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ON A SPECIAL REPRESENTATION OF THE EFFECTIVE ACTION

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Abstract

A recently obtained special representation of the effective action is reviewed in order to discuss its impact on the consideration of the effective action of Euclidean Yang-Mills theory.

1. Introduction

This is the third of three contributions (also see [1], [2]) dealing with Euclidean Yang-Mills theory in constant background fields. The investigation presented here is part of a collaboration together with H.J. Kaiser and E. Wieczorek on the above mentioned subject. For previous results the reader is refered to [3].
In [1], [2] it has been shown that the explicit construction of gluon and ghost Green's functions of Euclidean SU(2)-Yang-Mills theory considered in a constant colormagnetic and constant color-electric background leads to the conclusion that its effective action does not acquire any radiative correction to the imaginary part beyond the 1-loop contribution. This rather surprising result encourages us to reconsider the 1-loop imaginary part itself. Let us remind ourselfes that the well known 1-loop result in Yang-Mills theory [4], [5] has been obtained by exploiting the Gaussian approximation in the functional integral by neglecting the rules of its applicability - the quadratic kernel of the gluon action in a constant background has negative, i.e. unstable, modes. Therefore, the application of the standard formula for the Gaussian approximation has to be considered here more as a kind of recipe or working rule rather than as a theoretically completely well based procedure. In difference to the imaginary part of the QED effective action in the case of a constant electric field [6] which has clear physical implications the imaginary part of the effective action in Yang-Mills theory found in accordance to usual wisdom in 1-loop approximation is already present in the Euclidean version of the theory. For the above mentioned reasons the 1-loop imaginary part in the Yang-Mills effective action looks suspicious and deserves a fresh

look from an alternative point of view. In the following we are going to review a recently obtained special representation of the effective action [7] laying the groundwork for an alternative discussion of the 1-loop Yang-Mills effective action.

2. The effective action reconstruction formula

Usually the effective action is considered as the generating functional for 1PI Green's functions. This is a point of view which relies on the effective action as some given object. But in real life rather we have to calculate the effective action from other quantities first. So, for example, the effective action of a given theory can be reconstructed from the knowledge of the polarization operator in a background field. This relation will shortly be described in this section for the case of a renormalizable scalar theory. A comprehensive discussion of this relation can be found in [7].
Let us start with a classical action

$$\Gamma_{cl}[\varphi] = \int dx\, \mathcal{L}(\varphi)(x) \tag{2.1}$$

and consider a background field $\bar{\varphi}(x)$ which fulfills the classical equation of motion with respect to a given classical source $\bar{J}$.

$$\frac{\delta \Gamma_{cl}[\bar{\varphi}]}{\delta \bar{\varphi}(x)} = -\bar{J}(x) \tag{2.2}$$

The generating functionals of the theory are taken in presence of the background field $\bar{\varphi}(x)$.

$$Z[J,\bar{\varphi}] = \int D\varphi\; e^{i\Gamma_{cl}[\varphi+\bar{\varphi}] + i\int dx\, J(x)\varphi(x)} \tag{2.3}$$

$$W[J,\bar{\varphi}] = -i \ln Z[J,\bar{\varphi}] \tag{2.4}$$

$$\Gamma[\varphi',\bar{\varphi}] = W[J,\bar{\varphi}] - \int dx\, J(x)\varphi'(x) \tag{2.5}$$

The effective action is constructed here in the spirit of the background field method applied to a scalar theory (cf. e.g. [8]). The standard relations

$$\frac{\delta W[J,\bar{\varphi}]}{\delta J(x)} = \varphi'(x) \quad , \quad \frac{\delta \Gamma[\varphi',\bar{\varphi}]}{\delta \varphi'(x)} = -J(x) \qquad (2.6),\ (2.7)$$

$$\frac{\delta^2 W[J,\bar{\varphi}]}{\delta J(x)\,\delta J(x')} = \frac{\delta \varphi'(x)}{\delta J(x')} \quad , \quad \frac{\delta^2 \Gamma[\varphi',\bar{\varphi}]}{\delta \varphi'(x)\,\delta \varphi'(x')} = -\frac{\delta J(x)}{\delta \varphi'(x')} \qquad (2.8),\ (2.9)$$

as usual immediately yield the following formula.

