PHE 85-15

PROCEEDINGS

OF THE

XIX INTERNATIONAL SYMPOSIUM AHRENSHOOP

from 18 to 22 November 1985

Special Topics in Gauge Field Theories

INSTITUT FÜR HOCHENERGIEPHYSIK
AKADEMIE DER WISSENSCHAFTEN DER DDR
BERLIN-ZEUTHEN · DDR

December 1985

Radiative Corrections to the Casimir Pressure under the Influence of Temperature and External Fields

D. Robaschik
Sektion Physik der Karl-Marx-Universität Leipzig, DDR

K. Scharnhorst
Sektion Physik der Humboldt-Universität Berlin, DDR

E. Wieczorek
Institut für Hochenergiephysik AdW, Berlin-Zeuthen, DDR

1. We start with a review on a quantum field theoretical formalism for QED with boundary conditions developped earlier [1]. The boundary conditions considered here are that of the classical Casimir effect: There are two parallel ideally conducting infinitely thin plates S_i $\{x_3=a_i; i=1,2\}$ on which the appropriate components of the em field have to vanish:

$$\begin{matrix} E^{\parallel}=0 \\ B^{\perp}=0 \end{matrix} \quad \text{or} \quad n_\mu \partial_\nu \varepsilon^{\mu\nu\alpha\beta} A_\alpha(x)=0 \qquad \text{for} \quad x\in S_i .$$

The electron field should not obey boundary conditions.
The quantization of the em field is performed by means of a modified functional integral where the boundary conditions are introduced by the help of functional δ functions. Accordingly the generating functional reads

$$Z(j,\bar\eta,\eta)=C\int \mathcal{D}A_\mu \mathcal{D}B\, \mathcal{D}\bar\psi \mathcal{D}\psi \exp i\Big\{\int d^4x\,(\mathcal{L}+j_\mu A^\mu+\bar\eta\psi+\bar\psi\eta) + \sum_{K=1}^{2}\int dS_K(x) B^{K\alpha}(x) H_{\alpha\mu}(x,\partial_x) A^\mu(x) - \frac{1}{2\lambda}\sum_{K=1}^{2}\int dS_K(x)\,dS_K(y)\, B^{K\alpha}(x)\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta} D^c(x_\alpha-y_\alpha,0) B^{K\beta}(y)\Big\} . \tag{1}$$

Here $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2+\bar\psi(i\partial\!\!\!/-m+eA\!\!\!/)\psi$

is the usual Lagrangian of QCD with a gauge fixing term. The three component vector $B^{K\alpha}{}_{(\alpha=0,1,2)}$ denotes an auxiliary field which is defined on the plate K $(K=1,2)$ i.e. at $x_3=a_K$ only. Furthermore

$$dS_K=d^4x\,\delta(x_3-a_K)\,;\quad K=1,2$$
$$H_{\alpha\mu}(x,\partial_x)=-n^\lambda\varepsilon_{\lambda\alpha\mu\gamma}\frac{\partial}{\partial x_\gamma} \qquad n^\lambda=(0,0,0,1)$$
$$(x_\alpha)=(x_0,x_1,x_2)\,.$$

In writing down expression (1) the standard Faddeev-Popov determinant of QED and two similiar determinants corresponding to the gauge freedom in B^K have been omitted as beeing independent of the distance of the plates $a=|a_1-a_2|$.
Formula (1) serves as the starting point for usual T=0 QED as well as for QED at T=0 with boundary conditions. If the interaction part is transformed into functional derivatives as usual all the functional integrals are of Gaussian type.

$$Z(j,\bar\eta,\eta)=C\exp i\int d^4x\,\mathcal{L}_{int}\Big(\frac{\delta}{i\delta j},\frac{\delta}{i\delta\eta},\frac{\delta}{i\delta\bar\eta}\Big)\cdot \exp -i\int d^4x\,d^4y\,\bar\eta S^c\eta\cdot\tilde Z(j) \tag{2}$$

$$\tilde Z=\int\mathcal{D}B\,\mathcal{D}A\,\exp i\Big\{\int d^4x\,\big(\tfrac12 A_\mu K^{\mu\nu}A_\nu+\xi_\mu A^\mu\big) - \frac{1}{2\lambda}\sum_K\int dS_K(x)\,dS_K(y)\,B^{K\alpha}(x)\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}D^c(x_\alpha-y_\alpha,0)B^{K\beta}(y)\Big\} \tag{3}$$

with
$$\xi_\mu=j_\mu+\sum_K\delta(x_3-a_K)B^{K\alpha}H_{\alpha\mu}(x,\partial_x)$$
$$K^{\mu\nu}=g^{\mu\nu}\Box-(1-\tfrac{1}{\alpha})\partial^\mu\partial^\nu\,.$$

