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PROPAGATORS OF NON-ABELIAN GAUGE THEORY IN BACKGROUND FIELDS

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Abstract

Gauge and ghost field propagators in a homogeneous background are investigated. The gauge propagator is determined for arbitrary values of the gauge parameter.

1. Introduction

Non-Abelian gauge theories in classical backgrounds offer a lot of interesting theoretical problems. Whereas the effective action to 1-loop order has been studied over a rather long period /1/ the apparatus of gauge field theory in backgrounds (propagators, radiative corrections, renormalization terms etc.) has not been developed to the same extent as for QED in backgrounds /2/. Motivated by the intriguing problem of the physical interpretation of the imaginary part of the 1-loop effective Lagrangian we have evaluated /3/ the propagators of Euclidean SU(2) gauge theory in a homogeneous background (in appropriate background gauge, of course). This opens the possibility to calculate the radiative corrections of background QFT and especially the 2-loop contributions to L_{eff}. For this purpose a precise knowledge of the short-distance and the large distance behaviour of the propagator is necessary, both being not as obvious as in the zero field case.

In Section 2 a construction of the gauge propagator for arbitrary values of the gauge fixing parameter is presented without specifying the gauge group or the background field.

In Section 3 the short- and large-distance behaviour of the special propagators obtained earlier /3/ are determined. This justifies the construction of the general propagators of Section 2 and will allow to elucidate the structure of the renormalization terms of the theory.

2. General construction of the gauge field propagator in covariant gauges

The Lagrangian for a gauge theory in the presence of a classical background field B_μ is obtained from the usual Lagrangian by splitting the gauge field into a quantum field a_μ and the background field B_μ together with choosing the gauge fixing term in covariant form, e.g.

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}\left(D_\mu(B)\, a_\mu\right)^2 . \tag{2.1}$$

If B_μ is a solution of the classical field equations then the Lagrangian starts with the quadratic term

$$\mathcal{L} = \frac{1}{2}\, a_\mu^a K_{\mu\nu}^{ab} a_\nu^b + \frac{1}{2}\, \bar{c}^a K^{ab} c^b \tag{2.2}$$

where
$$\begin{aligned} \underset{x}{K}{}_{\mu\nu}^{ab} &= \left(\underset{x}{D}^2 \delta_{\mu\nu} + \left(\tfrac{1}{\alpha}-1\right) \underset{x}{D}_\mu \underset{x}{D}_\nu\right)^{ab} + 2g f^{abc} F_{\mu\nu}^c(x) \\ \underset{x}{K}{}^{ab} &= (\underset{x}{D}^2)^{ab} \\ \underset{x}{D}{}_\mu^{ab}(B) &= \delta^{ab} \frac{\partial}{\partial x_\mu} + g f^{abc} B_\mu^c(x) . \end{aligned} \tag{2.3}$$

Here and in the following Euclidean QFT is always understood. Gauge and ghost field propagators are defined by

$$\underset{x}{K}{}_{\lambda\mu}^{ab}(\alpha)\, G_{\mu\nu}^{bc}(x,y,\alpha) = -\delta^{ac}\delta_{\lambda\nu}\,\delta(x-y), \qquad \underset{x}{K}{}^{ab} G^{bc}(x,y) = -\delta^{ac}\,\delta(x-y). \tag{2.4}$$

Without specifying the gauge group or the background field we derive a relation which allows to express the general gauge propagator (i.e. for arbitrary values of α) in terms of the special propagator (i.e. for $\alpha=1$).

From the identity (valid if $D_\mu^{ab} F_{\mu\nu}^b = 0$)

$$D_\mu^{ab}\left\{ \delta_{\mu\nu}(D^2)^{bc} + \left(\tfrac{1}{\alpha}-1\right)(D_\mu D_\nu)^{bc} + 2g f^{bcd} F_{\mu\nu}^d \right\} = \frac{1}{\alpha}\,(D^2 D_\nu)^{ac} \tag{2.5}$$

and from (2.4) we obtain

$$\underset{x}{D}{}_\lambda^{ab} \underset{x}{K}{}_{\lambda\mu}^{bc}(\alpha)\, G_{\mu\nu}^{cd}(x,y,\alpha) = -\underset{y}{D}{}_\lambda^{ab}\,\delta^{bd}\delta_{\lambda\nu}\,\delta(x-y) = \frac{1}{\alpha}\,(\underset{x}{D}^2 D_\mu)^{ac}\, G_{\mu\nu}^{cd}(x,y,\alpha). \tag{2.6}$$

