HUTP-87/A087

Background Field Method and Effective Action

K. Scharnhorst *

Lyman Laboratory of Physics
Harvard University
Cambridge, Massachusetts 02138

Quantum field theory in background fields is considered in order to set up a relation between the polarization operator in a background field and the effective action. Within that approach an application to Yang-Mills theory is discussed.

* On leave of absence from Humboldt University Berlin, Sektion Physik, GDR.

12/87

1. Introduction

The effective action method is one of the basic and universal tools in quantum field theory. The pioneering work of *Euler* and *Heisenberg* [1] as well as that of *Schwinger* [2] dealing with the effective Lagrangian in QED can be regarded as the starting point of that development.Within a systematic quantum field theoretic setting the effective action was established at the beginning of the 1970s ([3],[4],[5]). Since then, the effective action method has found broad application in studying such issues such as symmetry breaking, the phase structure of quantum field theoretic models, fermionisation/bosonisation and anomalies.

In the present paper we will give a description of quantum field theory within the background field method. The main investigation is done for a scalar theory, but generalisation to rather complicated models is relatively straightforward. The case of gauge theories deserves some special attention and is therefore also considered. In principle we are following the standard lines but from a slightly different point of view. Basically it agrees with the formulation of *Dittrich* and *Reuter* given in [6].

The organization of the paper is as follows. In chapter 2 we present the general setting and derive as our main result a relation between the effective action and the polarization operator in a background field. Chapter 3 discusses briefly the renormalization issue, while in chapter 4 emphasis is placed on the consideration of some aspects of the 1-loop effective action. Finally in chapter 5 we apply the method to Yang-Mills theory in background fields taking into account some problems specific for gauge theories. Two Appendices have been included in order to make the main chapters readable.

2. The General Relation between the Effective Action and the Polarization Operator in a Background Field

Consider a local, scalar, renormalizable quantum field theory with the classical action

$$\Gamma_{cl}[\varphi] = \int dx\ \mathcal{L}(\varphi)(x) \ . \tag{2.1}$$

First we are going to define the effective action. The definition differs slightly from the usual one, but in principle we will follow the standard lines. Consider a background field $\bar\varphi(x)$, that fulfills the classical equation of motion with respect to a given classical source $\bar J$, i.e.

$$\frac{\delta\Gamma_{cl}[\bar\varphi]}{\delta\bar\varphi(x)} = -\bar J(x) \ . \tag{2.2}$$

We define the generating functional of the Greens functions in the background field $\bar\varphi(x)$:

$$Z[J,\bar\varphi] = \int D\varphi\ e^{i\Gamma_{cl}[\varphi+\bar\varphi]+i\int dx J(x)\varphi(x)} \ . \tag{2.3}$$

The source couples to the quantum field only and the physical Greens functions should be calculated at the point $J = \bar J$ rather than at $J = 0$, as is obvious from the physical point of view (quantum fields acting in a classical environment, possibly generated by themselves in a non-perturbative way; for instance $\bar J$ may represent some bound state of the system). Then the generating functional of the connected Greens functions is given by

$$W[J,\bar\varphi] = -i \ln Z[J,\bar\varphi] \ . \tag{2.4}$$

As usual we define the complete effective action by a Legendre transformation

$$\Gamma[\varphi',\bar\varphi] = W[J,\bar\varphi] - \int dx J(x)\varphi'(x), \tag{2.5}$$

where

$$\frac{\delta W[J,\bar\varphi]}{\delta J(x)} = \varphi'(x) \tag{2.6}$$

defines the field $\varphi'(x)$. Due to (2.6) φ' and J are related and φ' can be considered as functional of J. Applying the standard procedure we find

$$\frac{\delta\Gamma[\varphi',\bar\varphi]}{\delta\varphi'(z)} = \int dx\ \frac{\delta W[J,\bar\varphi]}{\delta J(x)}\frac{\delta J(x)}{\delta\varphi'(z)} - J(z) - \int dx\ \frac{\delta J(x)}{\delta\varphi'(z)}\varphi'(x), \tag{2.7}$$

$$\frac{\delta\Gamma[\varphi',\bar\varphi]}{\delta\varphi'(z)} = -J(z) \ . \tag{2.8}$$

Differentiating (2.6) and (2.8) with respect to J and φ' respectively we get

$$\frac{\delta^2 W[J,\bar\varphi]}{\delta J(x)\,\delta J(x')} = \frac{\delta\varphi'(x)}{\delta J(x')}\,, \tag{2.9}$$

$$\frac{\delta^2\Gamma[\varphi',\bar\varphi]}{\delta\varphi'(x)\,\delta\varphi'(x')} = -\frac{\delta J(x)}{\delta\varphi'(x')}\,. \tag{2.10}$$

From (2.9) and (2.10) we conclude

$$\int dz\, \frac{\delta^2\Gamma[\varphi',\bar\varphi]}{\delta\varphi'(x)\,\delta\varphi'(z)}\,\frac{\delta^2 W[J,\bar\varphi]}{\delta J(z)\,\delta J(x')} = -\delta(x-x')\,. \tag{2.11}$$

We define by

$$\frac{\delta^2\Gamma_{cl}[\bar\varphi]}{\delta\bar\varphi(x)\,\delta\bar\varphi(x')} = -K_{cl}(\bar\varphi;x)\,\delta(x-x') \tag{2.12}$$

the quadratic kernel $K_{cl}(\bar\varphi;x)$ of the classical action of the field φ in the background field $\bar\varphi$, which is a local operator. The free connected propagator $G_o(\bar\varphi;x,x')$ of the φ-field in the background $\bar\varphi$ is given by

$$K_{cl}(\bar\varphi;x)\,G_o(\bar\varphi;x,x') = \delta(x-x')\,, \tag{2.13}$$

where

$$G_o(\bar\varphi;x,x') = i\langle T(\varphi(x)\varphi(x'))\rangle_{\bar\varphi,conn.}\,. \tag{2.14}$$

Of course $G_o(\bar\varphi;x,x')$ coincides in the case $\bar\varphi(x)\equiv 0$ with the usual Greens function.

Now we are ready to describe the perturbation theory for quantum field theoretic calculations. For this purpose we split the argument of the exponential in (2.3) into 4 pieces in the following way.

$$\begin{aligned}\Gamma_{cl}[\varphi+\bar\varphi] = \Gamma_{cl}[\bar\varphi] &+ \int dx\left(\frac{\delta\Gamma_{cl}[\bar\varphi]}{\delta\bar\varphi(x)} + J(x)\right)\varphi(x)\\ &- \frac{1}{2}\int dx\,\varphi(x)K_{cl}(\bar\varphi;x)\varphi(x) \;+\; \Gamma_I[\varphi,\bar\varphi]\end{aligned} \tag{2.15}$$

The first term represents the classical action of the background field $\bar\varphi$, the second is the source term, while the third term defines the quadratic part of the quantum field action. In $\Gamma_I[\varphi,\bar\varphi]$ all contributions higher than quadratic in φ are collected. We can define the generating functional for the non-interacting quantum field theory in the background $\bar\varphi$ (i.e. the quantum field interaction is ignored keeping the background field interaction). The free Z-functional is given by the following expression.

$$Z_o[J',\bar\varphi] = e^{i\Gamma_{cl}[\bar\varphi]}\int D\varphi\; e^{-\frac{i}{2}\int dx\,\varphi(x)K_{cl}(\bar\varphi;x)\varphi(x) + i\int dx J'(x)\varphi(x)} \tag{2.16}$$

$$= e^{i\Gamma_{cl}[\bar\varphi]}\left[\det K_{cl}(\bar\varphi;.)\right]^{-\frac{1}{2}} e^{\frac{i}{2}\int dx dx'\; J'(x)G_o(\bar\varphi;x,x')J'(x')}\,, \tag{2.17}$$

where we have redefined $J'(x) = J(x) + \frac{\delta\Gamma_{cl}[\bar\varphi]}{\delta\bar\varphi(x)}$. So we now have to calculate all quantities at the new $J'(x)\equiv 0$ (in the following, if not mentioned, we use this redefinition for all quantities). As usual we define

$$W_o[J',\bar\varphi] = -i\ln Z_o[J',\bar\varphi] \tag{2.18}$$

and

$$\Gamma_o[\varphi',\bar\varphi] = W_o[J',\bar\varphi] - \int dx\; J'(x)\varphi'(x)\,. \tag{2.19}$$

The complete generating functionals ((2.3),(2.4),(2.5)) can be deduced from the perturbation theoretic formula

$$Z[J',\bar\varphi] = e^{i\Gamma_I[\frac{\delta}{i\delta J'},\bar\varphi]}\; Z_o[J',\bar\varphi]\,. \tag{2.20}$$

Equipped with these formulas we find

$$G_o(\bar\varphi;x,x') = \left.\frac{\delta^2 W_o[J',\bar\varphi]}{\delta J'(x)\,\delta J'(x')}\right|_{J'=0}\,. \tag{2.21}$$

The complete connected φ-field propagator in the background $\bar\varphi$ is given by

$$G(\bar\varphi;x,x') = \left.\frac{\delta^2 W[J',\bar\varphi]}{\delta J'(x)\,\delta J'(x')}\right|_{J'=0}\,. \tag{2.22}$$

As usual, this complete propagator can be written with the help of 1PI blocks as follows.

$$\begin{aligned}G(\bar\varphi;x,x') &= G_o(\bar\varphi;x,x') + \int dz dz'\; G_o(\bar\varphi;x,z)\Pi(\bar\varphi;z,z')G_o(\bar\varphi;z',x') +\\ &+ \int dy dy'\; dz dz'\; G_o(\bar\varphi;x,y)\Pi(\bar\varphi;y,y')\cdot\\ &\quad\cdot G_o(\bar\varphi;y',z)\Pi(\bar\varphi;z,z')G_o(\bar\varphi;z',x') + \ldots\end{aligned} \tag{2.23}$$

Here $\Pi(\bar{\varphi}; x, x')$ is the polarisation operator in the background field $\bar{\varphi}$, which coincides at $\bar{\varphi}(x) \equiv 0$ with the usual expression.