Integration over A leads to

$$\tilde Z=\big[\mathrm{Det}\,K^{\mu\nu}\big]^{-1/2}\exp\big(-\tfrac{i}{2}\textstyle\int jK^{-1}j\big)\int\mathcal{D}B\,\exp i\Big\{\frac12\sum_i\sum_j\int dS_i(x)\,dS_j(y)\,B^{i\alpha}(x)K^{ij}_{\alpha\beta}(x,y)B^{j\beta}(y) - \sum_K\int d^4x\,d^4y\,\delta(x_3-a_K)B^{K\alpha}(x)H_{\alpha\mu}(x,\partial_x)(K^{-1})_{\mu\nu}(x-y)j^\nu(y)\Big\} \tag{4}$$

Here $(K^{-1})_{\mu\nu} = D^c_{\mu\nu}$ is the well known photon propagator in covariant gauge. For the kernel $K^{ij}_{\alpha\beta}$ we find the explicit form

$$K^{ij}_{\alpha\beta}(x-y) = -\frac{i}{2}\int \frac{d^3\tilde{p}}{(2\pi)^3}\, e^{-i\tilde{p}(\tilde{x}-\tilde{y})}\,\Gamma(p)\,\cdot$$

$$\cdot \left\{ \left(g_{\alpha\beta} - \frac{p_\alpha p_\beta}{\Gamma^2}\right) h_{ij} + \frac{1}{\alpha}\frac{p_\alpha p_\beta}{\Gamma^2}\delta_{ij} \right\}, \quad \alpha,\beta = (0,1,2) \qquad (5)$$

with $\tilde{p} = (p_0, p_1, p_2)$, $\tilde{x} = (x_0, x_1, x_2)$, $\Gamma(p) = (p_0^2 - p_1^2 - p_2^2)^{1/2}$
(positive imaginary part understood)

and $$h_{ij} = e^{i\Gamma|a_i - a_j|} .$$

The following integration over B gives the final result

$$\tilde{Z}(j) = \left[\mathrm{Det}\, K^{ij}_{\alpha\beta}\right]^{-1/2} \exp -\frac{i}{2}\left\{\int d^4x\, d^4y\, j^\mu(x)\, {}^S D^c_{\mu\nu}(x,y)\, j^\nu(y)\right\} \qquad (6)$$

with the full photon propagator

$${}^S D^c_{\mu\nu}(x,y) = D^c_{\mu\nu}(x-y) + \bar{D}_{\mu\nu}(x,y) \qquad (7)$$

$$\bar{D}_{\mu\nu} = \sum_{i,j}\int dS_i(z)\, dS_j(z')\, D^c_{\mu\rho}(x-z)\, H^{\rho\alpha}(z,\partial_z)\,(K^{-1})^{ij}_{\alpha\beta}(z,z')\,\cdot$$

$$\cdot\, H^{\beta\rho'}(z',\partial_{z'})\, D^c_{\rho'\nu}(z'-y)$$

$$= \frac{1}{2i}\int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\Gamma(p)}\, P_{\mu\nu}(p)\, e^{i\tilde{p}(\tilde{x}-\tilde{y})}\, e^{i\Gamma|x_3 - a_i|}\,(h^{-1})_{ij}\, e^{i\Gamma|y_3 - a_j|}$$

$$P_{\mu\nu}(p) = \left\{ g_{\mu\nu} - \frac{p_\mu p_\nu}{\Gamma^2} \text{ for } \mu,\nu \neq 3 \;;\; 0 \text{ for } \mu = 3 \text{ or } \nu = 3 \right\}$$

Equs. (2), (6) and (7) determine the QED with boundary conditiones in perturbation theory. We note once more that all functional and Faddeev -Popov determinants which are independent of a have been omitted.