Provided that D^2 is without zero modes the relation

$$(D^2 D_\mu)^{ac}\, G_{\mu\nu}^{cd}(x,y,\alpha) = -\alpha\, \underset{x}{D}{}_\nu^{ad}\,\delta(x-y) \tag{2.7}$$

is equivalent to

$$\underset{x}{D}{}_\mu^{ab}\, G_{\mu\nu}^{bc}(x,y,\alpha)\, \underset{y}{D}{}_\nu^{cd} = -\alpha\,\delta^{ad}\,\delta(x-y) \tag{2.8}$$

which, however, must be understood as a functional equality with respect to y. Indeed, integration of (2.7) with the covariant derivative of a test function $f^a(y)$ gives

$$\begin{aligned} 0 &= \int dy \left\{ (\underset{x}{D}^2 D_\mu)^{ac}\, G_{\mu\nu}^{cd}(x,y,\alpha) + \alpha\, \underset{x}{D}{}_\nu^{ad}\,\delta(x-y) \right\} \underset{y}{D}{}_\nu^{de} f^e(y) \\ &= \int dy \left\{ (\underset{x}{D}^2 D_\mu)^{ac}\, G_{\mu\nu}^{cd}(x,y,\alpha)\, \underset{y}{D}{}_\nu^{de} f^e(y) + \alpha (\underset{x}{D}^2)^{ae}\,\delta(x-y)\, f^e(y) \right. \\ &= (\underset{x}{D}^2)^{ab} \int dy \left\{ \underset{x}{D}{}_\mu^{bc} G_{\mu\nu}^{cd}(x,y,\alpha)\, \underset{y}{D}{}_\nu^{de} + \alpha\,\delta^{be}\,\delta(x-y) \right\} f^e(y) \end{aligned} \tag{2.9}$$

which is equivalent to (2.8).

Along the same lines one verifies

$$\underset{y}{D}{}_\mu^{ab}\, G_{\mu\nu}^{bc}(x,y,\alpha) = -\alpha\, G^{ab}(x,y,\alpha)\, \underset{y}{D}{}_\nu^{bc} \tag{2.10}$$

(again as a functional with respect to y).

With the help of (2.8) it is easy to prove that

$$G_{\mu\nu}^{ab}(x,y,\alpha) = G_{\mu\nu}^{ab}(x,y,1) + (1-\alpha)\int dz\, G_{\mu\kappa}^{ac}(x,z,1) \left(\underset{z}{D}_\kappa D_\lambda\right)^{cd} G_{\lambda\nu}^{db}(z,y,1) \tag{2.11}$$

is a solution to (2.4). In order to apply (2.11) one has to assume, however, that $\underset{z}{D}_\mu G_{\mu\nu}(z,y)$ vanishes sufficiently fast for $z \to \infty$ as to allow partial integration.

Eq. (2.11) reduces the problem of finding the gauge propagator to the easier case $\alpha=1$. Here a warning should be kept in mind: Without knowledge of the long-distance behaviour of the special propagator $G_{\mu\nu}(x,y,1)$ the expression (2.11) will remain a formal one. At this point the further analysis depends on the specific example of background field. Only an explicit construction of $G_{\mu\nu}(x,y,1)$ will decide whether the integral in (2.11) is well-defined or not.

Note that for vanishing background field ($B_\mu=0$) the expression (2.11) reduces to the well-known form for the gauge propagator in a generalized Landau gauge.

3. Propagators for homogeneous background fields

Now we restrict ourselves to the gauge group SU(2) and the special background field

$$B_\mu^a(x) = -\tfrac{1}{2} F_{\mu\nu}^a x_\nu \qquad \varepsilon_{\mu\nu}^\perp = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{x}_\mu = \varepsilon_{\mu\nu}^\perp x_\nu$$
$$F_{\mu\nu}^c = B\,\delta^{c3}\varepsilon_{\mu\nu}^\perp \tag{3.1}$$

i.e. to a constant magnetic field with a 3^{rd} colour component only. We begin with the study of the gauge and ghost propagators for the special case $\alpha=1$. The kernels (2.3) take the form

$$K_{\mu\nu}^{ab} = \delta_{\mu\nu}(D^2)^{ab} + 2gB\,\varepsilon_{\mu\nu}^\perp\,\varepsilon^{ab3}$$
$$K^{ab} = (D^2)^{ab}$$
$$(D_x^2)^{ab} = \Delta\,\delta^{ab} - \left(\tfrac{gB}{2}\right)^2 x_\perp^2\,\delta_\perp^{ab} + gB\,(x_1\partial_2 - x_2\partial_1)\,\varepsilon^{ab3} \tag{3.2}$$

