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QUANTUM FIELD THEORETIC TREATMENT OF THE CASIMIR EFFECT AT T≠0 IMAGINARY TIME FORMALISM

D. Robaschik, K. Scharnhorst and E. Wieczorek

INSTITUT FÜR HOCHENERGIEPHYSIK
AKADEMIE DER WISSENSCHAFTEN DER DDR
BERLIN-ZEUTHEN · DDR

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Quantum Field Theoretic Treatment of the Casimir Effect at $T \neq 0$
(Imaginary Time Formulation)

D. Robaschik
Sektion Physik der Karl-Marx-Universität Leipzig, DDR
K. Scharnhorst
Sektion Physik der Humboldt Universität Berlin, DDR
E. Wieczorek
Institut für Hochenergiephysik der Akademie der Wissenschaften Zeuthen, DDR

The second order radiative corrections to the Casimir pressure at $T = 0$ are rederived in a more simple manner. Within the Matsubara formalism these corrections are obtained in the case $T \neq 0$. Explicit expressions are given for $a^{-1} \ll T \ll m$.

1. Introduction

Only a short time after the first considerations of the Casimir effect [1] it has been pointed out that thermal effects have to be taken into account [2]. In the mean time this problem has been investigated by several authors [3], [4]. These investigations are based on intuitive arguments or operate with Green functions for the field strength directly. Until quite recently a general quantum field theoretic treatment has been developped [5]. It clarifies the reduction onto the two physical degrees of freedom of the radiation field, and allows the calculation of higher order corrections to the Casimir pressure. In the present paper this method combined with the Matsubara formalism is applied to the Casimir effect within a heat bath.

The present paper is organized as follows. In the second section the generating functional for the complete Green functions at T = 0 and thereby an expression for the free energy are derived. Section 3 presents an alternative calculation of the Casimir pressure for zero temperature via the free energy. Section 4 is concerned with the case T ≠ 0. The obtained results will be discussed in the final section 5.

2. Z-Functional and Free Energy

The generating functional Z for QED with boundary conditions $n_\mu \partial_\nu \varepsilon^{\mu\nu\alpha\beta} A_\alpha(x) = 0$ on two parallel plates at $x_3 = a_1$ and $x_3 = a_2$ has been derived in [5]

$$Z_B(j,\bar\eta,\eta) = C \int DA\, DB\, D\bar\psi\, D\psi \,\exp i \Big\{ \int d^4x \left(\mathcal{L}(x) + j_\mu A^\mu + \bar\eta\psi + \bar\psi\eta \right) + \sum_{k=1}^{2} \int dS_k(x)\, B^{k\alpha}(x)\, H_{\alpha\mu}(x,\partial_x) A^\mu(x) - \frac{1}{2\lambda} \sum_k \int dS_k(x)\, dS_k(y)\, B^{k\alpha}(x) \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} D^c(x_k - y_k, 0)\, B^{k\beta}(y) \Big\} . \tag{2.1}$$

Here

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar\psi (i\hat\partial - m + e\hat{A}) \psi$$

is the usual Lagrangian of QED with a gauge fixing term [8] $B^{k\alpha}$ ($\alpha = 0,1,2$) denotes an auxiliary field which is defined on the plate k (k = 1,2) i. e. at $x_3 = a_k$ only and

$$dS_k = d^4x\, \delta(x_3 - a_k) \qquad k = 1,2 ,$$

$$H_{\alpha\mu}(x,\partial_x) = -n^\lambda \varepsilon_{\lambda\alpha\mu}{}^{\gamma} \frac{\partial}{\partial x^\gamma} , \tag{2.2}$$

$$n^\lambda = (0,0,0,1) .$$

In writing down expression (2.1) the standard Fadeev-Popov determinants of QED and two similar determinants corresponding to the gauge freedom in $B^{\kappa\kappa}$ have been omitted as beeing independent of the distance of the plates $a = |a_2 - a_1|$.

Formula (2.1) serves as the starting point for usual $T = 0$ QED as well as for QED at $T \neq 0$. If we choose for the latter case the Matsubara formulation, one has to continue to imaginary time and take into account the appropriate periodicity conditions. In the following we are interested in the a-dependent quantities only, so that the Faddeev-Popov determinants though depending on temperature now are without influence.