2. Casimir pressure at non-zero temperature.

For the investigation of the Casimir effect at $T \neq 0$ we choose the Matsubara formalism. This means we have to start with an euclidean functional satisfying appropriate boundary conditions. Without reviewing this further we shall note the necessary modi-

fications only. As general rules we have to take into account

$$x_0 = -i\tau \;, \quad 0 \leq \tau \leq \beta = \frac{1}{T} \qquad p_0 = ip_4$$

$p_4 = \frac{2\pi n}{\beta}$ for boson momenta

$$\int\frac{dp_4\, d^3p}{(2\pi)^4} \Longrightarrow \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}$$

$$\Gamma(p) = i\gamma \;, \qquad \gamma = (p_4^2 + p_1^2 + p_2^2)^{1/2} .$$

The (real) photon propagator for non-zero temperature QED with boundary conditions is given by (compare (7))

$${}^S D^\beta_{\mu\nu}(x,y) = D^\beta_{\mu\nu}(x-y) + \bar{D}^\beta_{\mu\nu}$$

$$D^\beta_{\mu\nu}(x-y) = \frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\{\delta_{\mu\nu} - \dots\}\frac{e^{-ip(x-y)}}{p^2} \;, \quad p^2 = \sum_1^4 p_i^2$$

$$\bar{D}^\beta_{\mu\nu}(x,y) = -\frac{1}{2\beta}\sum_n\int\frac{dp_1\, dp_2}{(2\pi)^2}\, P_{\mu\nu}(p)\, e^{-i\sum_{1,2,4} p_i(x_i - y_i)} .$$

$$\cdot\,\frac{1}{\gamma}\, e^{-|x_3 - a_i|\gamma}\,(h^{-1})_{ij}\, e^{-|y_3 - a_j|\gamma} \qquad (8)$$

with

$$P_{\mu\nu}(p) = \left\{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{\gamma^2} \text{ for } \mu \neq 3, \nu \neq 3 ;\; 0 \text{ for } \mu = 3 \text{ or } \nu = 3\right\}.$$

The relevant physical quantity is the free energy per unit surface which is connected with the Z functional by $F = -\frac{1}{\beta V_2}\log Z$.
From equs. (2), (6) we obtain up to order e^2 and neglecting a- independent terms

$$\log Z = -\frac{1}{2}\log \mathrm{Det}\, K^{ij}_{\alpha\beta} - \frac{1}{2}\int dx_E\, dy_E\, \bar{D}^\beta_{\mu\nu}(x,y)\, \Pi^\beta_{\mu\nu}(x-y) . \qquad (9)$$

$\Pi^\beta_{\mu\nu}$ is the polarization tensor of QED at $T \neq 0$.
The lowest order contribution from $\log \mathrm{Det} K^{ij}_{\alpha\beta}$ can be evaluated easily taking into account that $K^{ij}_{\alpha\beta}$ depends on the difference $\tilde{x} - \tilde{y}$ only which allows to apply the usual Fourier transform technique

$$\log \mathrm{Det}\, K^{ij}_{\alpha\beta} = \mathrm{Tr}\log K^{ij}_{\alpha\beta}$$

$$= V_2 \sum_n \int\frac{dp_1\, dp_2}{(2\pi)^2}\log \det_{(ij)(\alpha\beta)} \tilde{K}^{ij}_{\alpha\beta}(p) .$$

In spite of the fact that $\tilde{K}$ depends on the gauge fixing parameter λ there is no a-dependent contribution to $\log \mathrm{Det}\, K$ from λ. This follows from a variation with respect to $1/\lambda$:

$$\delta \log \det \tilde{K} \approx \log \det (1+\tilde{K}^{-1}\delta K) = \log \det\left[\delta_{\alpha\beta}\delta_{ij} + \lambda \frac{p_\alpha p_\beta}{\gamma^2}\delta_{ij}\,\delta(1/\lambda)\right].$$

So one may choose the most convenient value of λ, e.g. $\lambda=1$ and obtains, up to an a-independent term (for details see [2])

$$\log \mathrm{Det}\, K^{ij}_{\alpha\beta}\Big/_{\lambda=1} = V_2 \sum_n \int \frac{dp_1 dp_2}{(2\pi)^2} \underset{\alpha\beta}{\mathrm{tr}}\, \underset{ij}{\mathrm{tr}} \log\Big[\left[\delta_{\alpha\beta}\delta_{ij} - \left(\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{\gamma^2}\right)(\delta_{ij} - h_{ij})\right]\Big]$$