In the following we use the notation

$$x_\perp^2 = x_1^2 + x_2^2 \qquad \delta_{\mu\nu}^\perp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \delta_{\mu\nu}^\parallel = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$x_\parallel^2 = x_3^2 + x_4^2$$
$$\delta_\perp^{ab} = \delta^{a1}\delta^{b1} + \delta^{a2}\delta^{b2} \tag{3.3}$$

The propagators corresponding to (2.4) and (3.2) have been evaluated in /3/ with the result

$$G^{ab}(x,y) = \phi^{ab}(x,y)\,D^o(x-y) + \frac{\delta^{a3}\delta^{b3}}{4\pi^2(x-y)^2} \tag{3.4}$$

$$G_{\mu\nu}^{ab}(x,y) = \phi^{ab}(x,y)\left[\delta_{\mu\nu}^\parallel D^o(x-y) + \delta_{\mu\nu}^\perp D^p(x-y)\right] + \tilde{\phi}^{ab}(x,y)\,\varepsilon_{\mu\nu}^\perp D^m(x-y) + \frac{\delta^{a3}\delta^{b3}}{4\pi^2(x-y)^2}\,\delta_{\mu\nu} \tag{3.5}$$

where

$$D^p = \frac{D^+ + D^-}{2}, \qquad D^m = \frac{D^+ - D^-}{2} \tag{3.6}$$

$$D^o(x-y) = \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\int_0^\infty\frac{d\tau}{\cosh\tau}\,e^{-\frac{k_\parallel^2\tau}{gB} - \frac{k_\perp^2}{gB}\tanh\tau} \tag{3.7}$$

$$D^-(x-y) = \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\int_0^\infty\frac{d\tau}{\cosh\tau}\,e^{-2\tau}\,e^{-\frac{k_\parallel^2\tau}{gB} - \frac{k_\perp^2}{gB}\tanh\tau} \tag{3.8}$$

$$D^+(x-y) = \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,e^{-\frac{k_\perp^2}{gB}} \times \left\{\int_0^\infty\frac{d\tau}{\cosh\tau}\,e^{\left(2-\frac{k_\parallel^2}{gB}\right)\tau}\left[e^{\frac{k_\perp^2}{gB}(1-\tanh\tau)} - 1 - e^{-2\tau}\right] + \frac{2}{\frac{k_\parallel^2}{gB}-1}\right\} \tag{3.9}$$

and

$$\phi^{ab}(x,y) = \delta_\perp^{ab}\cos\varrho(x,y) + \varepsilon^{ab3}\sin\varrho(x,y)$$
$$\tilde{\phi}^{ab}(x,y) = -\delta_\perp^{ab}\sin\varrho(x,y) + \varepsilon^{ab3}\cos\varrho(x,y)$$
$$\varrho(x,y) = \frac{gB}{2}(x_1y_2 - x_2y_1). \tag{3.10}$$

By the way, ϕ^{ab} is nothing else but the phase factor $\exp ig\int_y^x d\xi_\mu B_\mu$ in the adjoint representation, taken along the straight line connecting x and y. The pole term in (3.9) corresponds to the well-known negative mode of the kernel $K_{\mu\nu}^{ab}(\alpha=1)$ /1/. For convenience the derivation of (3.4), (3.5) is outlined in the Appendix.