Let us first study Z as it is written in Minkowski space

$$Z_B(j,\bar{\eta},\eta) = C \; \exp i\int d^4x\, \mathcal{L}_{int}\left(\frac{\delta}{\delta i j}, \frac{\delta}{\delta i \eta}, \frac{\delta}{\delta i \bar{\eta}}\right) \cdot \exp -i\int d^4x\, d^4y\, \bar{\eta} S^c \eta \quad \cdot \tilde{Z} \qquad (2.3)$$

The Fermion determinant has been included into the normalization factor C.

$$\tilde{Z} = \int DB\, DA\, \exp i\left\{\int d^4x \left(\tfrac{1}{2} A_\mu K^{\mu\nu} A_\nu + \mathcal{J}_\mu A^\mu\right) - \frac{1}{2\lambda}\sum_\kappa \int dS_\kappa(x)\, dS_\kappa(y)\, B^{\kappa\alpha}_{(x)} \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y_\beta} D^c(x_\kappa - y_\kappa, 0)\, B^{\kappa\beta}(y)\right\} \qquad (2.4)$$

with

$$\mathcal{J}_\mu = j_\mu(x) + \sum_\kappa \delta(x_3 - a_\kappa)\, B^{\kappa\kappa}\, H_{\kappa\mu}(x,\partial_x)$$

$$K^{\mu\nu} = g^{\mu\nu}\Box - \partial^\mu\partial^\nu\left(1 - \tfrac{1}{\alpha}\right) \qquad (2.5)$$

Gaussian integration over A_μ leads to

$$\tilde{Z} = \left[\mathrm{Det}\, K^{\mu\nu}\right]^{-\frac{1}{2}} \exp\left(-\tfrac{i}{2}\, j K^{-1} j\right) \int DB \exp i\Big\{ \frac{1}{2}\sum_i\sum_j \int dS_i(x)\, dS_j(y)\, B^{i\alpha}_{(x)} K^{ij}_{\alpha\beta} B^{j\beta}(y) - \sum_\kappa \int d^4x\, d^4y\, \delta(x_3 - a_\kappa)\, B^{\kappa\alpha}_{(x)} H_{\alpha\mu}(x,\partial_x) \left(K^{-1}\right)_{\mu\nu}(x-y)\, j^\nu(y)\Big\} \qquad (2.6)$$

Here $(K^{-1})_{\mu\nu} = D^c_{\mu\nu}$ is the well known photon propagator in covariant gauge. For the kerenel $K^{ij}_{\alpha\beta}$ we find the explicit form

$$K^{ij}_{\alpha\beta}(x-y) = \frac{-i}{2}\int \frac{d^3\tilde{p}}{(2\pi)^3}\, e^{-i\tilde{p}(\widetilde{x-y})}\, \Gamma(p) \cdot \left\{\left(g_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}\right) h_{ij} + \frac{1}{\lambda}\frac{p_\alpha p_\beta}{p^2}\,\delta_{ij}\right\} \qquad (2.7)$$

with $\tilde{p} = (p_0, p_1, p_2)$, $\tilde{x} = (x_0, x_1, x_2)$, $\Gamma(p) = (p_0^2 - p_1^2 - p_2^2)^{+\frac{1}{2}}$ (positive imaginary part understood)

$$h_{ij} = e^{i\Gamma|a_i - a_j|} \qquad \text{and } \alpha,\beta = (0,1,2).$$

The following integration over B gives the final result

$$\tilde{Z} = \left[\mathrm{Det}\, K^{\mu\nu}\right]^{-\frac{1}{2}} \left[\mathrm{Det}\, K^{ij}_{\alpha\beta}\right]^{-\frac{1}{2}} \cdot \exp -\frac{i}{2}\left\{\int d^4x\, d^4y\, j^\mu(x)\, {}^S D^c_{\mu\nu}(x,y)\, j^\nu(y)\right\} \qquad (2.8)$$

with the full photon propagator

$$^{S}D^{c}_{\mu\nu}(x,y) = D^{c}_{\mu\nu}(x-y) + \bar{D}_{\mu\nu}(x,y)$$

$$\bar{D}_{\mu\nu}(x,y) = \sum_{i,j}\int d\Sigma_i(z)\, d\Sigma_j(z')\, D^{c}_{\mu\rho}(x-z)\, H^{\rho\alpha}(z,\partial_z)\,(K^{-1})^{ij}_{\alpha\beta}(z,z') \cdot H^{\beta\rho'}(z',\partial_{z'})\, D^{c}_{\rho'\nu}(z'-y) \tag{2.9}$$

$$= \frac{1}{2i}\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\Gamma(p)} P_{\mu\nu}(p)\, e^{i\tilde{p}(\widetilde{x-y})}\, e^{i\Gamma|x_3-a_i|} \cdot k^{-1}_{ij}\, e^{i\Gamma|y_3-a_j|},$$

$$P_{\mu\nu}(p) = \left\{ g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \text{ for } \mu,\nu\neq 3\, ;\ 0 \text{ for } \mu=3 \text{ or } \nu=3 \right\}$$

Eqs. (2.3), (2.8) and (2.9) determine the QED with boundary conditions in perturbation theory.