$$= 2 V_2 \sum_n \int \frac{dp_1 dp_2}{(2\pi)^2} \log(1-e^{-2\gamma a})$$

The evaluation of this integral is straightforward. By the help of the useful formula [4]

$$\sum_{k,m=1}^{\infty} \frac{\xi^4}{[k^2+m^2\xi^2]^2} = -\frac{\pi^4}{180}\xi^4 + \frac{\pi}{4}\xi^2 \sum_{k=1}^{\infty}\Big[\xi \frac{1}{k^3}\frac{\cosh k\pi\xi^{-1}}{\sinh k\pi\xi^{-1}} + \frac{\pi}{k^2}\frac{1}{\sinh^2 k\pi\xi^{-1}}\Big]$$

$$= -\frac{\pi^4}{180} + \frac{\pi}{4}\xi \sum_{k=1}^{\infty}\Big[\frac{1}{k^3}\frac{\cosh k\pi\xi}{\sinh k\pi\xi} + \xi\frac{\pi}{k^2}\frac{1}{\sinh^2 k\pi\xi}\Big] \qquad (10a)$$

we write the o. order contribution to the free energy

$${}^0F(a,\beta) = \frac{\pi^2}{45}\frac{a}{\beta^4} - \frac{\pi^2}{720}\frac{1}{a^3} - \frac{1}{2\pi\beta^3}\sum_{k=1}^{\infty}\Big[\frac{1}{k^3}\frac{\cosh \frac{k\pi\beta}{2a}}{\sinh \frac{k\pi\beta}{2a}} + \frac{\left(\frac{\pi\beta}{2a}\right)}{k^2}\frac{1}{\sinh \frac{k\pi\beta}{2a}}\Big] \qquad (10b)$$

(compare [4] and [6]).
The 2.order contribution has been evaluated [2] using

$$\Pi^{\beta}_{\mu\nu}(x-y) = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\tilde{\Pi}^{\beta}_{\mu\nu}(k)\, e^{-ik(x-y)}$$

and

$$\int dx_3\, dy_3\, e^{-\gamma|x_3-a_l|}(h^{-1})_{kl}\, e^{-\gamma|y_3-a_k|}\, e^{-ik_3(x_3-y_3)} \qquad (10c)$$

$$= \frac{4\gamma^2_{(p)}}{p^2 \sinh a\gamma}(\cosh a\gamma - \cos a k_3)$$

so that ${}^2F(a,\beta)$ takes the form

$${}^2F(a,\beta) = -\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\tilde{\Pi}^{\beta}_{\mu\nu}(k)\, P_{\mu\nu}(k)\, \frac{\gamma}{k^4 \sinh a\gamma}(\cosh a\gamma - \cos a k_3),$$

$$k = \{k_4^2+k_1^2+k_2^2+k_3^2\}^{1/2}, \quad \gamma = \{k_4^2+k_1^2+k_2^2\}^{1/2} \qquad (11)$$

For $\tilde{\Pi}^{\beta}_{\mu\nu}$ we refer to results from the literature [5]

$$\tilde{\Pi}^{\beta}_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)A + \left(\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu + k_\nu u_\mu}{uk} + \frac{u_\mu u_\nu}{(uk)^2}k^2\right)B$$

$$A = -k^2\Pi_0 + A^\beta, \quad \frac{\vec{k}^2}{k_4^2}B = \frac{k^2}{\vec{k}^2}\Pi_{44} - A,$$

$$\Pi_{44} = -\vec{k}^2\Pi_0 + \Pi^{\beta}_{44}, \quad u = (1,0,0,0)$$

Here Π_0 is a contribution with the same stucture as in T=0 QED

$$\Pi_0(k^2) = \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z^2)\log\left[1+(1-z^2)\frac{k^2}{4m^2}\right]. \qquad (12)$$

The detailed stucture of A^β and Π^{β}_{44} is not necessary for the following. It is sufficient to know that A^β and Π^{β}_{44} both vanish for $k\to\infty$ and behave (due to the statistical weight of the electron which appears in the integral representation) as $\exp(-m/T)$ for $m/T \gg 1$. Therefore ${}^2F$ can be expressed as

$${}^2F(a,\beta) = \frac{2}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\frac{\Pi_0(k^2)}{k^2}\frac{\gamma}{\sinh a\gamma}(e^{-a\gamma} - \cos a k_3)$$

$$-\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\left\{\left(2-\frac{k_1^2+k_2^2}{\gamma^2}\cdot\frac{k^2}{\vec{k}^2}\right)A^\beta + \frac{k_1^2+k_2^2}{\gamma^2}\frac{k^4}{(\vec{k}^2)^2}\Pi^{\beta}_{44}\right\}\cdot$$

$$\cdot\ \frac{\gamma}{k^4 \sinh a\gamma}(\cosh a\gamma - \cos a k_3) \qquad (13)$$