$$\int dz\, \frac{\delta^2 \Gamma[\varphi',\bar{\varphi}]}{\delta \varphi'(x)\,\delta \varphi'(z)} \, \frac{\delta^2 W[J,\bar{\varphi}]}{\delta J(z)\,\delta J(x')} = -\,\delta(x-x') \qquad (2.10)$$

This relation plays a crucial role in the derivation of the effective action reconstruction formula.
We define by

$$\frac{\delta^2 \Gamma_{cl}[\bar{\varphi}]}{\delta \bar{\varphi}(x)\,\delta \bar{\varphi}(x')} = -\,K_{cl}(\bar{\varphi};x)\,\delta(x-x') \qquad (2.11)$$

the quadratic kernel $K_{cl}(\bar{\varphi};x)$ of the classical action of the field φ in the background field $\bar{\varphi}(x)$, which is a local operator. The free connected propagator $G_0(\bar{\varphi};x,x')$ of the φ-field in the background $\bar{\varphi}$ is given by

$$K_{cl}(\bar{\varphi};x)\, G_0(\bar{\varphi};x,x') = \delta(x-x') \qquad (2.12)$$

In order to build up the perturbation theory we split the exponential in (2.3) in 4 pieces.

$$\Gamma_{cl}[\varphi+\bar{\varphi}] + \int dx\, J(x)\varphi(x) = \Gamma_{cl}[\bar{\varphi}] + \int dx\, J'(x)\varphi(x) - \frac{1}{2}\int dx\, \varphi(x) K_{cl}(\bar{\varphi};x)\varphi(x) + \Gamma_I[\varphi,\bar{\varphi}] \qquad (2.13)$$

$$J'(x) = J(x) + \frac{\delta \Gamma_{cl}[\bar{\varphi}]}{\delta \bar{\varphi}(x)} \qquad (2.14)$$

On the basis of (2.12), (2.13) the construction of the perturbation theory is obvious and will not be discussed here. Clearly, Green's functions have to be calculated at $J'(x) = 0$.
Let us sketch now the derivation of the effective action reconstruction formula. The starting point is the well known mathematical fact that a sufficiently well behaved functional Φ of some function φ can be reconstructed from its value at $\varphi = 0$, the first functional derivative of Φ at $\varphi = 0$ and the second functional derivative of Φ.

$$\Phi[\varphi] = \Phi[0] + \int dz\, \frac{\delta \Phi[\varphi']}{\delta \varphi'(z)}\Big|_{\varphi'=0} \varphi(z) + \int_0^1 d\tau\,(1-\tau) \int dz\, dz'\, \varphi(z)\, \frac{\delta^2 \Phi}{\delta \varphi(z)\,\delta \varphi(z')}[\tau\varphi]\, \varphi(z') \qquad (2.15)$$

(2.15) is applied now to the effective action. From perturbation theory we know the following relation

$$\frac{\delta^2 \Gamma[0,\bar{\varphi}]}{\delta \bar{\varphi}(x)\,\delta \bar{\varphi}(x')} = -\left[K_{cl}(\bar{\varphi};x)\,\delta(x-x') - \Pi(\bar{\varphi};x,x')\right] \qquad (2.16)$$

where $\Pi(\bar{\varphi};x,x')$ is the polarization operator in the background field $\bar{\varphi}$. For $\bar{\varphi} = 0$ Π coincides with the usual expression.
So, we get

$$\Gamma[0,\bar{\varphi}] = \Gamma[0,0] + \int dz\, \frac{\delta \Gamma[0,\varphi']}{\delta \varphi'(z)}\Big|_{\varphi'=0} \bar{\varphi}(z) - \int_0^1 d\tau\,(1-\tau) \int dz\, dz'\, \bar{\varphi}(z)\left[K_{cl}(\tau\bar{\varphi};z)\,\delta(z-z') - \Pi(\tau\bar{\varphi};z,z')\right]\bar{\varphi}(z') \qquad (2.17)$$

Taking into account the condition

$$\frac{\delta \Gamma[0,\varphi']}{\delta \varphi'(z)}\Big|_{\varphi'=0} = 0 \qquad (2.18)$$

and writing $\Gamma[\bar{\varphi}]$ for $\Gamma[0,\bar{\varphi}]$ we reach at the desired effective action reconstruction formula.

