Results for the strong coupling lattice Schwinger model with Wilson fermions from a study of the equivalent loop model

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Salmhofer has demonstrated the equivalence of the strong coupling lattice Schwinger model with Wilson fermions to a self-avoiding loop model on the square lattice with a bending rigidity $\eta = 1/2$. The present paper applies two approximate analytical methods to the investigation of critical properties of the self-avoiding loop model for variable η, discusses their validity, and makes a comparison with known Monte Carlo results. One method is based on the independent loop approximation used in the literature for studying phase transitions in polymers, liquid helium, and cosmic strings. The second method relies on the known exact solution of the self-avoiding loop model with $\eta = 1/\sqrt{2}$. The present investigation confirms recent findings that the strong coupling lattice Schwinger model becomes critical for $\kappa_{cr} \simeq 0.38 - 0.39$. The phase transition is of second order and lies in the Ising model universality class. Finally, the central charge of the model at criticality is discussed and predicted to be $c = 1/2$. [S0556-2821(97)04316-6]

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I. INTRODUCTION

Recently, the strong coupling lattice Schwinger model with Wilson fermions ($N_f = 1$) has received some attention [1–6] following work by Salmhofer [7] who has shown that it is equivalent to a certain eight-vertex model (a seven-vertex model, more precisely) which also can be understood as a self-avoiding loop model on the square lattice with a bending rigidity $\eta = 1/2$ and monomer weight $z = (2\kappa)^{-2}$. Beyond its toy character, interest in the lattice Schwinger model [two-dimensional QED (QED_2)] mainly derives from the similarity of some of its major features with those of QCD in four dimensions. However, because the result of Salmhofer [7] is related to the polymer (hopping parameter) expansion of the fermion determinant [8,9] the strong coupling Schwinger model is also interesting from the point of view of the dynamical fermion problem within lattice gauge theory. While some investigations have been devoted to the polymer expansion of the fermion determinant in the case of staggered fermions [10–14], to date, almost no attention has been paid to the corresponding case of Wilson fermions [15] due to the additional difficulties involved in general, as the larger number of Grassmann variables per lattice site and the explicit breaking of chiral symmetry. However, while in the strong coupling limit the system with staggered fermions (QCD, QED) reduces to a pure monomer-dimer system [16], the same is not true for Wilson fermions as the investigation of Salmhofer [7] demonstrates. The equivalence of the strong coupling lattice Schwinger model with Wilson fermions to a self-avoiding loop model enables certain methods used in other branches of physics, e.g., in condensed matter physics (polymers, defect-mediated phase transitions) and in cosmic string physics, to be exploited in its investigation [17–22]. At the same time, its equivalence (in another language) to some eight-vertex model [23] makes further results available.

Self-avoiding loop models [24–26] have a long history due to their prominent role in polymer physics as well as their inherent attractiveness as a simple problem of non-Markovian nature. In addition, systems of closed noncrossing lines or systems, which can be approximated by them, appear in a variety of contexts ranging from condensed matter physics through cosmology to quantum field theory which generates common interest for appropriate model building [27,17]. Recently, quantum field theoretic methods have been exploited to study the critical behavior of self-avoiding loop models in two dimensions [28–31]. Somewhat less attention has been paid so far to the self-avoiding loop model with a variable bending rigidity (while for open chains with bending rigidity a number of investigations exists, e.g., [32] and references therein). Beyond the work of Müser and Rys [33,34], certain insight in this direction has been obtained in connection with the study of two-dimensional vesicles [35–37].

From the point of view of the eight-vertex model, a general solution to the self-avoiding loop model with a variable bending rigidity on the square lattice is not known. However, for the special case $\eta = 1/\sqrt{2}$, the free-fermion condition [38,39] is satisfied and it can be solved exactly [40–42]. This way, one point on the critical line of the the self-avoiding loop model with a variable bending rigidity is known exactly and, consequently, one may use analytic perturbative methods to approximately find the critical line in its neighborhood.

The plan of the paper is as follows. In Sec. II we briefly review the relevant facts concerning the lattice Schwinger model with Wilson fermions and discuss the relation of recent studies of it [4–6] to the earlier Monte Carlo (MC) results of Müser and Rys [33,34]. Section III is devoted to the approximate analytical study of the self-avoiding loop

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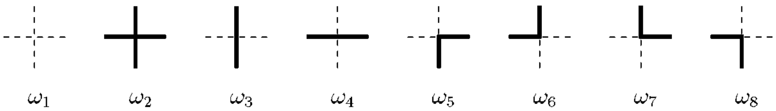

FIG. 1. Vertices of the eight-vertex model and their weights [cf. Eqs. (4)–(7)].

model (SALM) with a variable bending rigidity by means of the independent loop approximation. Section IV then explores the application of the exact solution of the SALM with a bending rigidity $\eta = 1/\sqrt{2}$ to the study of the critical behavior of the SALM in a neighborhood of this point in the relevant parameter space. Section V finally discusses the picture emerging from the present investigation paying special attention to the central charge of the SALM with a variable bending rigidity on the critical line.