Let us now pause a moment and mention some points for the further discussion. First of all the effective action (2.5) has the following symmetry under $\bar{\varphi} \longrightarrow \bar{\varphi} + \tilde{\varphi}$ (this includes $\bar{J}[\bar{\varphi}] \rightarrow \bar{J}[\bar{\varphi} + \tilde{\varphi}]$) for an arbitrary $\tilde{\varphi}$.

$$\Gamma[\varphi', \bar{\varphi}] = \Gamma[\varphi' - \tilde{\varphi}, \bar{\varphi} + \tilde{\varphi}] \tag{2.24}$$

This actually means that a usual quantum field theory having a nontrivial vacuum characterized by some $\varphi' \neq 0$ can be represented by a "rather trivial" quantum field theory with $\varphi' = 0$ acting in a nontrivial background characterized by $\bar{\varphi} \neq 0$ (and $\bar{J}[\bar{\varphi}]$). So this nontrivial background $\bar{\varphi}$ (and its source $\bar{J}$) can be regarded as some quantum condensate (the quantum fields in effect are moving in their own condensate; in that sense classical physics can be regarded as quantum condensate physics).

Second, the definition of the effective action at $\bar{\varphi} = 0$ completely agrees with the usual one, where $\varphi' \neq 0$ and $\bar{\varphi} = 0$. But in the following on the basis of (2.24) we will adopt another point of view. We consider $\bar{\varphi} \neq 0$ and assume $\varphi' = 0$, because every nonzero φ' can be absorbed into $\bar{\varphi}$ due to (2.24). Taking into account (2.24) we have for (2.8)

$$\frac{\delta\Gamma[\varphi', \bar{\varphi}]}{\delta\varphi'(z)} = \frac{\delta\Gamma[\varphi', \bar{\varphi}]}{\delta\bar{\varphi}(z)} = -J'(z) - \bar{J}(z) \quad , \tag{2.25}$$

where $J'(z) = J(z) - \bar{J}(z)$ (remembering the shift in the genuine $J(z)$ we did in (2.16)). So it is reasonable to calculate all the nontrivial vacuum properties of a quantum field theory by considering a corresponding quantum field theory with $\varphi' = 0$ at $J' = 0$ in a nontrivial background $\bar{\varphi}$ respecting

$$\frac{\delta\Gamma[0, \bar{\varphi}]}{\delta\bar{\varphi}(z)} = -\bar{J}(z) \; . \tag{2.26}$$

Finally, the background field $\bar{\varphi}$ has to be a stationary point of the quantum correction to the classical action of $\bar{\varphi}$.

$$\frac{\delta}{\delta\bar{\varphi}(z)}\left[\Gamma[0, \bar{\varphi}] - \Gamma_{cl}[\bar{\varphi}]\right] = 0 \tag{2.27}$$

Due to the points mentioned we have the relation

$$\Gamma[\bar{\varphi}, 0] = \Gamma[0, \bar{\varphi}] = W[0, \bar{\varphi}] = W[J, 0] \; . \tag{2.28}$$

Of course these nonperturbative considerations have to be followed in perturbative calculations (i.e. with respect to the genuine quantum field interactions) order by order. In all the above considerations we have adopted the point of view that all quantities are renormalized (or at least regularized) ones, so that the final answers are finite. We will comment later on the renormalization peculiarities for quantum field theory in background fields.

Let us now continue our original discussion. Consider the second functional derivative of $\Gamma[0, \bar{\varphi}]$ with respect to $\bar{\varphi}$. Remembering (2.12) it is given by the sum of a classical term and some quantum correction $K_{qu}(\bar{\varphi}; x, x')$.

$$\frac{\delta^2\Gamma[0, \bar{\varphi}]}{\delta\bar{\varphi}(x)\,\delta\bar{\varphi}(x')} = -\left[K_{cl}(\bar{\varphi}; x)\,\delta(x - x') + K_{qu}(\bar{\varphi}; x, x')\right] \tag{2.29}$$

We now introduce the expressions (2.22),(2.23) and (2.29) into formula (2.11) and take into account relation (2.13). We get

$$\begin{aligned}\int dz\; \left[K_{cl}(\bar{\varphi}; x)\,\delta(x - z) + K_{qu}(\bar{\varphi}; x, z)\right]\left[G_o(\bar{\varphi}; z, x') + \right.\\ \left. + \int dy dy'\; G_o(\bar{\varphi}; z, y)\Pi(\bar{\varphi}; y, y')G_o(\bar{\varphi}; y', x') + \ldots\right] = \delta(x - x') \quad ,\end{aligned} \tag{2.30}$$

$$\int dz\; \left[K_{qu}(\bar{\varphi}; x, z) + \Pi(\bar{\varphi}; x, z)\right]G(\bar{\varphi}; z, x') = 0 \; . \tag{2.31}$$

Due to the invertibility of the complete propagator we find

$$K_{qu}(\bar{\varphi}; x, x') = -\Pi(\bar{\varphi}; x, x') \; . \tag{2.32}$$

So we have the relation

$$\frac{\delta^2\Gamma[0, \bar{\varphi}]}{\delta\bar{\varphi}(x)\,\delta\bar{\varphi}(x')} = -\left[K_{cl}(\bar{\varphi}; x)\,\delta(x - x') - \Pi(\bar{\varphi}; x, x')\right] \quad . \tag{2.33}$$

Applying formula (A 15) from Appendix A to $\Gamma[0,\bar{\varphi}]$ we find

$$\Gamma[0,\bar{\varphi}] = \Gamma[0,0] + \int dz \left.\frac{\delta\Gamma[0,\varphi']}{\delta\varphi'(z)}\right|_{\varphi'=0} \bar{\varphi}(z) -$$
$$- \int_0^1 d\tau(1-\tau) \int dz dz' \, \bar{\varphi}(z) \left[K_{cl}(\tau\bar{\varphi};z)\, \delta(z-z') - \Pi(\tau\bar{\varphi};z,z')\right] \bar{\varphi}(z') \, . \tag{2.34}$$

Assuming $\Gamma_{cl}[0]=0$ (in a situation where $\Gamma_{cl}[0]\neq 0$ this term is necessary to restore $\Gamma_{cl}[\bar{\varphi}]$; then in (2.36) one has to reset $\Gamma[0] \longrightarrow \Gamma[0]-\Gamma_{cl}[0]$) and taking into account

$$\left.\frac{\delta\Gamma[0,\varphi']}{\delta\varphi'(z)}\right|_{\varphi'=0} = 0 \tag{2.35}$$

(due to (2.26) such a term could arise only from the classical action and contributes only to its reconstruction) we apply (A 15) to (2.34) again and find finally (writing $\Gamma[0,\bar{\varphi}] = \Gamma[\bar{\varphi}]$ in the following)

$$\Gamma[\bar{\varphi}] = \Gamma[0] + \Gamma_{cl}[\bar{\varphi}] + \int_0^1 d\tau(1-\tau) \int dz dz' \, \bar{\varphi}(z)\Pi(\tau\bar{\varphi};z;z')\bar{\varphi}(z') \quad . \tag{2.36}$$

Formula (2.36) can be rewritten equally as

$$\Gamma[\bar{\varphi}] = \Gamma[0] - \int_0^1 d\tau(1-\tau) \int dz dz' \, \bar{\varphi}(z) G^{-1}(\tau\bar{\varphi};z,z')\bar{\varphi}(z') \; , \tag{2.37}$$

where $G^{-1}(\bar{\varphi};z,z')$ is the inverse of the complete connected propagator $G(\bar{\varphi};z,z')$.

So we have arrived at a general relation between the effective action and the polarization operator in a background field, which might actually prove useful to calculate the effective action within some approximation. On the basis of the above definition of the polarization operator in the background field one can always make contact with the usual quantum field theory by performing the limit $\bar{\varphi} \to 0$. So, in actual calculations the background field expressions can be tested by comparing them with the zero-field results.

Generally one should keep in mind that the background field $\bar{\varphi}$ has to be a function, which mathematically behaves sufficiently smoothly for our purposes. Otherwise one faces additional problems which we will not discuss here.

Although we have derived the formulas for the scalar case, the generalisation to rather complicated models is straightforward and without serious complications. Of course, some peculiarities may occur as they do in gauge theories. We will discuss this problem later on. In the case of a gauge theory the identification $\bar{\varphi} \leftrightarrow \bar{B}^a_\mu$ (the gauge potentials) should be made; finally of course Γ depends on gauge invariant objects only.

It is obvious that on the basis of formula (2.36) only those contributions to the effective action which are field dependent can be calculated from the polarization operator. Contributions which depend on other parameters like temperature or boundary conditions only and are contained in $\Gamma[0]$ have to be calculated separately.

Assume now a situation where in a certain quantum field theoretic model there is more than only one field (of course qualitatively different fields) expected to have a nonvanishing vacuum expectation value. This set of fields we represent by $\{\varphi,\psi,...\}$ and write $\Gamma = \Gamma[\bar{\varphi},\bar{\psi},...]$. The procedure is the following. Calculate $\Gamma = \Gamma[\bar{\varphi},0,0,...]$ by the described method (i.e. in all calculations $\bar{\varphi}\neq 0$ is taken into account only, while all other background fields are set to zero). Then, allow the next field to have a nonvanishing expectation value, let us say $\bar{\psi}\neq 0$. So, by calculating the polarization operator for the ψ-field in the background $\bar{\varphi}\neq 0, \bar{\psi}\neq 0$ (the other background fields are set to zero again) we will almost arrive at $\Gamma[\bar{\varphi},\bar{\psi},0,0,...]$. This process can be continued. For technical reasons, however, it may prove useful in this case to include the classical source in the classical action. In order to obtain the desired final result one has to remember, that in deriving (2.36) $\frac{\delta\Gamma[\varphi']}{\delta\varphi'}|_{\varphi'=0}$ has been discarded for rather good reasons. But the arguments that have been applied there do not work in the case of different nonvanishing background fields. So, for this term one has to set up a special analysis following the method used here. Of course, one has to consider a special subclass of diagrams only. We will not give any general formula, because the analysis within a special model may yield a quicker and simpler answer (e.g. due to symmetry arguments), than when dealing with the most general case.