Beeing interested in the vacuum-vacuum transition amplitude ($T = 0$) or the free energy ($T \neq 0$) we need the Z-functional for vanishing sources

$$\log Z_B = -\frac{1}{2}\log \operatorname{Det} K^{ij}_{\alpha\beta} + \log\left(1 - \frac{e^2}{2}\int d^4x\, d^4y \langle T : \bar{\psi}\gamma_\mu\psi A^\mu_{(x)} :: \bar{\psi}\gamma_\nu\psi A^\nu_{(y)} : \rangle + \cdots\right) + a\text{-independent terms}. \tag{2.10}$$

Up to order e^2 we get

$$\log Z_B = -\frac{1}{2}\log \operatorname{Det} K^{ij}_{\alpha\beta} - \frac{e^2}{2}\int d^4x\, d^4y \langle T : \bar{\psi}\gamma_\mu\psi A^\mu :: \bar{\psi}\gamma_\nu\psi A^\nu \rangle$$

$$= -\frac{1}{2}\log \operatorname{Det} K^{ij} + \frac{1}{2}\int d^4x\, d^4y\, \bar{D}_{\mu\nu}(x,y)\, \Pi_{\mu\nu}(x-y) + a\text{ - independent terms.} \tag{2.11}$$

This formula is a convienent basis for a treatment of the Casimir force for $T = 0$ as well as for $T \neq 0$.

3. The Casimir Pressure at $T = 0$

In this section we reproduce in a more elegant manner the earlier obtained results for the $T = 0$ case.

Let us first calculate $\log \operatorname{Det} K^{ij}_{\alpha\beta}$ using the explicit expression (2.7). For the following it is essential, that eq. (2.7) can be written in the form

$$K^{ij}_{\alpha\beta}(x-y) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\, e^{-i\tilde{p}(\widetilde{x-y})}\left(g_{\alpha\beta}\delta^{ij} - \tilde{M}^{ij}_{\alpha\beta}(p)\right)$$

so that

$$\log \operatorname{Det} K^{ij}_{\alpha\beta} = \operatorname{tr}\log K^{ij}_{\alpha\beta}(x-y)$$

$$= \operatorname{tr}_{xy}\operatorname{tr}_{\alpha\beta}\operatorname{tr}_{ij}\left\{-M(x-y) - \frac{1}{2}dx_1\, M(x-x_1)\, M(x_1-y) \cdots\right\}$$

$$= \tilde{V}\operatorname{tr}_{\alpha\beta}\operatorname{tr}_{ij}\int\frac{d^3\tilde{p}}{(2\pi)^3}\log\left(\delta^{ij}_{\alpha\beta} - \tilde{M}^{ij}_{\alpha\beta}(p)\right) \tag{3.1}$$

$$= \tilde{V}\int\frac{d^3\tilde{p}}{(2\pi)^3}\log \det_{\substack{ij\\ \alpha\beta}} \tilde{K}^{ij}_{\alpha\beta}(p)\,, \qquad \tilde{V} = V_{(2)}\cdot T\,.$$

In spite of the fact that $\tilde{K}$ depends on the gauge fixing parameter λ there is no a-dependent contribution to $\log \det K$ from λ. This can be seen by varying $\frac{1}{\lambda}$

$$\delta \log\det\tilde{K} = \log\det(\tilde{K}+\delta\tilde{K}) - \log\det\tilde{K} = \log\det\left(1 + \tilde{K}^{-1}\delta\tilde{K}\right).$$

Noting that

$$\tilde{K}^{-1\,ij}_{\alpha\beta} = \frac{2i}{\Gamma}\left\{ \left(g_{\alpha\beta} - \frac{p_\alpha p_\beta}{\Gamma^2}\right) h^{-1}_{ij} + \lambda \frac{p_\alpha p_\beta}{\Gamma^2}\delta_{ij} \right\}$$

and

$$\delta\tilde{K} = \frac{p_\alpha p_\beta}{\Gamma}\delta_{ij}\,\delta\left(\frac{1}{\lambda}\right)$$

$$h^{-1}_{ij} = \frac{i}{2\sin\Gamma a}\begin{pmatrix} e^{-i\Gamma a} & -1 \\ -1 & e^{-i\Gamma a}\end{pmatrix} \qquad (3.2)$$

and therefore

$$1 + \tilde{K}^{-1}\delta\tilde{K} = g_{\alpha\beta}\delta_{ij} + \lambda\frac{p_\alpha p_\beta}{\Gamma^2}\delta_{ij}\,\delta\left(\frac{1}{\lambda}\right)$$

is independent of a. So one may choose the most convienent value of λ, e.g. $\lambda = 1$.