The properties of D^0, D^-, and D^+ can be studied most conveniently by using the following representation.
Defining the function

$$I(a,b) = \int_0^\infty \frac{d\tau}{\cosh\tau}\, e^{-a\tau - b\tanh\tau}\,, \qquad (\mathrm{Re}\ a > -1) \tag{3.11}$$

and its analytic continuation

$$\begin{aligned} I(a,b) = e^{-b} \int_0^\infty \frac{d\tau}{\cosh\tau}\, e^{-a\tau}\left[e^{b(1-\tanh\tau)} - 1 - e^{-2\tau}\right] \\ + \frac{2e^{-b}}{a+1}\,, \qquad (\mathrm{Re}\ a > -3) \end{aligned} \tag{3.12}$$

we use the variables

$$a = \frac{k_\parallel^2}{gB}\,, \qquad b = \frac{k_\perp^2}{gB} \tag{3.13}$$

and write

$$\begin{aligned} D^0(x-y) &= \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, I(a,b) \\ D^-(x-y) &= \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, I(a+2,b) \\ D^+(x-y) &= \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, I(a-2,b). \end{aligned} \tag{3.14}$$

The analytic properties of I(a,b) follow from the formula

$$I(a,b) = 2e^{-b} \sum_{r=0}^\infty (-1)^r \frac{L_r(2b)}{a+2r+1}\,. \tag{3.15}$$

This results from (3.11) if we substitute $\exp(-2\tau)=z$ and apply the generating function for the Laguerre polynomials

$$\frac{1}{1+z}\, e^{\frac{2bz}{1+z}} = \sum_{r=0}^\infty (-z)^r L_r(2b). \tag{3.16}$$

Instead of a single pole at $k^2=0$ the gauge field propagator $G^{ab}_{\mu\nu}$ has a tower of poles at $k_\parallel^2 = gB, -gB, -3gB$, etc. but it is an entire function with respect to $k_\perp^2$.
The same applies to the ghost propagator (with the exception that the first pole at $k_\parallel^2=gB$ is absent).

The short-distance singularities of D^0 and D^- can be obtained by performing directly the Fourier transform in (3.7), (3.8)

$$\begin{aligned} D^0(x-y) &= \frac{gB}{16\pi^2}\int_0^\infty \frac{d\tau}{\tau\sinh\tau}\, e^{-\frac{\lambda}{\tau} - \mu\,\mathrm{ctanh}\,\tau} \\ D^-(x-y) &= \frac{gB}{16\pi^2}\int_0^\infty \frac{d\tau}{\tau\sinh\tau}\, e^{-2\tau-\frac{\lambda}{\tau} - \mu\,\mathrm{ctanh}\,\tau} \end{aligned} \tag{3.17}$$

where

$$\lambda = \frac{gB}{4}(x_\parallel - y_\parallel)^2, \qquad \mu = \frac{gB}{4}(x_\perp - y_\perp)^2. \tag{3.18}$$

Since the short-distance singularities originate from the region $\tau \approx 0$ one has to subtract the first singular terms with respect to τ, e.g. $e^{-2\tau}(\tau\sinh\tau)^{-1} = \tau^{-2} - 2\tau^{-1} + O(1)$ and to apply the formulas

$$\int_0^\infty \frac{d\tau}{\tau^2}\, e^{-\lambda/\tau} = \frac{1}{\lambda}\,, \qquad \int_0^1 \frac{d\tau}{\tau}\, e^{-\lambda/\tau} = -\ln\lambda - C + O(\lambda).$$

Thereby we arrive at

$$D^0(x-y) \longrightarrow \frac{1}{4\pi^2}\,\frac{1}{(x-y)^2} + \text{finite terms} \tag{3.19}$$

$$D^-(x-y) \longrightarrow \frac{1}{4\pi^2}\,\frac{1}{(x-y)^2} + \ln\left(\frac{gB(x-y)^2}{4}\right) + \text{finite terms} \tag{3.20}$$

for $(x-y)^2 \to 0$.

The determination of the short-distance singularities of D^+ demands a little more care. We split, quite arbitrarily, the τ integration at $\tau=1$. The integral $\int_1^\infty d\tau\ldots$ does not contribute to the short-distance singularities. The remaining terms can be written as

$$\begin{aligned} D^+(x-y) &= \frac{1}{gB}\int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)} \Big\{ \int_0^1 \frac{d\tau\, e^{2\tau}}{\cosh\tau}\, e^{-a\tau - b\tanh\tau} \\ &\quad -2e^{-b}\left[\int_0^1 d\tau\, e^{(1-a)\tau} - \frac{1}{(a-1)}\right] + \text{finite terms} \\ &= \frac{gB}{16\pi^2}\int_0^1 \frac{d\tau\, e^{2\tau}}{\tau\sinh\tau}\, e^{-\frac{\lambda}{\tau} - \mu\,\mathrm{ctanh}\,\tau} \\ &\quad + \frac{2}{gB}\int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\,\frac{e^{1-a-b}}{a-1} + \text{finite terms} \end{aligned} \tag{3.21}$$

