitative level this is not the case.

Above results demonstrate that renormalization in the presence of a singular external field can not be as simple as it is in well behaved constant or periodic background fields. In the case under study one may simply define a new parameter $b = f(a)$ and then using a counter term linear in b. This might be interpreted as a kind of nonlinear renormalization of our initial parameter a. But whether such a procedure can consistently be defined at the 1-loop level for all diagrams (this is under study by the present authors now) and eventually even beyond the 1-loop approximation remains open so far.

We would like to thank M. Bordag and D. V. Shirkov for interesting discussions and suggestions.

References

1. J. Schwinger, *Phys. Rev.* **82**(1951) 664.
2. D. Hennig and D. Robaschik, *Karl Marx University Leipzig preprint KMU-NTZ-89-09.*
3. M. Bordag, D. Hennig and D. Robaschik, *Karl Marx University Leipzig preprint* KMU-NTZ-89-16.
4. J. C. Collins, *Renormalization*, (Cambridge University Press, Cambridge, 1984), ch. 11.4..