In the first term an a-independent quantity has been subtracted according to $\frac{\cosh a\gamma}{\sinh a\gamma} = \frac{e^{-a\gamma}}{\sinh a\gamma} - 1$ so that this term is uv convergent. The second term is convergent because of the asymptotic properties of A^β and Π^{β}_{44}. Furthermore we take into account

$$m/T = m\beta \gg 1$$

for all temperatures which are physically meaningful in connection with the Casimir effect. Then the second term in (13) is exponenentially suppressed and will be neglected. Inserting (12) into (13) and doing the k_3 integration we arrive at

$$
{}^2F(a,\beta) = \frac{e^2}{\beta(2\pi^3)} \sum_{n=-\infty}^{+\infty} \int_{\frac{2\pi}{\beta}|n|}^{\infty} ds\, s^2 \int_0^1 dz\,(1-z^2) \cdot
$$

$$
\cdot \int_{2m/\sqrt{1-z^2}}^{\infty} dt\, \frac{1}{t\sqrt{t^2+s^2}}\, \frac{e^{-as} - e^{-a\sqrt{t^2+s^2}}}{\sinh as}
$$

Now one should remember that in any realistic case the plate distance a is much larger than the Compton wave length of the electron. Therefore using

$$
am \gg 1
$$

we have (up to terms of the order e^{-am})

$$
{}^2F(a,\beta) = \frac{1}{(2\pi)^3} \frac{e^2}{\beta a^2} \sum_{-\infty}^{\infty} \int_{\frac{2\pi a}{\beta}|n|} d\sigma\, \sigma^2 \frac{e^{-\sigma}}{\sinh \sigma} \int_0^1 dz\,(1-z^2) \int_{\frac{2ma}{\sqrt{1-z^2}}}^{\infty} d\tau\, \tau^{-2}
$$

$$
{}^2F(a,\beta) = \frac{1}{(2\pi)^2} \frac{3}{128} \frac{e^2}{\beta m a^3} \sum_{k=1}^{\infty} \left[\frac{1}{k^3} \frac{\cosh(2\pi a k/\beta)}{\sinh(2\pi a k/\beta)} + \right.
$$

$$
\left. + \left(\frac{2\pi a}{\beta}\right) \frac{1}{k^2 \sinh^2(2\pi a k/\beta)} + \left(\frac{2\pi a}{\beta}\right)^2 \frac{\cosh(2\pi a k/\beta)}{k \sinh^3(2\pi a k/\beta)} \right] \qquad (14)
$$

(for details compare [3]). By means of a further transformation formula of the type (10a) we write

$$
{}^2F(a,\beta) = \frac{\pi^2}{720} \frac{1}{a^3} \left(\frac{9}{128\pi} \frac{e^2}{ma} \right) + \frac{\pi^2}{45} \frac{a}{\beta^4} \left(\frac{3}{128\pi} \frac{e^2}{ma} \right)
$$

$$
- \frac{\pi}{4} \frac{1}{\beta a^2} \left(\frac{3}{128\pi} \frac{e^2}{ma} \right) \sum_{k=1}^{\infty} \frac{1}{k} \frac{\cosh(k\pi\beta/2a)}{\sinh^3(k\pi\beta/2a)} \qquad (15)
$$

The fully equivalent representations (14) and (15) have been derived assuming $1/a \ll m$ and $1/\beta \ll m$. They are convenient for studying low and high temperature limits, respectively. Collecting the results (10b), (14) and (15) we find the free energy per unit surface and the Casimir pressure $p = -\frac{dF}{da}$ (up to order e^2)

$$
F(a,\beta) = -\frac{\pi^2}{720} \frac{1}{a^3} \left[1 - \frac{9}{128\pi} \frac{e^2}{ma} \right] + \frac{\pi^2}{45} \frac{a}{\beta^4} \,,
$$

$$
p(a,\beta) = -\frac{\pi^2}{240} \frac{1}{a^4} \left[1 + \frac{16}{3} \frac{a^4}{\beta^4} - \frac{3}{32\pi} \frac{e^2}{ma} \right] \qquad (16)
$$