3. Renormalization within the Background Field Method

The main features of the renormalization procedure within the background field method can be recognized from formula (2.3). Reshifting in that formula $\varphi_o \to \varphi_o - \bar{\varphi}_o$ (the quantities with the subscript 'o' are the bare ones) we have

$$Z[J_o, \bar{\varphi}_o] = \int D\varphi_o \, e^{i\Gamma_{cl}[\varphi_o] + i\int dx \, J_o(x)\varphi_o(x)} \, e^{-i\int dx \, J_o(x)\bar{\varphi}_o(x)} \; . \qquad (3.1)$$

One gets finite answers at $\bar{\varphi} \equiv 0$ using the standard renormalization expressing the bare quantities by the renormalized ones (coupling constants and masses are denoted by g_i and m_i respectively).

$$\begin{aligned} \varphi_o &= Z^{\frac{1}{2}} \varphi \\ J_o &= Z^{-\frac{1}{2}} J \\ g_{oi} &= Z_{g_i} g_i \\ m_{oi} &= Z_{m_i} m_i \end{aligned} \qquad (3.2)$$

In the same way one obtains finite answers at $\bar{\varphi} \neq 0$ by expressing in addition the bare $\bar{\varphi}_o$ and J_o by the renormalized ones, where the Z- factors are dictated by (3.2).

$$\begin{aligned} \bar{\varphi}_o &= Z^{\frac{1}{2}} \bar{\varphi} \\ J_o &= Z^{-\frac{1}{2}} J \end{aligned} \qquad (3.3)$$

In every loop calculation on the basis of formula (2.20), one has to take into account that new counter terms associated with the background field appear, which will lead to finite expressions. The source renormalization does not actually matter, because we are calculating all quantities from (2.20) at vanishing source. Of course the renormalized $\bar{\varphi}$ is related to the renormalized J.

From formula (2.36), it is clear that the renormalization conditions for the effective action and those for the usual polarization operator (even in the background $\bar{\varphi} \equiv 0$) are closely related to each other. This relation has to be studied for the special theory one is working on. In principle, if the polarization operator in the chosen background $\bar{\varphi}$, which has been renormalized at some mass scale μ, is given within some loop approximation, one has to calculate the effective action and may recover by varying μ the relation between the renormalization conditions for the effective action and the polarization operator itself.

4. Determinants and the Background Field Method

We will now specialize our considerations to the calculation of the 1-loop effective action. Due to (2.36) it is given by the 1-loop approximation of the polarization operator in the given background field.

$$\begin{aligned} \Gamma[\bar{\varphi}]|_{1-loop} =& \Gamma[0]|_{1-loop} + \Gamma_{cl}[\bar{\varphi}] + \\ &+ \int_0^1 d\tau (1-\tau) \int dz dz' \; \bar{\varphi}(z) \Pi^{(1)}(\tau\bar{\varphi}; z, z') \bar{\varphi}(z') \end{aligned} \qquad (4.1)$$

On the other hand from (2.17)-(2.19) and (2.28) we know (and it is well known in general) that

$$\begin{aligned} \Gamma[\bar{\varphi}]|_{1-loop} =& \Gamma_o[0, \bar{\varphi}] = W_o[0, \bar{\varphi}] \\ =& \Gamma_{cl}[\bar{\varphi}] + \frac{i}{2} \ln \det K_{cl}(\bar{\varphi}; .) \; . \end{aligned} \qquad (4.2)$$

Taking into account that $(\Gamma_{cl}[0] = 0)$

$$\Gamma[0]|_{1-loop} = \frac{i}{2} \ln \det K_{cl}(0; .) \; , \qquad (4.3)$$

we find the following relation:

$$\det K_{cl}(\bar{\varphi}; .) = \\ = e^{-2i \int_0^1 d\tau (1-\tau) \int dz dz' \; \bar{\varphi}(z) \Pi^{(1)}(\tau\bar{\varphi}; z, z') \bar{\varphi}(z')} \det K_{cl}(0; .) \; . \qquad (4.4)$$

Of course this relation has to be understood as a renormalized (or at least regularized) one in the sense of the renormalization of the effective action we already discussed. Relation (4.4) governs the background field dependence of a given determinant. In order to detect non-field dependent contributions to $\det K_{cl}(\bar{\varphi}; .)$ one has to calculate the determinant in the zero-field case explicitly. For example, this concerns temperature dependence [7] or dependence from boundary conditions as in the Casimir effect configuration [8].

In order to illustrate our result (4.4) we regard the scalar φ^4-theory for a constant background field $\bar{\varphi}(x) \equiv \bar{\varphi} = \text{const.}$ in 4D Minkowski space-time. The Lagrangian is defined by

$$\mathcal{L}(\varphi)(x) = \frac{1}{2}\partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{m^2}{2}\varphi^2(x) - \frac{\lambda}{4!}\varphi^4(x) \; . \qquad (4.5)$$

So we have

$$K_{cl}(\bar{\varphi};x) = \Box_x + m^2 + \frac{\lambda}{2}\bar{\varphi}^2 \,, \tag{4.6}$$

$$G_o(\bar{\varphi};x-x') = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2-m^2-\frac{\lambda}{2}\bar{\varphi}^2+i\epsilon} \,, \tag{4.7}$$

$$\Gamma_I[\varphi,\bar{\varphi}] = \int d^4x\, \mathcal{L}_I(\varphi,\bar{\varphi})(x) \,, \tag{4.8}$$

$$\mathcal{L}_I(\varphi,\bar{\varphi})(x) = -\frac{\lambda}{3!}\bar{\varphi}\,\varphi^3(x) - \frac{\lambda}{4!}\varphi^4(x) \,. \tag{4.9}$$

We will give the regularised quantities only leaving aside the renormalization for the moment. For simplicity we are using a momentum cut-off at $k^2 = -\Lambda^2$. We have

$$\begin{aligned} \ln\det{}_\Lambda K_{cl}(\bar{\varphi};.) - \ln\det{}_\Lambda K_{cl}(0;.) &= \mathrm{tr}_\Lambda \ln\ K_{cl}(\bar{\varphi};.) - \mathrm{tr}_\Lambda \ln\ K_{cl}(0;.) \\ &= V^4 \int_\Lambda \frac{d^4k}{(2\pi)^4} \ln \frac{-k^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2-i\epsilon}{-k^2+m^2-i\epsilon} \,. \end{aligned} \tag{4.10}$$

Performing a Wick-rotation $k_o \to ik_4$ we get

$$\ln\det{}_\Lambda K_{cl}(\bar{\varphi};.) - \ln\det{}_\Lambda K_{cl}(0;.) = iV^4 \int_\Lambda \frac{d^4k_E}{(2\pi)^4} \ln \frac{k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2} \,. \tag{4.11}$$

The 1-loop polarization operator in the background field $\bar{\varphi}$ can be represented pictorially by

$\Pi^{(1)}(\bar{\varphi};x-x') = 12$  $x=x'$ $+\ 18$ x  x' $-$ c.t. (4.12)

displaying the weights explicitly. Leaving aside the counter terms for $\Pi^{(1)}$ and using the Feynman rules according to (4.8), (4.9) we find:

$$\begin{aligned} I &= -2i\int_0^1 d\tau(1-\tau)\int d^4z d^4z'\ \bar{\varphi}(z)\Pi^{(1)}(\tau\bar{\varphi};z-z')\bar{\varphi}(z') \\ &= -2iV^4\bar{\varphi}^2\int_0^1 d\tau(1-\tau)\int d^4z\ \Pi^{(1)}(\tau\bar{\varphi};z) \qquad\qquad (4.12) \\ &= -2iV^4\bar{\varphi}^2\int_0^1 d\tau(1-\tau)\left\{-12i\frac{\lambda}{4!}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2-\frac{\lambda}{2}\tau^2\bar{\varphi}^2+i\epsilon} - \right. \\ &\qquad \left. - 18i\left(\frac{\lambda}{3!}\tau\bar{\varphi}\right)^2\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2-\frac{\lambda}{2}\tau^2\bar{\varphi}^2+i\epsilon)^2}\right\} \qquad\qquad (4.13) \\ &= -2iV^4\bar{\varphi}^2\int_0^1 d\tau(1-\tau)\left\{\frac{\lambda}{2}\int_\Lambda \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2+\frac{\lambda}{2}\tau^2\bar{\varphi}^2} - \right. \\ &\qquad \left. - \frac{\lambda^2}{2}\tau^2\bar{\varphi}^2\int_\Lambda \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m^2+\frac{\lambda}{2}\tau^2\bar{\varphi}^2)^2}\right\} \qquad\qquad (4.14) \end{aligned}$$

Performing the τ-integration now we get [9]

$$\begin{aligned} I = iV^4\int_\Lambda \frac{d^4k_E}{(2\pi)^4}\Bigg\{\lambda\bar{\varphi}^2\Bigg(&\frac{1}{\sqrt{\frac{\lambda}{2}\bar{\varphi}^2(k_E^2+m^2)}}\arctan\sqrt{\frac{\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2}} - \\ &- \frac{1}{\lambda\bar{\varphi}^2}\ln\frac{k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2}\Bigg) - \lambda^2\bar{\varphi}^4\Bigg(\frac{-1}{\lambda\bar{\varphi}^2(k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2)} + \\ &+ \frac{1}{\lambda\bar{\varphi}^2\sqrt{\frac{\lambda}{2}\bar{\varphi}^2(k_E^2+m^2)}}\arctan\sqrt{\frac{\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2}} - \\ &- \frac{k_E^2+m^2}{\frac{\lambda^2}{2}\bar{\varphi}^4(k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2)} + \frac{1}{\frac{\lambda^2}{2}\bar{\varphi}^4} - \frac{1}{\frac{\lambda}{2}\bar{\varphi}^4}\ln\frac{k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2}\Bigg)\Bigg\} \,. \end{aligned} \tag{4.15}$$

And we obtain finally

$$\begin{aligned} -2i\int_0^1 d\tau(1-\tau)\int d^4zd^4z'\ \bar{\varphi}(z)\Pi^{(1)}(\tau\bar{\varphi};z-z')\bar{\varphi}(z') \\ = iV^4\int_\Lambda \frac{d^4k_E}{(2\pi)^4}\ln\frac{k_E^2+m^2+\frac{\lambda}{2}\bar{\varphi}^2}{k_E^2+m^2} \,. \end{aligned} \tag{4.16}$$

This result agrees completely with (4.11). So we have seen that for a constant background field $\bar{\varphi}$ in the φ^4-theory in 4D space-time the relation (4.4) holds at the regularized level and of course is equally valid at the renormalized level.