for $(x-y)^2 \to 0$. We have used the abbreviations introduced in (3.13), (3.18). Obviously the second term in (3.21) is also finite so that we get

$$D^+(x-y) = \frac{1}{4\pi^2}\frac{1}{(x-y)^2} - \frac{gB}{8\pi^2}\ln\left(\frac{gB(x-y)^2}{4}\right) + \text{finite terms} \tag{3.22}$$

for $(x-y)^2 \to 0$.

From (3.5), (3.6) with (3.2o), (3.22) we obtain finally for the gauge field propagator at $(x-y)^2 \to 0$

$$G^{ab}_{\mu\nu}(x,y) \to \left(\phi^{ab}(x,y) + \delta^{a3}\delta^{b3}\right)\frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2} - \tilde{\phi}^{ab}\varepsilon^{\perp}_{\mu\nu}\frac{gB}{8\pi^2}\ln\left(\frac{gB(x-y)^2}{4}\right) + \text{fin.t.} \tag{3.23}$$

Whereas the short-distance properties are essential for the structure of the renormalization parts of the QFT in a classical background (to be studied in a forthcoming paper) it is the large-distance behaviour which decides whether the construction (2.11) makes sense or not.

The large-distance behaviour of D^0 and D^- can be read off from the Fourier transform of (3.15)

$$\begin{aligned} D^0(x-y) &= \frac{gB}{4\pi^2} e^{-\mu} \sum_{r=0}^{\infty} L_r(2\mu)\, K_0\left(2\sqrt{(2r+1)\lambda}\right) \\ &\sim \frac{gB}{8}\pi^{-3/2}\lambda^{-1/4} e^{-2\sqrt{\lambda}-\mu}, \qquad (\lambda \to \infty) \end{aligned} \tag{3.24}$$

$$\begin{aligned} D^-(x-y) &= \frac{gB}{4\pi^2} e^{-\mu} \sum_{r=0}^{\infty} L_r(2\mu)\, K_0\left(2\sqrt{(2r+3)\lambda}\right) \\ &\sim \frac{gB}{8}\pi^{-3/2}(3\lambda)^{-1/4} e^{-2\sqrt{3\lambda}-\mu}, \qquad (\lambda \to \infty) \end{aligned} \tag{3.25}$$

In the case of D^+ the pole term (see (3.9)) gives rise to an oscillatory behaviour of the leading term

$$\begin{aligned} \frac{2}{gB}\int\frac{d^4k}{(2\pi)^4} e^{ikx}\frac{e^{-b}}{a-1\pm i\varepsilon} &= \frac{gB}{8\pi^2} e^{-\mu}\int_0^{\infty}\frac{dt}{t} e^{\mp i\left(t+\frac{\lambda}{t}\right)} \\ &= \mp\frac{i}{8\pi} gB\, e^{-\mu} H_0^{(2)}(2\sqrt{\lambda}) \sim \frac{gB}{8}\pi^{-3/2}\lambda^{-1/4} e^{\mp i\left(2\sqrt{\lambda}+\frac{\pi}{4}\right)-\mu}, \qquad (\lambda \to \infty) \end{aligned} \tag{3.26}$$

It follows

$$D^+(x-y) = \frac{gB}{4\pi^2} e^{-\mu}\left\{\mp\frac{i\pi}{2} H_0^{(2)}(2\sqrt{\lambda}) + \sum_{r=1}^{\infty} L_r(2\mu)\, K_0\left(2\sqrt{(2r-1)\lambda}\right)\right\}. \tag{3.27}$$