if $\quad 1/\beta \ll 1/a \ll m$

and

$$
F(a,\beta) = -\frac{\zeta(3)}{8\pi} \frac{1}{\beta a^2} \left[1 - \frac{3}{64\pi} \frac{e^2}{ma} \right] \,,
$$

$$
p(a,\beta) = -\frac{\zeta(3)}{4\pi} \frac{1}{\beta a^3} \left[1 - \frac{9}{128\pi} \frac{e^2}{ma} \right] \qquad (17)
$$

if $\quad 1/a \ll 1/\beta \ll m$.

The latter condition selects temperatures which are high compared with $1/a$ but nevertheless small compared with the electron mass. It is fulfilled eg. for $a \approx 10^{-4}\,\text{cm}$ and $T \sim 10^3\,{}^\circ K$. Formula (17) shows an interesting aspect if we introduce the necessary dimensional constants c, $\hbar$ and k (Boltzmann constant):

$$
p(a,T) = -\frac{\zeta(3)}{4\pi} \left(1 - \frac{9}{128\pi} \frac{e^2}{\hbar c} \cdot \frac{\hbar}{mca} \right) \frac{kT}{a^3}
$$

The Planck constant $\hbar$ drops out in spite of the fact that expressions (16) and (17) without any doubt are of quantum theoretical origin.

The same results have been obtained [3] by combining our method of quantization (1) with a recently proposed real time thermo-field formalism [7].

3. Constant external fields.

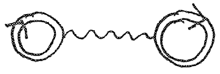
The contribution of external fields to the Casimir pressure will be estimated at zero temperature. The most natural starting point would be the vacuum expectation value of the energy-momentum tensor $\langle 0|T_{00}|0\rangle$. Following this line one would run into considerable difficulties because of the complicated structure of the electron propagator in constant external fields [8]. Fortunately our earlier experience in evaluating the Casimir pressure at zero temperature without fields [1,2,3] tells us that there are two absolutely equivalent procedures: either via the energy-momentum tensor or via the Z functional. By direct evaluation of $\log \mathcal{D}et\, K^{ij}_{\alpha\beta}$ and expression (9) at T=0 and comparison with the results of [1] we come to the conclusion

$$E_{vac}(a) = \int d^3x \langle 0|T_{00}|0\rangle = i \lim_{\Delta x_0 \to \infty} \frac{1}{\Delta x_0} \log Z . \qquad (18)$$

This equation is valid up to contributions which are independent of a . It is just the evaluation via Z which is the convenient one in case of external fields.

Switching on the external field is without any effect on our quantization of the radiation field in presence of conducting plates. The electron propagator in (2) , however, has to be replaced by the propagator in the background field. In case of constant electric or magnetic fields to which consideration is restricted the electron propagator S^{ext} and the polarization tensor $\Pi^{ext}_{\mu\nu}$ are explicitely given in literature [8].

The field dependent contibutions to the Casimir pressure start with order e^2. As usual the tadpole term to $\log Z$



vanishes because of $tr(\gamma^\mu S^{ext}_{(x,x)})=0$. Therefore the 2. order contribution depending on the distance and on the external field reads

$$\log Z = \frac{1}{2}\int d^4x\, d^4y\, \Pi^{ext}_{\mu\nu}(x-y)\, \overline{D_{\mu\nu}(x,y)} \qquad (19)$$

or, using a formula analogous to (10c)

$$\log Z = -\frac{1}{(2\pi)^4}\int d^3\tilde{x} \int d^4k\, \tilde{\Pi}^{ext}_{\mu\nu}(k)\, P_{\mu\nu}(k) .$$

$$\frac{\Gamma(k)}{[\Gamma^2(k)-k_3^2]^2}\, \frac{\cos a\Gamma(k) - \cos a k_3}{\sin a\Gamma} \qquad (20)$$