Having studied this little example we can now proceed further in the discussion of relation (4.4). Our result (4.4) is not only a pure quantum field theoretic one and it seems that there is some general functional analytic, differential geometric and topological background behind this expression. Given an operator in some external bosonic field $\bar{\varphi}$ taking into account relation (2.12) one can reconstruct a Lagrangian with the help of formula (A 15) and build up the corresponding quantum field theory in the background field $\not{\phi}$. Then relation (4.4) allows the calculation of the functional determinant of this operator within some quantum field theoretic regularisation procedure. Of course this only works, if the coupling of the field $\not{\phi}$ in the given operator admits a bosonic field theory within $\not{\phi}$-type fields.

On the other hand, if one is given some Dirac-type operator in a background field $\bar{A}$, one has to construct an appropriate quantum field theory, where the fermionic action is given essentially by the Dirac-type operator in question. The coupling to the quantum field A is also dictated by the Dirac-type operator. One now has to construct the fermionic contribution to the proper A-field 1-loop 2-point function in the background $\bar{A}$ (i.e. the fermionic contribution to the A-field polarization operator in the background $\bar{A}$ at 1-loop level). Of course one has to take into account that relation (4.4) in the case of a Dirac-type kernel $K_{cl}(\bar{A};.) = D(\bar{A})$ reads (indices suppressed)

$$\det D(\bar{A}) = e^{\, i \int_0^1 d\tau (1-\tau) \int dz dz' \bar{A}(z) \Pi^{(1)}(\tau \bar{A}; z, z') \bar{A}(z')} \det D(0) \quad . \quad (4.17)$$

Then formula (4.17) gives the answer for the considered determinant.

For example this kind of determinant has been considered by many authors in the calculation of local anomalies (chiral, non-abelian) by extracting the desired information out of a perturbation theoretic representation of the polarization operator for a vector field A in a given background. To be specific, let the determinant be

$$\det(\not{\partial} - i \not{\bar{A}} - i \not{\bar{V}} \gamma_5) \quad . \quad (4.18)$$

So in light of (4.17), studying the anomaly means studying the behaviour of the vector field A-polarisation operator in the background $\bar{V}$ and $\bar{A}$. It seems, that the anomaly is a characteristic feature of the short distance behaviour of the A-field polarization operator in a background $\bar{V}$ in the vicinity of $\bar{V} = 0$ (and $\bar{A} \neq 0$ in the non-abelian case). Of course here one should bear in mind the remarks made at the end of chapter 2.

A nice example for the combination of the relations (4.4) and (4.17) is provided by the 1-loop contribution to the effective action in Yang-Mills theory. The 1-loop contribution to the polarization operator in a Yang-Mills background field is given by the following diagrams.

(4.19)

At the same time the 1-loop contribution to the effective action is given by the determinants of the quadratic kernels of the gluon and ghost action in the given Yang-Mills background $\bar{B}$ (for notion see chapter 5).

$$\Gamma^{(1)}[\bar{B}] = \frac{i}{2} \ln \det K^{ab}_{\mu\nu}(\bar{B}; \alpha; .) - i \ln \det K^{ab}(\bar{B}; .) \qquad (4.20)$$

The gluon determinant corresponds to the first two diagrams in (4.19), the ghost determinant to the third. The diagrams are just coming as they should with the right relative weights and signs. The inclusion of quarks is straightforward.

Obviously there is a direct relation between the independence of the 1-loop Yang-Mills effective action of a wide class of gauge conditions (this has been shown in [10] for background fields obeying the classical equation of motion for vanishing source) and the properties of the polarization operator Π, which has been studied in Appendix B for the case of the covariant background gauge only. It should be expected that the results found for the 1-loop Yang-Mills effective action can be generalized to the whole effective action by studying the Ward-identities for the Yang-Mills theory in an arbitrary background for a rather general class of gauge conditions. This might be done in the future.

5. The Effective Action in Yang-Mills Theory

The investigation of the Yang-Mills effective action has gained a lot of attention since this topic was established by *Batalin*, *Matinyan* and *Savvidi* [11] in calculating the 1-loop effective Lagrangian of a SU(2)-Yang-Mills theory in a constant background. We do not intend to review the subject here, but try to apply the setting described in chapter 2 in order to obtain some information about the Yang-Mills effective action. The relevant formulation of the background field method for Yang-Mills theory which we will use was given in [12].

Let us start with a string of definitions in order to define the notion. We are working in 4D Minkowski space-time. The classical action of the Yang-Mills theory is given by the expression

$$\Gamma_{cl}[A] = -\frac{1}{4}\int d^4x \; F^a_{\mu\nu}(A)F^{a\,\mu\nu}(A) \quad . \tag{5.1}$$

The quantized version of the Yang-Mills theory in the covariant background gauge which we are using is defined by the action

$$\begin{aligned}\Gamma'_{cl}[A,\bar{B},\bar{c},c] =& \\ = \Gamma_{cl}[A] \;+\; &\int d^4x \left\{ -\frac{\alpha}{2}(D^{ab}_\mu(\bar{B})(A^{b\mu}-\bar{B}^{b\mu}))(D^{ac}_\nu(\bar{B})(A^{c\nu}-\bar{B}^{c\nu})) \;+\right. \\ &\left. +\bar{c}^a D^{ab}_\sigma(\bar{B})D^{bc\,\sigma}(A)\; c^b \right\}\end{aligned} \tag{5.2}$$

$$F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{5.3}$$

$$D^{ab}_\mu(A) = \delta^{ab}\partial_\mu + g f^{abc} A^c_\mu \tag{5.4}$$

α is the gauge parameter and $A^a_\mu, \bar{c}^a, c^a$ are the gauge potential and the ghost fields respectively. The f^{abc} are the totally antisymmetric structure constant of the gauge group.

We will assume the Yang-Mills field to act in a background field $\bar{A}$. So the generating functional of the Greens functions at $\bar{A} = \bar{B}$ is given by

$$Z[J,\eta,\bar{\eta},\bar{B}] = \int DA^a_\mu D\bar{c}^a Dc^a \; e^{i\Gamma'_{cl}[A+\bar{B},\bar{B},\bar{c},c] + i\int d^4x\; (J^a_\mu(x)A^{a\mu}(x)+\bar{c}^a(x)\eta^a(x)+\bar{\eta}^a(x)c^a(x))} \;. \tag{5.4}$$

The free Z-functional is defined by the following expression.

$$\begin{aligned}Z_o[J',\eta,\bar{\eta},\bar{B}] =& \\ = \;\; & e^{i\Gamma_{cl}[\bar{B}]}\int DA^a_\mu D\bar{c}^a Dc^a \; e^{\frac{i}{2}\int d^4x[A^{a\mu}(x)K^{ab}_{\mu\nu}(\bar{B};\alpha;x)A^{b\nu}(x)+J'^a_\mu(x)A^{a\mu}(x)]} \;. \\ & \cdot\, e^{i\int d^4x[\bar{c}^a(x)K^{ab}(\bar{B};x)c^b(x)+\bar{\eta}^a(x)c^a(x)+\bar{c}^a(x)\eta^a(x)]}\end{aligned} \tag{5.5}$$

$$\begin{aligned}= \;\; & e^{i\Gamma_{cl}[\bar{B}]}\; \det K^{ab}(\bar{B};.)\; [\det K^{ab}_{\mu\nu}(\bar{B};\alpha;.)]^{-\frac{1}{2}} \;\cdot \\ & \cdot\, e^{-\frac{i}{2}\int d^4x d^4x'\; J'^{a\mu}(x)G_o{}^{ab}_{\mu\nu}(\bar{B};\alpha;x,x')J'^{b\nu}(x')} \;. \\ & \cdot\, e^{-i\int d^4x d^4x'\; \bar{\eta}^a(x)G^{ab}_o(\bar{B};x,x')\eta^b(x')}\end{aligned} \tag{5.6}$$

Here we used the following notion. $K^{ab}_{\mu\nu}(\bar{B};\alpha;x)$ is the quadratic kernel of the gluon action in the background $\bar{B}$, i.e.

$$\frac{\delta^2\Gamma'_{cl}[A,\bar{B},0,0]}{\delta A^{a\mu}(x)\;\delta A^{b\nu}(x')}\Big|_{A=\bar{B}} = K^{ab}_{\mu\nu}(\bar{B};\alpha;x)\delta^{(4)}(x-x') \;\;, \tag{5.7}$$

$$K^{ab}_{\mu\nu}(\bar{B};\alpha;x) = g_{\mu\nu}D^{ac}_\sigma(\bar{B})D^{cb\,\sigma}(\bar{B}) - (1-\alpha)D^{ac}_\mu(\bar{B})D^{cb}_\nu(\bar{B}) + 2gf^{abc}F^c_{\mu\nu}(\bar{B}) \;\;, \tag{5.8}$$

where $K^{ab}(\bar{B};x)$ is the quadratic kernel of the ghost action in the background $\bar{B}$, i.e.

$$\frac{\delta^2\Gamma'_{cl}[\bar{B},\bar{B},\bar{c},c]}{\delta c^b(x')\;\delta\bar{c}^a(x)} = K^{ab}(\bar{B};x)\delta^{(4)}(x-x') \tag{5.9}$$

$$K^{ab}(\bar{B};x) = D^{ac}_\sigma(\bar{B})D^{cb\,\sigma}(\bar{B}) \tag{5.10}$$

The free connected gluon propagator in the background $\bar{B}^a_\mu$ fulfills the equation

$$K^{ab}_{\mu\nu}(\bar{B};\alpha;x,x')G^{ba'\;\nu}_{o\;\;\;\;\;\mu'}(\bar{B};\alpha;x,x') = g_{\mu\mu'}\delta^{aa'}\delta^{(4)}(x-x') \;\;, \tag{5.11}$$

while the free connected ghost propagator satisfies

$$K^{ab}(\bar{B};x)G^{ba'}_o(\bar{B};x,x') = \delta^{aa'}\delta^{(4)}(x-x') \tag{5.12}$$

(for an explicit construction both of these propagators in the case of a special constant SU(2)-background see reference [13]). The background field $\bar{B}^a_\mu$ respects the equation of motion

$$\frac{\delta\Gamma_{cl}[\bar{B}]}{\delta\bar{B}^{a\nu}(x)} = D^{ab\;\mu}(\bar{B})F^b_{\mu\nu}(\bar{B})(x) = -J^a_\nu(x) \;\;, \tag{5.13}$$

where of course

$$D^{ab\;\nu}(\bar{B})J^b_\nu(x) = 0 \quad . \tag{5.14}$$

So we define $J'^a_\nu(x) = J^a_\nu(x) - J^a_\mu(x)$ and calculate all quantities at $J'^a_\nu \equiv 0$. The perturbation theory is defined on the basis of (5.6) by

$$Z[J',\eta,\bar\eta,B] = e^{i\Gamma'_I[\frac{\delta}{i\delta J'},B,\frac{-\delta}{i\delta\eta},\frac{\delta}{i\delta\bar\eta}]}\ Z_o[J',\eta,\bar\eta,B] \quad . \tag{5.15}$$