Finally we obtain from (3.4) and (3.5) with (3.24-27) the leading large-distance behaviour

$$G^{ab}(x,y) \sim \frac{(gB)^{3/4}}{2(2\pi)^{3/2}}\left[(x_{\parallel}-y_{\parallel})^2\right]^{-1/4}\phi^{ab}(x,y) \times e^{-\sqrt{gB(x_{\parallel}-y_{\parallel})^2} - \frac{gB}{4}(x_{\perp}-y_{\perp})^2} + \frac{\delta^{a3}\delta^{b3}}{4\pi^2(x-y)^2} \tag{3.28}$$

$$G^{ab}_{\mu\nu}(x,y) \sim \frac{1\mp i}{8\pi^{1/2}}(gB)^{3/4}\left[(x_{\parallel}-y_{\parallel})^2\right]^{-1/4}\left[\phi^{ab}(x,y)\delta^{\perp}_{\mu\nu} + \tilde{\phi}^{ab}(x,y)\varepsilon^{\perp}_{\mu\nu}\right] \times e^{\mp i\sqrt{gB(x_{\parallel}-y_{\parallel})^2} - \frac{gB}{4}(x_{\perp}-y_{\perp})^2} + \frac{\delta^{a3}\delta^{b3}\delta_{\mu\nu}}{4\pi^2(x-y)^2}. \tag{3.29}$$

Here a remark is in order. In (3.26) the pole term has been specified as a distribution with a non-zero imaginary part in accordance with the appearance of an imaginary part in the 1-loop effective action. Obviously the dominant term (3.29) behaves for $(x_{\parallel}-y_{\parallel})^2 \to \infty$, $(x_{\perp}-y_{\perp})^2$ fix, worse than the zero field propagator. In principle this could generate problems for the construction (2.11) of the general propagator; indeed a superficial inspection indicates divergence of the z integral in (2.11). It turns out, however, that due to the structure of the phase factors (3.1o) the dangerous term in (3.29) will be projected to zero

$$D^{ab}_{\mu\,x}\left[\phi^{bc}(x,y)\delta^{\perp}_{\mu\nu} + \tilde{\phi}^{bc}(x,y)\varepsilon^{\perp}_{\mu\nu}\right] e^{-\frac{gB}{4}(x_{\perp}-y_{\perp})^2} H_0^{(2)}\left(\sqrt{gB(x_{\parallel}-y_{\parallel})^2}\right) = 0. \tag{3.3o}$$

We conclude that for the case of a homogeneous background field eq.(2.11) is a well-defined expression which determines the gauge field propagator for arbitrary values of the gauge parameter.

To evaluate the integral occuring in (2.11)

$$\mathcal{U}^{ae}_{\varkappa\nu}(x,y) = \int d^4z \; G^{ab}_{\varkappa\lambda}(x,z,1)\, \underset{z}{D}^{bc}_{\lambda}\, \underset{z}{D}^{cd}_{\mu}\; G^{de}_{\mu\nu}(z,y,1) \tag{3.31}$$

we insert $G^{ab}_{\varkappa\lambda}(x,z,1) \equiv G^{ab}_{\varkappa\lambda}(x,z)$ and $G^{de}_{\mu\nu}(z,y,1) \equiv G^{de}_{\mu\nu}(z,y)$ from (3.5) and use partial integration to let $\underset{z}{D}^{bc}_{\lambda}$ operate on $G^{ab}_{\varkappa\lambda}(x,z,1)$

$$\begin{aligned}
\mathcal{U}^{ae}_{\varkappa\nu}(x,y) = &- \int dz \Big\{ \phi^{ac}(x,z)\Big[\frac{\partial D^{o}(x-z)}{\partial z^{\parallel}_{\varkappa}} + \frac{\partial D^{p}(x-z)}{\partial z^{\perp}_{\varkappa}} + \frac{gB}{2}(z^{\perp}_{\varkappa}-x^{\perp}_{\varkappa})D^{m}(x-z)\Big] + \\
&\tilde{\phi}^{ac}\Big[\frac{gB}{2}(\tilde{z}_{\varkappa}-\tilde{x}_{\varkappa})D^{p}(x-z) + \frac{\partial D^{m}(x-z)}{\partial \tilde{z}_{\varkappa}}\Big] + \delta^{a3}\delta^{c3}\frac{\partial}{\partial z_{\varkappa}}\Delta^{-1}(x-z)\Big\} - \\
&\Big\{\phi^{ce}(z,y)\Big[\frac{\partial D^{o}(z-y)}{\partial z^{\parallel}_{\nu}} + \frac{\partial D^{p}(z-y)}{\partial z^{\perp}_{\nu}} + \frac{gB}{2}(z^{\perp}_{\nu}-y^{\perp}_{\nu})D^{m}(z-y)\Big] + \\
&\tilde{\phi}^{ce}(z,y)\Big[\frac{gB}{2}(\tilde{y}_{\nu}-\tilde{z}_{\nu})D^{p}(z-y) - \frac{\partial D^{m}(z-y)}{\partial \tilde{z}_{\nu}}\Big] + \delta^{c3}\delta^{e3}\frac{\partial}{\partial z_{\nu}}\Delta^{-1}(z-y)\Big\}.
\end{aligned} \tag{3.32}$$