Further evaluation makes use of the explicit expression for the renormalized polarization tensor $\tilde{\Pi}^{ext}_{\mu\nu}$ corresponding to a constant magnetic field [8]

$$\tilde{\Pi}^{ext}_{\mu\nu} = \frac{e^2}{16\pi^2}\int_0^\infty \frac{ds}{s}\int_{-1}^{+1} d\nu \Big\{ e^{-i\varphi s}\big[(g_{\mu\nu}k^2 - k_\mu k_\nu)N_0 \qquad (21)$$
$$- (g^{\parallel}_{\mu\nu}k_\parallel^2 - k^\parallel_\mu k^\parallel_\nu)N_1 + (g^{\perp}_{\mu\nu}k^{\perp 2} - k^\perp_\mu k^\perp_\nu)N_2\big]$$
$$- e^{-ism^2}(1-\nu^2)(g_{\mu\nu}k^2 - k_\mu k_\nu)\Big\}$$

with

$$N_0 = \frac{z}{\sin z}(\cos z\nu - \nu \cot z \cdot \sin z\nu) \qquad (22)$$
$$N_1 = N_0 - z\cot z(1-\nu^2)$$
$$N_2 = -N_0 + \frac{2z}{\sin^3 z}(\cos z\nu - \cos z)$$
$$\varphi = m^2 - \frac{1}{4}(1-\nu^2)k_\parallel^2 - \frac{\cos z\nu - \cos z}{2z\sin z}k_\perp^2$$
$$z = eBs .$$

If for definiteness B is choosen parallel to X_1 axis one has $k^\parallel = (k_0, k_1, 0, 0)$, $k^\perp = (0,0,k_2,k_3)$ and corresponding to our metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$: $g^\parallel_{\mu\nu} = \mathrm{diag}\,(1,-1,0,0)$, $g^\perp_{\mu\nu} = (0,0,-1,-1)$.

In spite of this rather complicated structure further progress can be achieved in special cases.

After introduction of the dimensionless variables $q_\mu = a k_\mu$, $z = eBs$

the first exponential in (21) becomes

$$\exp\{-i\xi s\} = \exp -i\left\{\frac{m^2}{eB} - \frac{1}{eBa^2}\left(\frac{1-\nu^2}{4}q_{\parallel}^2 - \frac{\cos z\nu - \cos z}{2z\sin z}q_{\perp}^2\right)\right\}z . \quad (23)$$

Now it is crucial to take into account the two conditions which are fulfilled in any realistic experiment

$$a \gg 1/m \quad , \quad B \ll \frac{m^2}{e} \approx 10^{18}\,[A/m] . \quad (24)$$

The inequalities (24) leave room to choose B according to

$$1/a^2 \ll eB \ll m^2 \quad (25)$$

Let us therefore restrict consideration to the limit of the largest available fields (which nevertheless remain small in comparison with the critical Schwinger field (24)). This leads to an important simplification of (23):

$$\exp(-i\xi s) \Longrightarrow \exp\left(-\frac{im^2}{eB}z\right)$$

We write the resulting expression for $\log Z$ in a form where two of the integrations have been Wick-rotated $q_0 \to iq_4, z \to -iz$:

$$\log Z = \frac{-ie^2}{4(2\pi)^6 a^3}\int d^3\tilde{x}\int d^4q_E \frac{\gamma(q)}{q_E^4}\,\frac{e^{-\gamma(q)} - e^{iq_3}}{\sinh\gamma(q)} . \quad (26)$$

$$\cdot\int_0^\infty \frac{dz}{z}\int_{-1}^{+1} d\nu\, e^{-\frac{m^2}{eB}z}\Big[-2q_E^2 N_0(-iz) + (q_4^2+q_1^2)N_1(-iz)$$

$$- \frac{q_3^2(q_4^2+q_1^2)}{\gamma^2}N_2(-iz) + 2q_E^2(1-\nu^2)\Big]$$

$q_E^2 = q_4^2+q_1^2+q_2^2+q_3^2$, $\gamma^2 = q_4^2+q_1^2+q_2^2$.

Note that in transforming (20) into (26) again an a-independent quantity has been subtracted. Performing the q_3 integrals like

$$\int_{-\infty}^{+\infty} dq_3 \frac{e^{-\gamma} - e^{iq_3}}{q_3^2+\gamma^2} = 0, \quad \int_{-\infty}^{+\infty} dq_3 \frac{e^{-\gamma} - e^{iq_3}}{(q_3^2+\gamma^2)^2} = -\frac{\pi}{2}\frac{e^{-\gamma}}{\gamma^2} \quad etc.$$

and doing the elementary ν integration we obtain

$$\log Z = \frac{ie^2}{4(2\pi)^6}\frac{\pi}{2a^3}\int d^3\tilde{x}\int dq_4\,dq_1\,dq_2\,\frac{e^{-\gamma}(q_4^2+q_1^2)}{\gamma\sinh\gamma} .$$

$$\cdot\int_0^\infty \frac{dz}{z} e^{-\frac{m^2}{eB}z}\left\{-\frac{4}{3}z\coth z - \frac{4}{\sinh^3 z} + \frac{4z\coth z}{\sinh^2 z}\right\} . \quad (27)$$