The effective action is obtained by

$$\Gamma[B] = W[0,0,0,B] \quad , \tag{5.16}$$

where

$$W[J',\eta,\bar\eta,B] = -i\ln Z[J',\eta,\bar\eta,B] \quad . \tag{5.17}$$

The free functionals W_o, Γ_o are defined in analogy to chapter 2 by

$$W_o[J',\eta,\bar\eta,B] = -i\ln Z_o[J',\eta,\bar\eta,B] \quad , \tag{5.18}$$

$$\Gamma_o[B] = W_o[0,0,0,B] \quad . \tag{5.19}$$

The complete connected gluon and ghost propagator are given by the following expressions.

$$G^{ab}_{\mu\nu}(B;\alpha;x,x') = -\frac{\delta^2 W[J',0,0,B]}{\delta J'^{a\mu}(x)\ \delta J'^{b\nu}(x')}\Big|_{J'=0} \quad , \tag{5.20}$$

$$G^{ab}(B;x,x') = -\frac{\delta^2 W[0,\eta,\bar\eta,B]}{\delta\eta^b(x')\ \delta\bar\eta^a(x)}\Big|_{\eta=\bar\eta=0} \quad . \tag{5.21}$$

The complete connected gluon propagator in the background B^a_μ can be written with the help of the polarisation operator in the background field B^a_μ.

$$\begin{aligned} G^{ab}_{\mu\nu}&(B;\alpha;x,x') = \\ &= G_o{}^{ab}_{\mu\nu}(B;\alpha;x,x') + \\ &+\int d^4z d^4z'\ G_o{}^{ac}_{\mu\lambda}(B;\alpha;x,z)\Pi^{cd\ \lambda\kappa}(B;z,z')G_o{}^{db}_{\kappa\nu}(B;\alpha;z',x') + \ldots \end{aligned} \tag{5.22}$$

After this introductory part we now are ready to discuss our main topic. Basically our intention is to relate the Yang-Mills effective action to the polarisation operator in the background field B^a_μ by the method described in chapter 2. But due to the gauge invariance of the effective action (5.16) at first glance we face a serious problem, because the second functional derivative has no inverse and looking naively at relation (2.11) yields questionable results.

Let us go back to our starting point (5.2) and have a closer look at the gauge condition. First let us assume we had shifted the gauge potential by $A^a_\mu \to A^a_\mu + \bar A^a_\mu$. Second let us modify slightly the gauge condition without affecting the Faddeev-Popov procedure, so leaving aside the ghost contribution for a moment. Then the gauge term reads

$$\sim \frac{\alpha}{2}(D^{ab}_\mu(B)(A^{b\mu} + \bar A^{b\mu} - B'^{b\mu}))(D^{ac}_\nu(B)(A^{c\nu} + \bar A^{c\nu} - B'^{c\nu})) \quad . \tag{5.23}$$

Applying our method described in chapter 2 we have to take into account the renormalization properties of the Yang-Mills theory. So consider the multiplicative renormalization of different relevant quantities. We have

$$\begin{aligned} A^{a\mu}_o &= Z_3^{\frac12}\ A^{a\mu} \quad , \\ \bar A^{a\mu}_o &= Z_3^{\frac12}\ \bar A^{a\mu} \quad , \\ B'^{a\mu}_o &= Z_3^{\frac12}\ B'^{a\mu} \quad , \\ B^{a\mu}_o &= Z_g^{-1}\ B^{a\mu} \quad , \\ g_o &= Z_g\ g \quad , \qquad Z_g = Z_1 Z_3^{-\frac32} \quad , \\ \alpha_o &= Z_3^{-1}\ \alpha \quad . \end{aligned} \tag{5.24}$$

Here Z_1 is the gluon 3-point function renormalization constant. The renormalization (5.24) is the only consistent one which admits a shift of the gluon field without conflicting with renormalization. But on the other hand the whole background field formalism, which heavily relies on $\bar A = B$, demands

$$Z_g^{-1} = Z_3^{\frac12} \quad . \tag{5.25}$$

So from (5.25) we find for the well known Ward-Takahashi identity

$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3} = \frac{\tilde Z_1}{\tilde Z_3} = 1 \tag{5.26}$$

(Z_4... gluon 4-point function renormalisation constant, $\tilde{Z}_1, \tilde{Z}_3$... ghost renormalisation constants). In principle $\frac{\tilde{Z}_1}{\tilde{Z}_3}$ is a function of the renormalized gauge parameter α, the renormalised coupling constant g, the renormalisation point μ and the regularization parameter. Condition (5.26) means that the gauge parameter α has to be chosen in such a way that (5.26) is respected. This is possible since $\alpha \neq 0$ is arbitrary and it can be done order by order. So practically the renormalized gauge parameter α has to be a function of the renormalized coupling constant $g(\mu)$ in order to make the whole background field calculation consistent. At the lowest order (i.e. 1-loop) this means $\alpha = -\frac{1}{3}$. Taking this into account we may almost forget (5.23) now. (5.25) makes everything consistent with the usual background field formalism. We have to calculate all quantities for arbitrary α and then to insert the right expression in accordance with (5.26). At this point we may raise the question, whether the independence of the effective action of α (at least proved for 1-loop and background fields respecting (5.13) for $J = 0$ [10],[11],[14]) has been lost. The answer is no, but obviously the application of the reconstruction method we are dealing with becomes highly complicated due to an interference with renormalisation.

Having gained some insight into the renormalisation peculiarities of the background field method in Yang-Mills theory now we have to answer the question concerning the second functional derivative of the effective action with respect to the background field. Recall (5.23) again (take $\bar{B} = \bar{B}'$). In order to get a gauge invariant effective action at the end one has to consider $\bar{A} = \bar{B}$. But in principle the effective action is a functional both of $\bar{A}$ and $\bar{B}$ independently. So we have to write the second functional derivative of Γ in the following way.

$$\begin{aligned}\frac{\delta^2\Gamma[\bar{B}]}{\delta\bar{B}^{a\mu}(x)\,\delta\bar{B}^{a'\mu'}(x')} = & \left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{A}^{a\mu}(x)\,\delta\bar{A}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}} + \\ & + \left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{A}^{a\mu}(x)\,\delta\bar{B}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}} + \\ & + \left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{B}^{a\mu}(x)\,\delta\bar{A}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}} + \\ & + \left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{B}^{a\mu}(x)\,\delta\bar{B}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}}\end{aligned} \quad (5.27)$$

To gain further insight let us calculate the first functional derivative of $\Gamma[(\bar{A},\bar{B})]$ with respect to $\bar{B}^{a'\mu'}(x')$. We will assume for all the following calculations in this chapter, that the background field $\bar{A}$ respects (5.13) for $J = 0$. Otherwise the Ward-identities we use (s. Appendix B) become highly complicated. We have

$$\frac{\delta\Gamma[(\bar{A},\bar{B})]}{\delta\bar{B}^{a'\mu'}(x')} = -i\frac{1}{Z[0,0,0,(\bar{A},\bar{B})]}\frac{\delta}{\delta\bar{B}^{a'\mu'}(x')}Z[0,0,0,(\bar{A},\bar{B})] \quad . \quad (5.28)$$

Taking into account the Ward-identities (B 5), (B 6) we find

$$\begin{aligned}&\frac{\delta\Gamma[(\bar{A},\bar{B})]}{\delta\bar{B}^{a'\mu'}(x')} = \\ &= -\alpha D^{a'b}_{\mu'}(\bar{A})_{x'}D^{ba}_{\mu}(\bar{B})_{x'}(\bar{A}^{a\mu}(x') - \bar{B}^{a\mu}(x')) - \\ &\quad - \alpha g f^{a'ac}(D^{ab}_{\mu}(\bar{B})_{x'}(\bar{A}^{b\mu}(x') - \bar{B}^{b\mu}(x')))\frac{\delta}{\delta J'^{c\mu'}(x')}W[J',0,0,(\bar{A},\bar{B})]\big|_{J'=J'_\alpha} \cdot\end{aligned} \quad (5.29)$$

Of course the functional derivatives have to be calculated at $J'^c_\nu(x) = J'^c_\nu(x)_\alpha$, where

$$J'^c_\nu(x)_\alpha = -\alpha D^{ca}_{\nu}(\bar{B})_x D^{ab}_{\mu}(\bar{B})_x(\bar{A}^{b\mu}(x) - \bar{B}^{b\mu}(x)) \quad . \quad (5.30)$$

Using the fact, that the vacuum expectation value of the quantum gluon field at $\bar{A} = \bar{B}$ vanishes from (5.29) we get for (5.27)

$$\begin{aligned}\frac{\delta^2\Gamma[\bar{B}]}{\delta\bar{B}^{a\mu}(x)\,\delta\bar{B}^{a'\mu'}(x')} = & \left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{A}^{a\mu}(x)\,\delta\bar{A}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}} - \\ & - \alpha D^{ab}_{\mu}(\bar{B})_x D^{ba'}_{\mu'}(\bar{B})_x\delta^{(4)}(x-x') \quad .\end{aligned} \quad (5.31)$$

Clearly only the second functional derivative of Γ with respect to the shifted mode $\bar{A}$ of the gluon field is related to the complete connected gluon propagator via (2.11). So we get

$$\left.\frac{\delta^2\Gamma[(\bar{A},\bar{B})]}{\delta\bar{A}^{a\mu}(x)\,\delta\bar{A}^{a'\mu'}(x')}\right|_{\bar{A}=\bar{B}} = K^{aa'}_{\mu\mu'}(\bar{B};\alpha;x)\delta^{(4)}(x-x') - \Pi^{aa'}_{\mu\mu'}(\bar{B};x,x') \quad . \quad (5.32)$$

Finally,

$$\frac{\delta^2\Gamma[\bar{B}]}{\delta\bar{B}^{a\mu}(x)\,\delta\bar{B}^{a'\mu'}(x')} = K^{aa'}_{\mu\mu'}(\bar{B};0;x)\delta^{(4)}(x-x') - \Pi^{aa'}_{\mu\mu'}(\bar{B};x,x') \quad . \quad (5.33)$$