$$\Gamma[\bar{\varphi}] = \Gamma[0] + \Gamma_{cl}[\bar{\varphi}] + \int_0^1 d\tau\,(1-\tau) \int dz\, dz'\, \bar{\varphi}(z)\, \Pi(\tau\bar{\varphi};z,z')\, \bar{\varphi}(z') \qquad (2.19)$$

This formula describes the reconstruction of the complete effective action from the complete polarization operator in a background field. (2.19) specified to the 1-loop approximation yields the following determinant formula.

$$\det K_{cl}(\bar{\varphi};\,.) =$$

$$= e^{-2i\int_0^1 d\tau(1-\tau)\int dz\,dz'\,\bar{\varphi}(z)\Pi^{(1)}(\tau\bar{\varphi};\,z,z')\bar{\varphi}(z')} \det K_{cl}(0;\,.) \qquad (2.20)$$

At the end of this section three comments are due. First, (2.19) clearly shows that the 1-loop term in the effective action is not somehow special as the usual belief is but fits completely into the perturbation theoretic representation of the effective action. Second, in some sense (2.20) stands for an alternative definition of the determinant of a given kernel in terms of quantum field theoretic building blocks, namely vertices and propagators. Third, (2.19) is a bilocal representation of the effective action which might be useful in discussing non-constant background fields.

3. An explicit example

In this section we are going to present a calculational example for relation (2.19). We consider the 1-loop contribution to the QED effective action in the case of a constant magnetic background field. The case of gauge theories is a little bit more advanced than the scalar case and needs some additional consideration. This discussion can be found in [7], here we state only the problem and the result of its investigation. In the case of a gauge theory dealing with a gauge invariant effective action relation (2.10) is no longer applicable, at least not in the naive sense. It can be shown by careful inspection that in the case of a gauge theory the effective action reconstruction formula (2.19) holds at least for background fields obeying the classical equation of motion for vanishing source.
We consider the following background field configuration in QED.

$$B_\mu(x) = -\frac{1}{2}F_{\mu\nu}x^\nu\,, \qquad F_{12} = -F_{21} = B \qquad (3.1)$$

All other field strength tensor components vanish. Fortunately, the 1-loop polarization operator in the magnetic background (3.1) has already been constructed and may be taken from the literature [9] (also cf. [10]).

$\Pi^{(1)}_{\mu\nu}$ = 

$$\tilde{\Pi}^{(1)}_{\mu\nu}(B,k)_{ren} = \frac{e^2}{8\pi^2}\int_0^\infty\frac{ds}{s}\int_{-1}^{1}\frac{dv}{2}\Big\{e^{-is\varphi_0}\big[(g_{\mu\nu}k^2-k_\mu k_\nu)N_0(z,v) -$$
$$-(g^{\parallel}_{\mu\nu}k^2_{\parallel}-k^{\parallel}_\mu k^{\parallel}_\nu)N_1(z,v)+(g^{\perp}_{\mu\nu}k^2_{\perp}-k^{\perp}_\mu k^{\perp}_\nu)N_2(z,v)\big] -$$
$$-\,e^{-ism^2}(1-v^2)(g_{\mu\nu}k^2-k_\mu k_\nu)\Big\}\,,$$
$$\varphi_0 = m^2+\frac{1}{4}(1-v^2)k^2_\parallel+\frac{\cos zv-\cos z}{2z\sin z}k^2_\perp\,,$$
$$z = eBs\,, \qquad \parallel\,\hat{=}\,0,3 \quad \perp\,\hat{=}\,1,2 \qquad (3.2)$$

Using (2.19) we find for the 1-loop contribution to the effective action the following expression.

$$V^4\mathcal{L}^{(1)}[B]_{ren} = -i\,[\ln\det(\not{0}-m)-\ln\det(\not{B}-m)]_{ren} \qquad (3.3)$$
$$= -\int_0^1 d\tau(1-\tau)\int d^4x\,d^4x'\,B^\mu(x)\Pi^{(1)}_{\mu\nu}(\tau B;\,x,x')_{ren}B^\nu(x') \qquad (3.4)$$
$$= -\int_0^1 d\tau(1-\tau)\int d^4x\,d^4x'\,B^\mu(x)\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-x')}\,\cdot$$
$$\cdot\,(g^\perp_{\mu\nu}k^2_\perp-k^\perp_\mu k^\perp_\nu)\tilde{\Pi}^{(1)\prime}(\tau B;k)_{ren}B^\nu(x') \qquad (3.5)$$