Leaving aside the factors $\frac{-i\Gamma}{2}$ in (2.7) and (3.1) we obtain using the projector properties of $g_{\alpha\beta} - \frac{p_\alpha p_\beta}{\Gamma^2}$

$$\log \mathrm{Det}\, K^{ij}_{\alpha\beta} = \tilde{V}\int\frac{d^3\tilde{p}}{(2\pi)^3}\,\mathop{tr}_{\alpha\beta}\,\mathop{tr}_{ij}\,\log\left\{\delta^\alpha_\beta\delta_{ij} - \left(\delta^\alpha_\beta - \frac{p_\alpha p^\beta}{\Gamma^2}\right)\left(\delta_{ij} - h_{ij}\right)\right\}$$

$$= \tilde{V}\int\frac{d^3\tilde{p}}{(2\pi)^3}\,\mathop{tr}_{\alpha\beta}\left(\delta^\alpha_\beta - \frac{p_\alpha p^\alpha}{\Gamma^2}\right)\mathop{tr}_{ij}\log h = 2\tilde{V}\int\frac{d^3\tilde{p}}{(2\pi)^3}\log\det h$$

$$= 2\tilde{V}\int\frac{d^3\tilde{p}}{(2\pi)^3}\log\left(1 - e^{2i\Gamma a}\right). \qquad (3.3)$$

After performing the Wick rotation ($p_0 = i p_4$, $\Gamma = i\gamma = i(p_4^2 + p_1^2 + p_2^2)^{1/2}$ we have in order e^0

$$\left(\log Z_B\right)_{e^0} = -i\tilde{V}\int\frac{dp_4\,dp_1\,dp_2}{(2\pi)^3}\log\left(1 - e^{-2\gamma a}\right) \qquad (3.4a)$$

$$= -i\tilde{V}\left(-\frac{\pi^2}{240}\,\frac{1}{3a^3}\right), \qquad (3.4b)$$

which is up to the factor $(-i\tilde{V})$ the well-known expression for the Casimir energy per unit area.

Before we turn to a discussion of this result we evaluate the second order contribution to log Z_B i.e. the second term in eq. (2.11). Taking into account (2.9) and the expression for the polarization operator [8]

$$\Pi_{\mu\nu}(z) = \left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\,\Pi(z^2)$$

$$\Pi(z^2) = \int\frac{d^4k}{(2\pi)^4}\,e^{-ikz}\,\tilde{\Pi}(k^2)$$

and

$$\bar{D}(z_1,z_2) = \frac{1}{2}g^{\mu\nu}\,\bar{D}_{\mu\nu}(z_1,z_2)$$

we can write

$$(\log Z_B)_{e^2} = \frac{1}{2}\int dz_1 dz_2\,\Pi_{\mu\nu}(z_1 - z_2)\,\bar{D}_{\mu\nu}(z_1,z_2) = \int dz_1 dz_2\,\Pi((z_1-z_2)^2)\,\Box_2\,\bar{D}(z_1,z_2)$$

$$= \int dz_1 dz_2\,\Pi((z_1-z_2)^2)\int\frac{d\tilde{q}}{(2\pi)^3}\,\delta(z_1^3 - a_i)\,(h^{-1})_{ij}\,e^{i\Gamma|z_2^3 - a_j|}\,e^{i\tilde{q}(\tilde{z}_1 - \tilde{z}_2)}$$

$$= \int d\tilde{z}_1\,d\tilde{y}\,dz_2^3\,\Pi(\tilde{y}^2 - (z_{23} - a_i)^2)\int\frac{d\tilde{q}}{(2\pi)^3}\,e^{i\tilde{q}\tilde{y}}\,(h^{-1})_{ij}\,e^{i\Gamma|z_2^3 - a_j|}$$

$$= -\frac{\tilde{V}}{(2\pi)^4}\int dk\,\tilde{\Pi}(k^2)\int_{-\infty}^{+\infty}dz_3\,e^{ik_3 z_3}\,e^{i\Gamma|z_3 - a_j + a_i|}\,(h^{-1})_{ij} \qquad (3.5)$$

A trivial but lengthy calculation which takes into account eq. (3.2) leads to

$$\int_{-\infty}^{+\infty} dz\, e^{ik_3 z}\, e^{i\Gamma|z+a_i-a_j|} (h^{-1})_{ij} = \tag{3.6}$$
$$= -\frac{2\Gamma}{k^2 \sin a\Gamma}\left(e^{ia\Gamma} - \cos a k_3\right) + \frac{4i\Gamma}{k^2}$$

so that finally (up to terms independent of a)

$$\left(\log Z_B\right)_{e^2} = 2\tilde{V}\int \frac{d^4k}{(2\pi)^4}\, \frac{\tilde{\Pi}(k^2)\,\Gamma}{k^2 \sin a\Gamma}\left(e^{ia\Gamma} - \cos a k_3\right) \tag{3.7}$$