Obviously the region $z \approx 0$ gives the dominant contrbution. This allows to approximate the expression in curly brackets by $(-44/45)z^2$ so that

$$\log Z = -\frac{i}{4}\frac{e^2}{(2\pi)^6}\cdot\frac{\pi}{2}\cdot\frac{44}{45}\cdot\frac{\pi^5}{45}\,\frac{1}{a^3}\left(\frac{eB}{m^2}\right)^2\int d^3\tilde{x} .$$

The vacuum energy per unit surface follows from (18)

$$E_{vac}(a) = \frac{11e^2}{2^7\cdot(45)^2}\left(\frac{eB}{m^2}\right)^2\frac{1}{a^3} \quad (28)$$

$$1/a^2 \ll eB \ll m^2 .$$

Let us briefly note further results.

For a magnetic field orthogonal to the plates, which, for real conducting plates and a constant field at least is a physically admissible configuration one obtains the same result up to a minus sign. An electric field field orthogonal to the plates can be dealt with by an appropriate transformation $B \to -iE$ [8]. The result is the following

$$E_{vac}(a) = \frac{11e^2}{2^7(45)^2}\left(\frac{eE}{m^2}\right)^2\frac{1}{a^3} - i\frac{\pi e^2}{4\cdot 2880}\,\frac{1}{a^3}\left(\frac{m^2}{eE}\right)^2 e^{-\frac{\pi m^2}{eE}} . \quad (29)$$

Also the distance dependent part of the vacuum energy has an imaginary part which signals pair creation processes!

It is interesting to compare our result (29) with the free space pair production rate in constant electric fields which is expressed in terms of the imaginary part of the effective

La grangian[9]

$$\mathfrak{Im}\,\mathcal{L}_{eff} = \frac{(eE)^2}{8\pi}\, e^{-\frac{\pi m^2}{eE}} \qquad (30)$$

Now one has to take into account that we have obtained a vacuum energy per unit surface or a quantity integrated over x_3 whereas (30) is an energy per unit volume. Let us therefore assume that the plate induced vacuum energy density is localized in a x_3 region of extension Na what is to be expected from physical reasons. The so estimated energy density $\rho(a) = E_{vac}/Na$ can now be compared with $\mathfrak{Im}\,\mathcal{L}_{eff}$:

$$\left| \frac{\mathfrak{Im}\,\rho(a)}{\mathfrak{Im}\,\mathcal{L}_{eff}} \right| \approx \frac{\pi^2}{1440}\, \frac{1}{(a^2 E)^2} \left(\frac{m^2}{eE}\right)^2 . \qquad (31)$$

It turns out that there are configurations (a,E) in accordance with our condition $1/a^2 \ll eE \ll m^2$ yielding an enhancement of the ratio (31) by many orders of magnitude. We conclude that neutral conducting plates can be of rather strong influence in the presence of electric fields. Nevertheless the pair creation rate remains far too small to be observable in conventional experimental set ups.

References

1 M.Bordag,D.Robaschik,E.Wieczorek Ann.Phys.(NY) 165(1985),192

2 D.Robaschik,K.Scharnhorst,E.Wieczorek Preprint PHE 85-9 Berlin-Zeuthen 1985

3 K.Scharnhorst,D.Robaschik,E.Wieczorek Preprint PHE 85-13 Berlin-Zeuthen 1985

4 L.S.Brown, G.I.Maclay Phys.Rev. 184(1969) , 1272

5 O.K.Kalashnikov Fortschr.Phys. 32(1984),525

6 J.Mehra Physica 37(1967),145

7 A.J.Niemi, G.W.Semenoff Ann.Phys.(NY) 152(1984),105

8 W.Dittrich, M.Reuter Effective Lagrangians in Quantum

Electrodynamics (Lecture Notes in Physics 220) , Springer, Berlin-Heidelberg- New York 1985

W.Y.Tsai, Phys.Rev.D10(1974),2699

9 J.Schwinger, Phys.Rev.82(1952),664