Here $\tilde{\Pi}^{(1)\prime}(\tau B;k)_{ren}$ represents some reduced part of (3.2) having taken into account the special structure of the background field (3.1). Now, the transversal structure in (3.5) may be used to reduce the gauge potential to the magnetic field itself. This allows to factorize out the infinite space-time volume. Then, performing the s- and τ-integrations we reach at the final expression.

$$V^4\mathcal{L}^{(1)}[B]_{ren} = -V^4B^2\int_0^1 d\tau(1-\tau)\,\tilde{\Pi}^{(1)\prime}(\tau B;0)_{ren} \qquad (3.6)$$
$$= -V^4B^2\int_0^1 d\tau(1-\tau)\frac{e^2}{8\pi^2}\int_0^\infty\frac{ds}{s}\int_{-1}^{1}\frac{dv}{2}\,e^{-m^2s}\,\cdot$$
$$\cdot\Big\{2e\tau Bs\,\frac{(\cosh(e\tau Bs)-\cosh(e\tau Bvs))}{\sinh^3(e\tau Bs)} - (1-v^2)\Big\} \qquad (3.7)$$
$$= -V^4B^2\frac{e^2}{8\pi^2}\int_0^\infty\frac{ds}{s^2}\,e^{-\frac{m^2}{eB}s}\Big[\coth s-\frac{s}{3}-\frac{1}{s}\Big] \qquad (3.8)$$

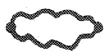
Our final expression (3.8) completely agrees with the result well known from literature [6] (also cf. [10]). This explicit calculation gives us confidence that relation (2.19) may be relied on even in the case of gauge theories (taking into account the above mentioned restriction).

4. Effective action in Euclidean Yang-Mills theory

In this final section let us turn attention to our main subject of interest - the effective action of Euclidean Yang-Mills theory. In [1], [2] the construction of gluon and ghost Green's functions for Euclidean SU(2)-Yang-Mills theory in the background

$$B^a_\mu(x) = -\frac{1}{2} F_{\mu\nu} x_\nu \delta^{a3}, \quad \begin{array}{l} F_{03} = -F_{30} = B', \\ F_{12} = -F_{21} = B \end{array} \tag{4.1}$$

has been presented (all components of the field strength tensor not listed in (4.1) vanish). On the basis of these building blocks (propagators and vertices) the 1-loop gluon polarization operator (4.2) in the background (4.1) may be constructed.

$\Pi^{(1)\,ab}_{\mu\nu}(B,B';x,x') = \frac{1}{2}$  $+\frac{1}{2}$  $-$  (4.2)

Although this is a quite tedious task and has not been done explicitely up to now it can be concluded that (4.2) is a purely real object inasmuch as all involved propagators (and vertices) have been shown to be purely real [1], [2]. $\Pi^{(1)ab}_{\mu\nu}(\tau B,\tau B';x,x')_{reg}$ is a well defined object for all $\tau \in [0,1]$. Therefore, the application of (2.19) leads to the conclusion that the 1-loop contribution (4.3) to the effective action of Euclidean Yang-Mills theory is purely real in contradiction to usual wisdom.

$$\Gamma^{(1)}[B,B']_{reg} = -\int_0^1 d\tau\,(1-\tau)\int d^4x\,d^4x'\, B^a_\mu(x)\,\Pi^{(1)\,ab}_{\mu\nu}(\tau B,\tau B';x,x')_{reg}\, B^b_\nu(x') \tag{4.3}$$

Where does the difference come from? As we already mentioned in the introduction the usual formula

$$\Gamma^{(1)} \sim \sum_n \ln \lambda_n \tag{4.4}$$

is applicable for bosonic kernels only if they are positive definite. In the presence of negative modes (4.4) has to be considered

merely as a kind of recipe only. Formula (2.19) is not necessarily connected with the functional integral approach and may equally be derived in some canonical quantization scheme. Furthermore, (2.19) is not invalidated in the presence of negative modes.
So, summarizing the results obtained we may conclude that the effective action of Euclidean Yang-Mills theory considered for a constant colormagnetic and colorelectric background does not exhibit any imaginary part. In order to be cautious let the final lesson be expressed by the message: don't trust the imaginary part in the effective action of Euclidean Yang-Mills theory.

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