Using (B 15) and (B 19) we see, that the result is gauge covariant.

$$D^{ba\ \mu}(\bar{B})_x \frac{\delta^2\Gamma[\bar{B}]}{\delta \bar{B}^{a\mu}(x)\ \delta \bar{B}^{a'\mu'}(x')} = 0 \tag{5.34}$$

Applying now our formula (2.34) we find for the effective action (5.16)

$$\begin{aligned}\Gamma[\bar{B}] =& \Gamma[0] + \\ &+ \int_0^1 d\tau(1-\tau) \int d^4z d^4z'\ \bar{B}^{a\mu}(z)[K^{ab}_{\mu\nu}(\tau\bar{B};0;z)\delta^{(4)}(z-z') - \\ &- \Pi^{ab}_{\mu\nu}(\tau\bar{B};z,z')]\bar{B}^{b\nu}(z') \quad .\end{aligned} \tag{5.35}$$

Taking into account the general relation (In principle it holds for arbitrary background fields but for zero modes of $K^{ab}_{\mu\nu}(\bar{B};\alpha;.)$ such as (5.47) only in a generalized sense. In this case (5.36) corrects for the neglected surface term coming from the partial integration performed in the classical action in order to obtain $K^{ab}_{\mu\nu}(\bar{B};\alpha;.)$.)

$$\begin{aligned}\int_0^1 d\tau(1-\tau)\int d^4x\ \bar{B}^{a\mu}(x)K^{ab}_{\mu\nu}(\tau\bar{B};0;x)\bar{B}^{b\nu}(x) &= \\ &= -\frac{1}{4}\int d^4x\ F^a_{\mu\nu}(\bar{B})F^{a\ \mu\nu}(\bar{B}) \\ &= \Gamma_{cl}[\bar{B}]\end{aligned} \tag{5.36}$$

we arrive at the desired expression for the Yang-Mills effective action in the background $\bar{B}^a_\mu$ (respecting the classical equation of motion (5.13) for vanishing source).

$$\begin{aligned}\Gamma[\bar{B}] =& \Gamma[0] + \Gamma_{cl}[\bar{B}] - \\ &- \int_0^1 d\tau(1-\tau)\int d^4z d^4z'\ \bar{B}^{a\mu}(z)\Pi^{ab}_{\mu\nu}(\tau\bar{B};z,z')\bar{B}^{b\nu}(z')\end{aligned} \tag{5.37}$$

From Appendix B, (B 18), we know the relation

$$D^{ab\ \mu}(\bar{B})_x\ \Pi^{ba'}_{\mu\mu'}(\bar{B};x,x') = 0 \quad . \tag{5.38}$$

Assuming that the polarization operator is to be renormalized at some mass scale $\bar{\mu}_o$, let us look for the structure of the counter terms for the polarization operator. Before going into detail remember the well known situation at $\bar{B}=0$. The polarization operator $\Pi^{aa'}_{\mu\mu'}(\bar{B};x,x')$ agrees completely in the case $\bar{B}=0$ with the usual field theoretic expression. We have the identity

$$\partial^\mu_x\ \Pi^{aa'}_{\mu\mu'}(x-x') = 0 \quad , \tag{5.39}$$

where $\Pi^{aa'}_{\mu\mu'}(x-x') = \Pi^{aa'}_{\mu\mu'}(0;x,x')$. Denoting the divergent part (to be precise, the counter term necessary to renormalize the polarization operator at a mass scale $\bar{\mu}_o$) by wavy brackets, we know for the usual unrenormalized polarization operator that

$$\left\{\Pi^{aa'}_{\mu\mu'}(x-x')\right\}_{\bar{\mu}^2_o} = (Z_3(\bar{\mu}^2_o)-1)\delta^{aa'}(g_{\mu\mu'}\Box_x - \partial_\mu\partial_{\mu'})\delta^{(4)}(x-x') \quad . \tag{5.40}$$

In the case of a nonvanishing background field the Ward-identity (5.38) (i.e. the generalization of (5.39)) dictates the structure of the counter terms. For the possible counter terms we have to take into account the requirements for gauge covariance, locality and dimension. With respect to (5.40) the following generalizations are straightforward.

$$\delta^{aa'}g_{\mu\mu'}\Box \longrightarrow g_{\mu\mu'}D^{ab}_\sigma(\bar{B})D^{ba'\ \sigma}(\bar{B}) \tag{5.41}$$

$$\delta^{aa'}\partial_\mu\partial_{\mu'} \longrightarrow D^{ab}_\mu(\bar{B})D^{ba'}_{\mu'}(\bar{B}) - \frac{1}{2}f^{aa'b}F^b_{\mu\mu'}(\bar{B}) \tag{5.42}$$

So on the basis of (5.38) we have to determine a local object $O^{aa'}_{\mu\mu'}$ obeying $O^{aa'}_{\mu\mu'}(x)\delta^{(4)}(x-x') = O^{aa'}_{\mu\mu'}(x')\delta^{(4)}(x'-x)$ which respects the equation

$$\begin{aligned}D^{ca\ \mu}(\bar{B})_x\ [g_{\mu\mu'}D^{ab}_\sigma(\bar{B})_x D^{ba'\ \sigma}(\bar{B})_x - D^{ab}_\mu(\bar{B})_x D^{ba'}_{\mu'}(\bar{B})_x + \\ + \frac{1}{2}f^{aa'b}F^b_{\mu\mu'}(\bar{B})_x + O^{aa'}_{\mu\mu'}(x)]\delta^{(4)}(x-x') = 0 \quad .\end{aligned} \tag{5.43}$$

We find

$$O^{aa'}_{\mu\mu'} = \frac{3}{2}f^{aa'b}F^b_{\mu\mu'}(\bar{B}) \tag{5.44}$$

up to a covariantly constant local object $O'^{aa'}_{\mu\mu'}$. As we will see immediately we have to set it to zero in the counter term structure. So we find for the divergent part of the polarization operator in the background $\bar{B}$

$$\left\{\Pi^{aa'}_{\mu\mu'}(\bar{B};x,x')\right\}_{\bar{\mu}^2_o} = (Z_3(\bar{\mu}^2_o)-1)\ K^{aa'}_{\mu\mu'}(\bar{B};0,x)\delta^{(4)}(x-x') \quad . \tag{5.45}$$

With the help of (5.36) we obtain immediately, that

$$\int_0^1 d\tau(1-\tau)\int d^4z d^4z'\ B^{a\mu}(z)\left\{\Pi^{aa'}_{\mu\mu'}(\tau\bar{B};z,z')\right\}_{\bar{\mu}_o^2} B^{a'\mu'}(z') = \\ = (Z_3(\bar{\mu}_o^2)-1)\ \Gamma_{cl}[\bar{B}] \tag{5.46}$$

holds. This is just the result predicted by renormalisability of the Yang-Mills theory. What can we learn from the last two expressions? Under a change of renormalization point from $\bar{\mu}_o^2$ to $\bar{\mu}^2$, both expressions can respond only by a change $Z_3(\bar{\mu}_o^2) \to Z_3(\bar{\mu}^2)$. This puts severe restrictions on the possible shape of the polarization operator and the effective action. On the other hand it turns out, that in every case the imaginary part of the effective action is finite and independent of the renormalization point. These arguments apply to QCD and QED too. Of course, (5.45) has been derived for background fields respecting the classical equation of motion at vanishing source only, but one has to expect that relaxing this condition does not alter the counter term structure.

Finally let us discuss in a qualitative way the Yang-Mills effective action for constant background fields. As usual we take the simplest configuration

$$\bar{B}^a_\mu(x) = -\frac{1}{2}n^a F_{\mu\nu}x^\nu \quad , \tag{5.47}$$

$$F_{\mu\nu} = B\ \epsilon^{(12)}_{\mu\nu} \quad , \quad n^a n^a = 1 \ , \tag{5.48}$$

$$(V^4)^{-1}\Gamma_{cl}[\bar{B}] = \mathcal{L}_{cl}[\bar{B}] = \frac{1}{2}\ B^2 \quad , \tag{5.49}$$

where n^a points in one direction in the color space and the antisymmetric tensor $\epsilon^{(12)}_{\mu\nu}$ has nonvanishing components only for $\mu,\nu = 1,2$, i.e. we choose a quasi-abelian colormagnetic field. For that configuration most explicit calculations of the 1-loop effective Lagrangian have been carried out. In the following all quantities are considered at the 1-loop level. Unfortunately the method we will use now fails beyond the 1-loop level for the following reason. Due to (5.26) beyond the 1-loop level even the unrenormalized gauge parameter α depends on the renormalization point μ, at which one intends to renormalise the theory.

From now on we consider the effective Lagrangian which is defined as the effective action divided by the infinite space-time volume. The real part

of the effective Lagrangian can always be renormalised at a given mass scale $(gB)^{\frac{1}{2}} = \bar{\mu}_o$ in such a way, that

$$\mathcal{L}[\bar{B}]\Big|_{gB=\bar{\mu}_o^2} = \mathcal{L}_{cl}[\bar{B}]\Big|_{gB=\bar{\mu}_o^2} = \frac{\bar{\mu}_o^4}{2g^2} \tag{5.50}$$

holds. The effective Lagrangian depends on B via the dimensionless parameter $\frac{gB}{\bar{\mu}_o^2}$. For convenience the coupling constant g has been included, because gB is RG-invariant (cf. (5.24), (5.25)). Now compare the 1-loop polarization operators renormalised at the mass scale $\bar{\mu}_o$ and an arbitrary mass scale $\bar{\mu}$ respectively. We have

$$\Pi^{aa'}_{\mu\mu'}(\bar{B};x,x')_{\bar{\mu}^2} - \Pi^{aa'}_{\mu\mu'}(\bar{B};x,x')_{\bar{\mu}_o^2} = (Z_3(\bar{\mu}_o^2)-Z_3(\bar{\mu}^2))\ K^{aa'}_{\mu\mu'}(\bar{B};0;x)\delta^{(4)}(x-x'). \tag{5.51}$$