As it is suggested by intuitive arguments one can connect Z_B with the vacuum energy

$$Z_B = e^{-i\Delta t E_{vac}}\Big|_{\Delta t\to\infty} = e^{-i\Delta t V_{(2)} E(a)}\Big|_{\Delta t\to\infty} \tag{3.8}$$

with $\Delta t\, V_{(2)}\big|_{\Delta t\to\infty} = \tilde{V}$ we therefore get

$$E(a) = \frac{i}{\tilde{V}} \log Z_B \tag{3.9}$$

and using eqs. (3.4) and (3.7) we obtain finally

$$E(a) = -\frac{\pi^2}{240}\frac{1}{3a^3} + 2i\int\frac{d^4k}{(2\pi)^4}\,\frac{\tilde{\Pi}(k^2)\,\Gamma}{k^2\sin a\Gamma}\left(e^{ia\Gamma} - \cos a k_3\right) \tag{3.10}$$

This result coincides with the result obtained in [5] based on the calculation of the energy-momentum tensor. It should be remarked that the present derivation is much more simple. On the other hand the original treatment [5] does not rely on arguments like (3.8).

4. The Casimir Pressure at $T \neq 0$

For the investigation of the Casimir effect at $T \neq 0$ we choose the Matsubara formalism. This means we have to start with a euclidean lagrangian and functional integrals satisfying appropriate boundary conditions. Without reviewing this further we shall note the necessary modifications only. As a general rule we have to take into account

$x_0 = -i\tau\,,\quad p_0 = ip_4$

$p_4 = \frac{2\pi n}{\beta}$ for boson lines,

$\int \frac{dp_4\, d^3p}{(2\pi)^4}\cdots \Rightarrow \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}$ in expressions given by Feynman diagrams and

$$\Gamma(p) = i\gamma = i\left(p_4^2+p_1^2+p_2^2\right)^{1/2}\,,\quad p^2 = \vec{p}^2 + p_4^2 .$$

Therefore the photon propagator for $T \neq 0$ QED with boundary conditions reads (compare (2.9))

$$\begin{aligned} {}^{S}D^{\beta}_{\mu\nu}(x,y) &= D^{\beta}_{\mu\nu}(x-y) + \bar{D}^{\beta}_{\mu\nu} \\ \bar{D}^{\beta}_{\mu\nu}(x,y) &= \frac{1}{2\beta}\sum_n \int\frac{dp_1\,dp_2}{(2\pi)^2}\, P_{\mu\nu}(p)\, e^{-i\sum_{i=4,1,2}(x-y)_i p_i} \\ &\quad\cdot \frac{1}{\gamma}\, e^{-|x_3-a_i|\gamma}\, (h^{-1})_{ij}\, e^{-\gamma|y_3-a_j|} \end{aligned} \tag{4.1}$$

with

$$P_{\mu\nu}(p) = \left\{ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{\gamma^2} \text{ for } \mu\neq 3, \nu\neq 3\,;\ 0 \text{ for } \mu=3 \text{ or } \nu=3 \right\} \tag{4.2}$$

The fermion propagator is the standard one. Here, however we need the polarization tensor only which can be taken from literature [6].

The relevant physical quantity is now the free energy per unit area F which is connected with log Z_B by the well-known relation

$$F = -\frac{1}{\beta V_{(2)}} \log Z_B$$

The zeroth order expression for log Z_B at T $\neq$ 0 can be obtained from eq. (3.4a) by the help of the necessary substitutions just mentioned. The reason is that the tensor structure of the kernel $\widetilde{K}^{-1}$ (eq. 3.2) is not destroyed by temperature effects. With $\widetilde{V} = -i\beta V_{(2)}$ we have

$$\left(\log Z_B\right)_{e^0} = -V_{(2)} \sum_n \int \frac{dp_1 dp_2}{(2\pi)^2} \log\left(1 - e^{-2\gamma a}\right) \qquad (4.3)$$

and consequently the free energy per unit area turns out to be

$$\left(F\right)_{e^0} = \frac{1}{\beta} \sum_n \int \frac{dp_1 dp_2}{(2\pi)^2} \log\left(1 - e^{-2\gamma a}\right) \qquad (4.4)$$