Now we apply the relations

$$\begin{aligned}
\phi^{ab}(x,z)\,\phi^{bc}(z,y) &= -\tilde{\phi}^{ab}(x,z)\,\tilde{\phi}^{bc}(z,y) = \phi^{ac}(x-y,z) \\
\phi^{ab}(x,z)\,\tilde{\phi}^{bc}(z,y) &= \tilde{\phi}^{ab}(x,z)\,\phi^{bc}(z,y) = \tilde{\phi}^{ac}(x-y,z)
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z^{\perp}_{\varkappa}} D^{p}(z) + \frac{gB}{2} z^{\perp}_{\varkappa} D^{m}(z) &= \frac{\partial}{\partial z^{\perp}_{\varkappa}} D^{o}(z) \\
\frac{gB}{2} z^{\perp}_{\varkappa} D^{p}(z) + \frac{\partial}{\partial z^{\perp}_{\varkappa}} D^{m}(z) &= -\frac{gB}{2} z^{\perp}_{\varkappa} D^{o}(z)
\end{aligned} \tag{3.34}$$

to arrive at

$$\begin{aligned}
\mathcal{U}^{ae}_{\varkappa\nu}(x,y) = &\int dz \left\{ \phi^{ae}(x-y,z)\, \mathrm{Re}\, P_{\varkappa\nu}(\xi,\eta) + \tilde{\phi}^{ae}(x-y,z)\, \mathrm{Im}\, P_{\varkappa\nu}(\xi,\eta) \right\} \\
&- \delta^{a3}\delta^{e3} \int \frac{d^4k}{(2\pi)^4}\, \frac{k_{\varkappa} k_{\nu}}{k^4}\, e^{ik(x-y)}
\end{aligned} \tag{3.35}$$

where

$$\xi = x-z, \qquad \eta = z-y \tag{3.36}$$

$$P_{\varkappa\nu}(\xi,\eta) = \Big(\frac{\partial}{\partial\xi_{\varkappa}} - i\,\frac{gB}{2}\tilde{\xi}_{\varkappa}\Big) D_o(\xi)\, \Big(\frac{\partial}{\partial\eta_{\nu}} + i\,\frac{gB}{2}\tilde{\eta}_{\nu}\Big) D_o(\eta). \tag{3.37}$$

It is easy to show that for real fg

$$\begin{aligned}
\int dz\, \phi^{ab}(x-y,z)\, f(x-z)\, g(z-y) &= \phi^{ab}(x,y)\, \mathrm{Re}\, Q + \tilde{\phi}^{ab}(x,y)\, \mathrm{Im}\, Q \\
\int dz\, \tilde{\phi}^{ab}(x-y,z)\, f(x-z)\, g(z-y) &= -\phi^{ab}(x,y)\, \mathrm{Im}\, Q + \tilde{\phi}^{ab}(x,y)\, \mathrm{Re}\, Q
\end{aligned} \tag{3.38}$$

where

$$Q = \int d\chi\, e^{i\xi\tilde{\eta}} f(\xi)\, g(\eta), \qquad \chi = \frac{\xi-\eta}{2}. \tag{3.39}$$

Using these relations to evaluate (3.35) we get

$$\begin{aligned}
\mathcal{U}^{ae}_{\varkappa\nu}(x,y) = &\,\phi^{ae}(x,y)\, \mathrm{Re}\, Q_{\varkappa\nu}(x-y) + \tilde{\phi}^{ae}(x,y)\, \mathrm{Im}\, Q_{\varkappa\nu}(x-y) \\
&- \delta^{a3}\delta^{e3}\int \frac{d^4k}{(2\pi)^4}\, \frac{k_{\varkappa} k_{\nu}}{k^4}\, e^{ik(x-y)}
\end{aligned} \tag{3.40}$$

with $Q_{\varkappa\nu}(x-y)$ given in terms of the function $P_{\varkappa\nu}(\xi,\eta)$ introduced in (3.37)

$$Q_{\varkappa\nu}(x-y) = \int d\chi\, e^{i\xi\tilde{\eta}} P_{\varkappa\nu}(\xi,\eta), \qquad \xi = \frac{x-y}{2}+\chi, \quad \eta = \frac{x-y}{2}-\chi. \tag{3.41}$$