Writing

$$\int_0^1 (1-\tau)\int d^4z d^4z'\ B^{a\mu}(z)\Pi^{aa'}_{\mu\mu'}(\tau\bar{B};z,z')_{\bar{\mu}^2}B^{a'\mu'}(z') = \frac{1}{2}\ B^2\ f(\frac{gB}{\bar{\mu}^2}) \tag{5.52}$$

due to (5.50) we obtain

$$Re\ f(\frac{gB}{\bar{\mu}^2}) = Z_3(gB) - Z_3(\bar{\mu}^2) \quad . \tag{5.53}$$

Finally recalling (5.25) we find for the real part of the 1-loop effective Lagrangian corresponding to the configuration (5.47)

$$\mathcal{L}[\bar{B}] = \frac{1}{2}\ B^2\ (1 - Z_g^{-2}(gB) + Z_g^{-2}(\bar{\mu}_o^2)) \quad . \tag{5.54}$$

Using the 1-loop result for Z_g (for the gauge group SU(N); Λ is the regularization parameter)

$$Z_g(\bar{\mu}_o^2) = 1 - \frac{g^2}{16\pi^2}\frac{11\ N}{6}\ln\ \frac{\Lambda^2}{\bar{\mu}_o^2} \tag{5.55}$$

we recover the well known real part of the 1-loop effective Lagrangian for the configuration (5.47).

$$\mathcal{L}[\bar{B}] = \frac{1}{2}\ B^2 + \frac{(gB)^2}{16\pi^2}\frac{11\ N}{12}\ln\ \frac{(gB)^2}{\bar{\mu}_o^4} \tag{5.57}$$

In an heuristic approach one may re-identify $\frac{1}{2}B^2$ with $-\frac{1}{4}F^a_{\mu\nu}(\bar{B})F^{a\ \mu\nu}(\bar{B})$ for constant fields, but this is heuristic only.

The argument given equally applies to the case of massless QCD (of course Z_g has to be modified) as well as to other massless theories, where the background field provides the only parameter with an appropriate dimension.

6. Conclusions

We have seen that it is possible directly to relate the polarisation operator in a background field to the effective action. Even in the case of gauge theories the special problems connected with the gauge invariance of the effective action can be treated in a consistent way. The established relation can be used to get deeper insight into the structure of the effective action and may be applied actually to calculate the effective action for special configurations. Eventually one may find it useful to apply this relation in studying anomalies and fermion-boson relations.

Acknowledgements

I would like to thank E. Wiecsorek for numerous enlightening discussions sharing his deep insight into quantum field theory and critical comments on a preliminary version of this paper. Without having the background of related joint work in Yang-Mills theory it would never have been possible to write this paper.

This work has been supported by a Fulbright stipend and in part by NSF grant PHY 82-15249.

Appendix A

Consider a functional Φ of the field $\varphi(x)$, that admits the following representation.

$$\Phi[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \frac{\delta^n \Phi[\varphi']}{\delta\varphi'(x_1)...\delta\varphi'(x_n)}\Big|_{\varphi'=0} \varphi(x_1)...\varphi(x_n) \qquad \text{(A 1)}$$

Now we take the second functional derivative of Φ for later use.

$$\frac{\delta^2\Phi[\varphi]}{\delta\varphi(z)\,\delta\varphi(z')} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \int dx_1 ... dx_{n-2} \cdot \frac{\delta^n \Phi[\varphi']}{\delta\varphi'(x_1)...\delta\varphi'(x_{n-2})\,\delta\varphi'(z)\,\delta\varphi'(z')}\Big|_{\varphi'=0} \cdot \varphi(x_1)...\varphi(x_{n-2}) \qquad \text{(A 2)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \cdot \left(\frac{\delta^n}{\delta\varphi'(x_1)...\delta\varphi'(x_n)} \frac{\delta^2\Phi}{\delta\varphi'(z)\,\delta\varphi'(z')}[\varphi']\right)\Big|_{\varphi'=0} \cdot \varphi(x_1)...\varphi(x_n) \qquad \text{(A 3)}$$

We define

$$\bar\Phi[\varphi,\tau] = \Phi[\tau\varphi] \ , \qquad \tau \in \mathrm{R} \ . \qquad \text{(A 4)}$$

One has

$$\Phi[\varphi] = \int_0^1 d\tau \, \frac{\partial\bar\Phi[\varphi,\tau]}{\partial\tau} + \Phi[0] \qquad \text{(A 5)}$$

and

$$\frac{\partial\bar\Phi[\varphi,\tau]}{\partial\tau} = \int_0^{\tau} d\tau' \, \frac{\partial^2\bar\Phi[\varphi,\tau']}{\partial\tau'^2} + \frac{\partial\bar\Phi[\varphi,\bar\tau]}{\partial\bar\tau}\Big|_{\bar\tau=0} \ . \qquad \text{(A 6)}$$

Putting this together we find

$$\Phi[\varphi] = \int_0^1 d\tau \int_0^{\tau} d\tau' \, \frac{\partial^2\bar\Phi[\varphi,\tau']}{\partial\tau'^2} + \frac{\partial\bar\Phi[\varphi,\bar\tau]}{\partial\bar\tau}\Big|_{\bar\tau=0} + \Phi[0] \ . \qquad \text{(A 7)}$$

At this point the following general formula is worth noting, where $f = f(\tau)$ is a ordinary sufficiently well behaved function of τ.

$$\int_0^1 d\tau \int_0^{\tau} d\tau' \, f(\tau') = \tau \int_0^{\tau} d\tau' \, f(\tau')\Big|_0^1 - \int_0^1 d\tau \, \tau f(\tau) = \int_0^1 d\tau (1-\tau) \, f(\tau) \qquad \text{(A 8)}$$

Now we apply formula (A 8) to (A 7) and find

$$\Phi[\varphi] = \int_0^1 d\tau (1-\tau) \, \frac{\partial^2\bar\Phi[\varphi,\tau]}{\partial\tau^2} + \frac{\partial\bar\Phi[\varphi,\bar\tau]}{\partial\bar\tau}\Big|_{\bar\tau=0} + \Phi[0] \ . \qquad \text{(A 9)}$$

Taking into account the definitions (A 4) and (A 1) we have

$$\bar\Phi[\varphi,\tau] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \frac{\delta^n \Phi[\varphi']}{\delta\varphi'(x_1)...\delta\varphi'(x_n)}\Big|_{\varphi'=0} \tau^n \varphi(x_1)...\varphi(x_n) \ . \qquad \text{(A 10)}$$

From (A 10) we see immediately

$$\frac{\partial\bar\Phi[\varphi,\bar\tau]}{\partial\bar\tau}\Big|_{\bar\tau=0} = \int dz \, \frac{\delta\Phi[\varphi']}{\delta\varphi'(z)}\Big|_{\varphi'=0} \varphi(z) \ . \qquad \text{(A 11)}$$

At last we now are looking for the second derivative of $\Phi[\varphi,\tau]$ with respect to τ. We find from (A 10)

$$\frac{\partial^2\Phi[\varphi,\tau]}{\partial\tau^2}=\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int dx_1...dx_n\cdot \cdot\frac{\delta^n\Phi[\varphi']}{\delta\varphi'(x_1)...\delta\varphi'(x_n)}\Big|_{\varphi'=0}\ \tau^{n-2}\varphi(x_1)...\varphi(x_n) \quad (A\ 12)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1...dx_n dz dz'\cdot \cdot\left(\frac{\delta^n}{\delta\varphi'(x_1)...\delta\varphi'(x_n)}\frac{\delta^2\Phi}{\delta\varphi'(z)\,\delta\varphi'(z')}[\varphi']\right)\Big|_{\varphi'=0}\cdot \cdot\ \tau^n\varphi(x_1)...\varphi(x_n)\varphi(z)\varphi(z')\ . \quad (A\ 13)$$

Comparing this expression with (A 3) we read

$$\frac{\partial^2\Phi[\varphi,\tau]}{\partial\tau^2}=\int dzdz'\ \varphi(z)\frac{\delta^2\Phi}{\delta\varphi(z)\,\delta\varphi(z')}[\tau\varphi]\ \varphi(z')\ . \quad (A\ 14)$$

So, finally from all that we arrive at our reconstruction formula.

$$\Phi[\varphi]=\Phi[0]+\int dz\ \frac{\delta\Phi[\varphi']}{\delta\varphi'(z)}\Big|_{\varphi'=0}\varphi(z)+ +\int_0^1 d\tau(1-\tau)\int dzdz'\ \varphi(z)\frac{\delta^2\Phi}{\delta\varphi(z)\,\delta\varphi(z')}[\tau\varphi]\ \varphi(z') \quad (A\ 15)$$

This general formula gives the reconstruction prescription for any functional $\Phi[\varphi]$ of the form (A 1), once one is given the second functional derivative of Φ, the first functional derivative of Φ at $\varphi=0$ and its value at $\varphi=0$.

7. Appendix B, [15]

In the following we will derive generalized Ward-identities for the Yang-Mills theory in a background $\bar{B}^a_\mu$. We are working within the covariant background gauge. As starting point we choose the well known expression for the generalized Ward-identity of usual Yang-Mills theory borrowed from reference [16]. It is

$$\int DA^a_\mu\ \det M_\Phi\left\{\alpha\Phi^a(A)(y)-\int d^4z\ J^c_\mu(z)D^{cb\,\mu}(A)_z M_\Phi^{-1\,ba}(A;z,y)\right\}\cdot \cdot\ e^{i\Gamma_{cl}[A]+i\int d^4x(-\frac{\alpha}{2}\Phi^a(A)\Phi^a(A)+J^a_\mu(x)A^{a\mu}(x))}=0\ . \quad (B\ 1)$$

The gauge term Φ^a is given by the following expression.

$$\Phi^a(A)=D^{ab}_\mu(\bar{B})(A^{b\mu}-\bar{B}^{b\mu}) \quad (B\ 2)$$

The Faddeev-Popov operator has the structure

$$M^{ab}=D^{ac}_\mu(\bar{B})D^{cb\,\mu}(A)\quad . \quad (B\ 3)$$

In order to come along with the background field formulation we perform a shift $A^a_\mu\to A^a_\mu+\bar{A}^a_\mu$ in (B 1) and multiply the expression by a factor

$$e^{-i\int d^4x\ J^a_\mu(x)\bar{A}^{a\mu}(x)}\quad . \quad (B\ 4)$$

Introducing the ghost representation for the FP determinant we find

$$\int DA^a_\mu\ D\bar{c}^a\ Dc^a\left\{\alpha D^{ab}_\mu(\bar{B})_y(A^{b\mu}(y)+\bar{A}^{b\mu}(y)-\bar{B}^{b\mu}(y))\ - -\int d^4z\ J^c_\mu(z)D^{cb\,\mu}(\bar{A}+A)_z M^{-1\,ba}(\bar{A}+A;z,y)\right\}\cdot \cdot\ e^{i\Gamma'_{cl}[A+\bar{A},\bar{B},\bar{c},c]+i\int d^4x\ J^a_\mu(x)A^{a\mu}(x)}=0\ . \quad (B\ 5)$$