This leads immediately to the Casimir pressure

$$\left(P\right)_{e^0} = -\frac{d}{da}\left(F\right)_{e^0} = -\frac{1}{8\pi a^3 \beta} \sum_{n=-\infty}^{+\infty} \int_{\frac{4\pi n a}{\beta}}^{\infty} dx \frac{x^2}{e^x - 1} \qquad (4.5)$$

$$= -\frac{\zeta(3)}{4\pi a^3 \beta} - \frac{1}{4\pi a^3 \beta} \sum_{n=1}^{\infty} \int_{\frac{4\pi n a}{\beta}}^{\infty} \frac{x^2}{e^x - 1}, \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

This formula is especially convienent for the determination of p in the region $aT = a/\beta \gg 1$. One obtains

$$p = -\frac{\zeta(3)}{4\pi a^3 \beta}\left(1 + O\left(e^{-\frac{4\pi a}{\beta}}\right)\right). \qquad (4.6)$$

in accordance with [3] and [4].

More difficult is the calculation of second order corrections to log Z_B at T $\neq$ 0. We consider from the beginning the euclidian expression corresponding to eq. (3.5)

$$\left(\log Z_B\right)_{e^2} = -\frac{1}{2} \int dx_E\, dy_E\, \Pi_{\mu\nu}(x-y)\, \bar{D}_{\mu\nu}(x,y). \qquad (4.7)$$

Using the Fourier representation

$$\Pi_{\mu\nu}(x-y) = \frac{1}{\beta} \sum_{k_4} \int \frac{d^3k}{(2\pi)^3} \widetilde{\Pi}_{\mu\nu}(k)\, e^{-ik(x-y)}$$

eq. (4.7) can be evaluated as follows

$$\left(\log Z_B\right)_{e^2} = -\frac{1}{2} V_{(2)} \sum_{k_4} \int \frac{d^3k}{(2\pi)^3} \widetilde{\Pi}_{\mu\nu}(k) \int dz_4\, dz_1\, dz_2\, e^{-i(k_4 z_4 + k_1 z_1 + k_2 z_2)}\, \cdot$$

$$\cdot \int dx_3\, dy_3\, \bar{D}_{\mu\nu}(\tilde{z}; x_3, y_3)\, e^{-ik_3(x_3 - y_3)}. \qquad (4.8)$$

By the help of

$$\int dx_3\, dy_3\, e^{-\gamma|x_3 - a_e|} (h^{-1})_{ue}\, e^{-\gamma|y_3 - a_k|}\, e^{-ik_3(x_3 - y_3)} = $$

$$= \frac{4\gamma^2}{k^2} \frac{\operatorname{ch} a\gamma - \cos a k_3}{\operatorname{sh} a\gamma} \qquad (4.9)$$

we get

$$\left(\log Z_B\right)_{e^2} = -\sum_{k_4} \int \frac{d^3k}{(2\pi)^3} \widetilde{\Pi}_{\mu\nu}(k)\, P_{\mu\nu}(k) \frac{\gamma}{(k^2)^2 \operatorname{sh} a\gamma} \left(\operatorname{ch} a\gamma - \cos a k_3\right).$$

For $\Pi_{\mu\nu}$ we apply the structure given in the literature [6]

$$\tilde{\Pi}_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) A + \left(\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu + k_\nu u_\mu}{uk} + \frac{u_\mu u_\nu}{(uk)} k^2\right) B \qquad u = (1,0,0,0)$$

With eq. (4.2) $\tilde{\Pi}_{\mu\nu}(k)\, P_{\mu\nu}$ takes the form

$$\tilde{\Pi}_{\mu\nu} P_{\mu\nu} = 2A + \frac{k^2}{k_4^2}\, \frac{k_1^2 + k_2^2}{\gamma^2}\, B .$$

Introducing the function $\tilde{\Pi}_{44}$ connected with B by the formula

$$\frac{\vec{k}^2 B}{k_4^2} = \frac{k^2}{\vec{k}^2} \tilde{\Pi}_{44} - A$$

and turning to F itself we find

$$(F)_{e^2} = -\frac{1}{\beta} \sum_{k_4} \int \frac{d^3k}{(2\pi)^3} \left\{ \left(2 - \frac{k_1^2 + k_2^2}{\gamma^2} \frac{k^2}{\vec{k}^2}\right) A + \frac{k_1^2 + k_2^2}{\gamma^2} \frac{(k^2)^2}{(\vec{k}^2)^2} \tilde{\Pi}_{44} \right\} \cdot \frac{\gamma}{(k^2)^2 \operatorname{sh} a\gamma} \left(\operatorname{ch} a\gamma - \cos a k_3\right). \qquad (4.10)$$