In the expressions (3.40), (3.41), (3.37) we have now a convenient form which provides a starting point for e.g. the study of the analytic properties of the gauge propagator $G^{ab}_{\mu\nu}(x,y,\alpha)$ in a homogeneous background for general gauge parameter α.

Appendix

The Appendix contains, based on /3/, a short derivation of the expressions (3.4) and (3.5) for the ghost and gauge propagators at α=1.

We diagonalize the kernels (3.2) in colour and space-time indices and arrive at the problem to invert the operators

$$h^{\pm} = -\Delta + \frac{g^2B^2}{4}(x_1^2+x_2^2) \mp igB\,(x_1\partial_2 - x_2\partial_1) \tag{A.1}$$

as well as $h^{+}\pm 2gB$ and $h^{-}\pm 2gB$. We introduce creation and annihilation operators $(z=\sqrt{gB/2}\;x)$

$$\begin{aligned}
d^{\dagger} &= \tfrac{1}{2}\left[-\partial_1 + z_1 + i(-\partial_2+z_2)\right], & d &= \tfrac{1}{2}\left[\partial_1 + z_1 - i(\partial_2+z_2)\right] \\
\tilde{d}^{\dagger} &= \tfrac{1}{2}\left[-\partial_1 + z_1 - i(-\partial_2+z_2)\right], & \tilde{d} &= \tfrac{1}{2}\left[\partial_1 + z_1 + i(\partial_2+z_2)\right] \\
\left[d, d^{\dagger}\right] &= \left[\tilde{d}, \tilde{d}^{\dagger}\right] = 1
\end{aligned} \tag{A.2}$$

Next we perform a Fourier transformation for the x_3, x_4 dependence and obtain the 2-dimensional harmonic oscillator Hamiltonians

$$h^+(x_1,x_2,k_\parallel) = k_\parallel^2 + gB(2\alpha^+\alpha+1)$$
$$h^-(x_1,x_2,k_\parallel) = k_\parallel^2 + gB(2\tilde{\alpha}^+\tilde{\alpha}+1) \qquad (A.3)$$

with eigenfunctions

$$u_{mn}(z_\perp) = \sqrt{\frac{gB}{2}}\,\frac{1}{4\pi^{3/2}\sqrt{m!\,n!}}\,e^{z_\perp^2/2} \times$$
$$\int d^2k_\perp\, e^{-k_\perp^2/4 + i k_\perp z_\perp}\left(\frac{-ik_1+k_2}{2}\right)^m\left(\frac{-ik_1-k_2}{2}\right)^n \qquad (A.4)$$

and (infinitely degenerate) eigenvalues

$$\lambda_{mn}^+ = k_\parallel^2 + (2m+1)gB, \qquad \lambda_{mn}^- = k_\parallel^2 + (2n+1)gB \qquad (A.5)$$

for h^+ and h^- respectively.

From (A.4) and (A.5) we calculate the inverses

$$[h^\pm(x,y)]^{-1} = \int\frac{d^2k_\parallel}{(2\pi)^2}\,e^{ik_\parallel(x_\parallel-y_\parallel)}\sum_{m,n} u_{mn}(x_\perp)\,u_{mn}^*(y_\perp)/\lambda_{mn}^\pm$$
$$= \frac{1}{gB}\,e^{\pm i\varphi(x,y)}\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,I(a,b) \qquad (A.6)$$

with $I(a,b)$ defined in (3.11-13).

The inverses of $h\pm 2gB$ simply involve the Fourier transforms of $I(a\pm 2,b)$ instead of $I(a,b)$ in (A.6). In /3/ we took care in the treatment of the negative mode during the calculation of the inverses of $h^\pm - 2gB$.

It remains to undo the diagonalizations to verify finally the expressions (3.4) and (3.5).

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