Taking a functional derivative with respect to $J^c_{\mu'}(x)$ and contracting with $gf^{aa'c}$ we get

$$\int DA^a_\mu D\bar{c}^a Dc^a\left\{\alpha g f^{aa'c}D^{ab}_\mu(\bar{B})_y(A^{b\mu}(y)+\bar{A}^{b\mu}(y)-\bar{B}^{b\mu}(y))A^{c\mu'}(x)\ + +igf^{aa'c}D^{cb\,\mu'}(\bar{A}+A)_x M^{-1\,ba}(\bar{A}+A;x,y)\ - -\int d^4z\ J^c_\mu(z)D^{cb\,\mu}(\bar{A}+A)_z M^{-1\,ba}(\bar{A}+A;z,y)gf^{aa'c}A^{c\mu'}(x)\right\}\cdot \cdot\ e^{i\Gamma'_{cl}[A+\bar{A},\bar{B},\bar{c},c]+i\int d^4x\ J^a_\mu(x)A^{a\mu}(x)}=0\ . \quad (B\ 6)$$

Having derived this formula which is needed in chapter 5 we consider in the following $\bar{A} = \bar{B}$ only. The perturbation theoretic formulation of (B 5) on the basis of (5.15) then reads (having redefined $J = J' + \bar{J}$)

$$\Big\{\alpha D_\mu^{ab}(\bar{B})_y \frac{\delta}{i\delta J_\mu'^b(y)} + i\int d^4z\, (J_\mu'^c(z) + \bar{J}_\mu^c(z)) D^{cb\,\mu}(\bar{B} + \frac{\delta}{i\delta J'})_z \frac{\delta}{i\delta\eta^a(y)} \frac{\delta}{i\delta\bar{\eta}^b(z)} \Big\} \cdot Z[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0} = 0 \ . \qquad \text{(B 7)}$$

Taking into account

$$\frac{\delta}{\delta\bar{\eta}^a(x)} W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0} = \frac{\delta}{\delta\eta^a(x)} W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0} = 0 \qquad \text{(B 8)}$$

from (B 7) and (5.17) we obtain for the connected Greens functions the relation

$$\Big\{\alpha D_\mu^{ab}(\bar{B})_y \frac{\delta}{\delta J_\mu'^b(y)} + \int d^4z\, (J_\mu'^c(z) + \bar{J}_\mu^c(z)) D^{cb\,\mu}(\bar{B} + \frac{\delta}{i\delta J'})_z \frac{\delta}{\delta\eta^a(y)} \frac{\delta}{\delta\bar{\eta}^b(z)} + \int d^4z\, (J_\mu'^c(z) + \bar{J}_\mu^c(z)) g f^{cbd} (\frac{\delta}{\delta J_\mu'^d(z)} W[J', 0, 0, \bar{B}]) \frac{\delta}{\delta\eta^a(y)} \frac{\delta}{\delta\bar{\eta}^b(z)} \Big\} \cdot W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0} = 0 \ . \qquad \text{(B 9)}$$

Taking into account (5.14) in the case $J' = 0$ (B 9) reduces to

$$\int d^4z\, \bar{J}_\mu^c(z) g f^{cbe} \frac{\delta}{\delta J_\mu'^e(z)} \frac{\delta}{\delta\eta^a(y)} \frac{\delta}{\delta\bar{\eta}^b(z)} \ W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0, J'=0} = 0 \qquad \text{(B 10)}$$

due to the vanishing vacuum expectation value of the gluon field.

$$\frac{\delta}{\delta J_\mu'^a(x)} W[J', 0, 0, \bar{B}]\big|_{J'=0} = 0 \qquad \text{(B 11)}$$

Operating on (B 9) with a functional derivative with respect to J' and taking into account (5.14) we find at $J' = 0$

$$\int d^4z \Big\{\alpha D^{ac\,\mu}(\bar{B})_y \delta^{(4)}(y - z) - G^{ab}(\bar{B}; y, z) g f^{bcd} \bar{J}^{d\mu}(z)\Big\} G_{\mu\nu}^{ca'}(\bar{B}; \alpha; z, y') = D_\nu^{a'b}(\bar{B} + \frac{\delta}{i\delta J'})_{y'} \ \frac{\delta}{\delta\eta^a(y)} \frac{\delta}{\delta\bar{\eta}^b(y')} \ W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0, J'=0} \ . \qquad \text{(B 12)}$$

Acting on (B 12) with a covariant derivative and taking into account the relation

$$D_\mu^{ab}(\bar{B})_{y'} D^{bc\,\mu}(\bar{B} + \frac{\delta}{i\delta J'})_{y'} \frac{\delta}{\delta\eta^{a'}(y)} \frac{\delta}{\delta\bar{\eta}^c(y')} W[J', \eta, \bar{\eta}, \bar{B}]\big|_{\eta=\bar{\eta}=0, J'=0} = -\delta^{aa'} \delta^{(4)}(y - y') \qquad \text{(B 13)}$$

we find the following Ward-identity for the complete connected gluon propagator in the background $\bar{B}_\mu^a$.

$$D^{a'b'\,\mu'}(\bar{B})_{y'} \int d^4z \Big\{\alpha D^{ac\,\mu}(\bar{B})_y \delta^{(4)}(y - z) - G^{ab}(\bar{B}; y, z) g f^{bcd} \bar{J}^{d\mu}(z)\Big\} G_{\mu\mu'}^{cb'}(\bar{B}; \alpha; z, y') = -\delta^{aa'} \delta^{(4)}(y - y') \qquad \text{(B 14)}$$

In principle we intend to derive from (B 14) the Ward-identity for the gluon polarization operator in the background $\bar{B}_\mu^a$. To find this for $\bar{J} \neq 0$ seems at the time being hardly possible due to the highly nonlinear character of (B 14). So we restrict ourselves to $\bar{J} = 0$, i.e. the background field should fulfill the classical equation of motion for vanishing source.

Now let us have a look at a useful relation for the free connected propagators, at first allowing an arbitrary background field. One has the relation [11]

$$D^{ab\,\mu}(\bar{B}) K_{\mu\mu'}^{ba'}(\bar{B}; \alpha; x) = \alpha K^{ab}(\bar{B}; x) D_{\mu'}^{ba'}(\bar{B}) - g f^{aa'c} \bar{J}_{\mu'}^c(x) \ . \qquad \text{(B 15)}$$

With the help of (5.11) and (5.12) we obtain

$$\int d^4z \Big\{\alpha D^{ac\,\mu}(\bar{B})_x \delta^{(4)}(x - z) - G_o^{ab}(\bar{B}; x, z) g f^{bcd} \bar{J}^{d\mu}(z)\Big\} \cdot G_{o\,\mu\mu'}^{\ ca'}(\bar{B}; \alpha; z, x') = -D_{\mu'}^{a'b}(\bar{B})_{x'} G_o^{ab}(\bar{B}; x, x') \ . \qquad \text{(B 16)}$$

Of course this could have been deduced from (B 12) too. Restricting the background field $\bar{B}_\mu^a$ to obey the classical equation of motion for a vanishing source we find

$$\alpha D^{ab\,\mu}(\bar{B})_x G_{o\,\mu\mu'}^{\ ba'}(\bar{B}; \alpha; x, x') = -D_{\mu'}^{a'b}(\bar{B})_{x'} G_o^{ab}(\bar{B}; x, x') \ . \qquad \text{(B 17)}$$

Introducing (5.22) into (B 14) and taking into account (B 17) we get for the polarization operator in the background $\bar{B}^a_\mu$ the following identity.

$$D^{ab\,\mu}(\bar{B})_x\,\Pi^{ba'}_{\mu\mu'}(\bar{B};x,x')=0 \qquad \text{(B 18)}$$

One should keep in mind that we have used the condition $\bar{A}=\bar{B}$ in deriving (B 18). Therefore α is subject to the condition (5.26).

References

[1] W. Heisenberg, H. Euler; Z. Phys. 98(1936)714.

[2] J. Schwinger; Phys. Rev. 82(1951)664.

[3] S. Coleman, E. Weinberg; Phys. Rev. D 7(1973)1888.

[4] R. Jackiw; Phys. Rev. D 9(1974)1686.

[5] L. Dolan, R. Jackiw; Phys. Rev. D 9(1974)3320.

[6] W. Dittrich, M. Reuter; Selected Topics in Gauge Theories. Lecture Notes in Physics No. 244. Springer-Verlag, Berlin -...-Tokyo 1986.

[7] C.W. Bernard; Phys. Rev. D 9(1974)3312.

[8] D. Robaschik, K. Scharnhorst, E. Wieczorek; Ann.Phys.(N.Y.)174(1987)401.

[9] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev; Integrals and Series. Vol. 1. Gordon and Breach Science Publ., New York 1986.

[10] K. Scharnhorst; Thesis. Humboldt University Berlin, cited material to appear elsewhere.

[11] I.A. Batalin, S.G. Matinyan, G.K. Savvidi; Yad. Fiz. 26(1977)407.

[12] L.F. Abbott; Nucl. Phys. B 185(1981)189.

[13] H. Kaiser, K. Scharnhorst, E. Wieczorek; in: Proc. XX. Int. Symp. Ahrenshoop 1986, IfH Berlin-Zeuthen preprint, PHE 87-. ; IfH Berlin-Zeuthen preprint, PHE 87-9; in: Proc. XXI. Int. Symp. Ahrenshoop/Sellin 1987, IfH Berlin-Zeuthen preprint to appear.

[14] T.H. Hansson, I. Zahed; Nucl. Phys. B 292 (1987)725.

[15] based on joint work together with E. Wieczorek.

[16] A.A. Slavnov, L.D. Faddeev; Gauge Fields: Introduction to Quantum Theory. Benjamin/Cummings Publ. Comp., London-...-Tokyo 1980.