Now we have to take into account the explicit structure of A and $\tilde{\Pi}_{44}$ [6]

$$A = -k^2 \Pi_0 + A^t , \quad \tilde{\Pi}_{44} = -\vec{k}^2 \Pi_0 + \Pi_{44}^t , \qquad (4.11)$$

$$A^t = \frac{e^2}{2\pi^2} \int_0^\infty \frac{\vec{k}^2}{\sqrt{\vec{k}^2 + m^2}} dk \cdot n_k^F \left(1 - \frac{p_4^2}{\vec{p}^2} - \frac{\vec{p}^4 - p_4^4 + 4(\vec{k}^2 + m^2)^2 p_4^2 + 4\vec{k}^2\vec{p}^2}{\;} \ln a^R \right.$$

$$\left. - \frac{i p_4 (\vec{k}^2 + m^2)^{1/2}}{2|\vec{k}||\vec{p}|^3} (\vec{p}^2 + p_4^2) \ln b^R \right), \qquad (4.12)$$

$$\Pi_{44}^t = +\frac{e^2}{\pi^2} \int_0^\infty \frac{dk}{(\vec{k}^2 + m^2)^{1/2}} \vec{k}^2 n_k^F \left(1 - \frac{\vec{p}^2 + p_4^2 - 4(\vec{k}^2 + m^2)}{8|\vec{k}||\vec{p}|} \ln a^R + i \frac{p_4 (\vec{k}^2 + m^2)^{1/2}}{2|\vec{p}||\vec{k}|} \ln b^R \right),$$

$$a^R = \frac{(\vec{p}^2 + p_4^2 + 2|\vec{p}||\vec{k}|)^2 + 4p_4^2(\vec{k}^2 + m^2)}{(\vec{p}^2 + p_4^2 - 2|\vec{p}||\vec{k}|)^2 + 4p_4^2(\vec{k}^2 + m^2)}, \quad b^R = \frac{(p^2)^2 - 4(|\vec{p}||\vec{k}| - i p_4 (\vec{k}^2 + m^2)^{1/2})^2}{(p^2)^2 - 4(|\vec{p}||\vec{k}| + i p_4 (\vec{k}^2 + m^2)^{1/2})^2},$$

$$n_k^F = 2\left(\exp \beta (\vec{k}^2 + m^2)^{1/2} + 1\right)^{-1} .$$

It is reasonable to treat the contribution of Π_0 to eq. (4.10) separately

$$(F)_{e^2, \Pi_0} = \frac{2}{\beta} \sum_{k_4} \int \frac{d^3k}{(2\pi)^3} \frac{\Pi_0(k^2)}{k^2} \frac{\gamma}{\operatorname{sh} a\gamma} \left(e^{-a\gamma} - \cos a k_3\right). \qquad (4.13)$$

Note that the expression (4.10) with Π_0 inserted would be uv divergent. In writing down (4.13) an uv - divergent but a- independent quantity has been subtracted according to

$$\frac{\operatorname{ch} a\gamma}{\operatorname{sh} a\gamma} = \frac{e^{-a\gamma}}{\operatorname{sh} a\gamma} + 1$$

so that the convergent expression (4.13) remains.

As in the T = 0 case a further explicit evaluation can be performed in suitable regions of the relevant parameters am and aT only. As in [5] let us assume $am \gg 1$ which is quite natural keeping in mind that the experimentally accessible distances are of the order of 10^{-4} cm and therefore $am \sim 10^7$. With respect to the other parameter aT we consider the region $aT \gg 1$ (note that $aT = O(1)$ for $a \sim 10^{-4}$ cm and $T \sim 10^3$ K). Taking into account $\gamma = \left(\left(\frac{2\pi n}{\beta}\right)^2 + k_1^2 + k_2^2\right)^{1/2}$ this allows us to restrict the summation in (4.13) to the n = 0 term because all other terms are of the order $e^{-a/\beta}$. The further evaluation of (4.13) is most conveniently done by performing the k_3-integration first and making use of the well-known analytic properties of Π_0

$$\int dk_3 \frac{\Pi_0(k_1^2 + k_2^2 + k_3^2)}{k_1^2 + k_2^2 + k_3^2} \left(e^{-a(k_1^2 + k_2^2)^{1/2}} - \cos a k_3\right) =$$

$$= -i\int_t^\infty dq \quad \frac{\Pi_0(-q^2+k_1^2+k_2^2+i\varepsilon) - \Pi_0(-q^2+k_1^2+k_2^2-i\varepsilon)}{q^2-k_1^2-k_2^2} \, . \tag{4.14}$$
$$\cdot\left(e^{-a(k_1^2+k_2^2)^{1/2}} - e^{-aq}\right).$$

Here $t = (4m^2+k_1^2+k_2^2)$ denotes the threshold of Π_0 in the q-plane. Obviously the leading contribution to (4.13) in the region $am \gg 1$ originates from $\gamma = (k_1^2+k_2^2)^{1/2} \approx 0$

Then (4.14) can be approximated as

$$-i\int_{2m}^{\infty} \frac{\Pi_0(-q^2+i\varepsilon) - \Pi_0(-q^2-i\varepsilon)}{q^2}\, dq \cdot e^{-a(k_1^2+k_2^2)^{1/2}} = \frac{3e^2}{8\cdot 16\, m}\, e^{-a(k_1^2+k_2^2)^{1/2}}$$

Insertion of this expression into (4.13) leads finally to

$$(F)_{e^2,T_0} = \frac{2}{\beta}\cdot\frac{3e^2}{8\cdot 16 m}\int \frac{dk_1 dk_2}{(2\pi)^3}\, \frac{(k_1^2+k_2^2)^{1/2}}{\sinh a(k_1^2+k_2^2)^{1/2}}\, e^{-a(k_1^2+k_2^2)^{1/2}}$$

$$= \frac{2\cdot 3}{8\cdot 16}\frac{e^2}{a^3\beta m}\frac{1}{(2\pi)^2}\int_0^\infty dx\, \frac{x^2 e^{-x}}{\sinh x} \tag{4.15}$$

$$= \frac{3e^2}{8\cdot 16\, a^3\beta m}\, \frac{\zeta(3)}{(2\pi)^2} \, .$$

Before turning to the Casimir preassure the contribution of the other parts A^t and Π_{44}^t of $\Pi_{\mu\nu}$ should be discussed shortly. Typical for both of them is the statistical factor $\left(e^{\beta(p^2+m^2)^{1/2}} + 1\right)^{-1}$ in the integral respresentations (4.12) which guarantees vanishing of A^t and Π_{44}^t in the uv region and therefore convergence of (4.10). Now in those parts where $k_4 \neq 0$ the same subtraction like that leading to (4.13) can be applied. Because of $\gamma > 0$ this gives contributions vanishing exponentially for $aT \gg 1$. The remaining

$k_4 = 0$ term reads explicitely

$$(F)_{e^2, A^t_1} = -\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\left(A^t + \Pi_{44}^t\right)\frac{\gamma_0}{(k^2)^2 \sinh a\gamma_0}\left(\cosh a\gamma_0 - \cos a k_3\right) \tag{4.16}$$

$$\gamma_0 = (k_1^2+k_2^2)^{1/2}, \quad k = (k_1^2+k_2^2+k_3^2)$$

By inspection of (4.16) it follows that this contribution is of order $e^{-\frac{m}{T}}$, i.e. negligible for all reasonable temperatures in connection with the Casimir experiment. Obviously the three conditions

$$aT \gg 1 \quad , \quad \frac{m}{T} \gg 1 \quad , \quad am \gg 1$$

are compatible among themselfes if T is restricted according to $\quad \frac{1}{a} \ll T \ll m$

Collecting now zeroth and second order contributions and inserting the necessary dimensional factors we finally obtain the Casimir pressure

$$p = -\frac{\zeta(3)}{4\pi}\left(1 - \frac{2}{8\cdot 16\pi}\,\frac{e^2}{\hbar c}\,\frac{\hbar}{amc}\right)\frac{kT}{a^3} \tag{4.17}$$

valid for

$$\frac{\hbar c}{a} \ll kT \ll mc^2 .$$

4. Discussion

The Casimir pressure of QED in a heat bath given in (4.17) has been derived from the free energy per unit area by the relation $p = -\frac{dF}{da}$ which is the generalization of $p = -\frac{dE}{da}$ for $T = 0$ (E energy perunit area) to isothermal conditions. In the alternative approach via the field theoretic pressure $\langle T_{33} \rangle$ it would be necessary to take the discontinuity of $\langle T_{33} \rangle$ across the plates to get the Casimir pressure.

From the methodical point of view it should be noted that the radiative corrections as in the zero temperature case are uv convergent and independent of the renormalization procedure of $\overline{T}_{\mu\nu}$, at least in second order of perturbation theory.

The result (4.17) applies to a version of high temperature limit, e.g. $T = O(10^3\ K)$ for $a = O(10^{-4}\ cm)$. Here the Matsubara formalism proves to be very convenient. In the opposite low temperature case the real time thermo-field formalism seems to be more appropriate reproducing the $T = 0$ result together with temperature corrections. Our method of deriving the photon propagator corresponding to boundary conditions can be easily generalized to dublett of thermo-fields [7].

A final comment concerns a question raised in [4]. Because of the absence of Planck's constant the zero order expression $p = -\frac{\zeta(3)}{4\pi} \frac{k T}{a^3}$ has been called there "a classical contribution of obscure origin". Also here the radiative corrections do not contain $\hbar$ in the considered limit!

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