II. THE STRONG COUPLING SCHWINGER MODEL WITH WILSON FERMIONS

The partition function Z_Λ of the Schwinger model with Wilson fermions (with Wilson parameter $r=1$) on a certain lattice Λ is given by the standard expression

$$Z_\Lambda = \int DUD\psi D\overline{\psi} e^{-S}, \tag{1}$$

where D denotes the multiple integration on the lattice. The action S is defined by

$$S = S_F + \beta S_G, \tag{2}$$

$$S_F = \sum_{x\in\Lambda}\left(\frac{1}{2}\sum_\mu\,[\overline{\psi}(x+\hat{\mu})(1+\gamma_\mu)U_\mu(x)\psi(x) + \overline{\psi}(x) \times(1-\gamma_\mu)U^\dagger_\mu(x)\psi(x+\hat{\mu})] - M\overline{\psi}(x)\psi(x)\right), \tag{3}$$

and $U_\mu = \exp[-iA_\mu]$, $M = 2+m$, $\beta = 1/g^2$. S_G is the standard Wilson action and the hopping parameter κ is given by $\kappa = 1/2M$. Salmhofer has shown [7] that in the strong (infinite) coupling limit $\beta = 0$ the partition function Z_Λ equals that of an eight-vertex model [more precisely, a seven-vertex model due to Eq. (5)] [23] with weights (cf. Fig. 1):

$$\omega_1 = z = \frac{1}{4\kappa^2} = M^2, \tag{4}$$

$$\omega_2 = 0, \tag{5}$$

$$\omega_3 = \omega_4 = 1, \tag{6}$$

$$\omega_5 = \omega_6 = \omega_7 = \omega_8 = \eta = \tfrac{1}{2}. \tag{7}$$

Consequently, one can write

$$Z_\Lambda = Z_\Lambda[z, \tfrac{1}{2}], \tag{8}$$

$$Z_\Lambda[z,\eta] = \sum_L z^{|\Lambda| - |L|}\eta^{C(L)}, \tag{9}$$

where L denotes any self-avoiding loop configuration, $|L|$ and $C(L)$ are the number of links and corners, respectively, a polymer configuration L is built of, and $|\Lambda|$ is the number of lattice points of the lattice Λ. $Z_\Lambda[z,\eta]$ is the partition function of a SALM with a monomer weight z and a bending rigidity η (the loop multiplicity/fugacity is 1 in this model). The same expression for Z_Λ can, of course, also be obtained for noncompact QED_2. It should be mentioned that the thermodynamic limit for a large class of models, to which the SALM belongs, has been studied in [43].

From the point of view of lattice field theory it is interesting to know the phase structure of the lattice Schwinger model. For free fermions ($\beta = \infty$), the critical value of the hopping parameter reads $\kappa_{cr}(\beta=\infty) = 1/4$. In order to pin down the critical line for $\beta < \infty$, it is of particular interest to know where it ends ($\beta = 0$). There is a critical point for $\kappa_{cr}(\beta=0) = \infty$ because then the strong coupling Schwinger model reduces to a six-vertex model whose behavior is known from its exact solution [7,23]. This point, however, is believed to be isolated and not to be the end point of the critical line starting at $\kappa_{cr}(\infty) = 1/4$ [3]. Recently, exact studies of the partition function of the strong coupling Schwinger model have been made on finite lattices [4,5]. It has been found $\kappa_{cr}(0) \simeq 0.38 - 0.39$ and that the phase transition is likely a continuous one (second order or higher) [5]. A very recent high precision MC study has confirmed these findings [characteristic signals for a second order phase transition at $\kappa_{cr}(0) = 0.3805(1)$ are found] [6].

It is worthwhile to compare the result obtained in [4–6] with the MC investigation of the SALM with a variable bending rigidity undertaken by Müser and Rys [33,34] (see also [44] for some computational background). Their investigation has been inspired by the generalized loop model of Rys and Helfrich [25]. Müser and Rys [33,34] employ a different parameter set to $\{z,\eta\}$ which we are going to describe first. Their language is thermodynamic in spirit and their parameter temperature and line stiffness $\{T,s\}$ are introduced the following way:

$$z = e^{(1-s)/T}, \tag{10}$$

$$\eta = e^{-s/T}, \tag{11}$$

which in turn entails

$$T = \frac{1}{\ln\frac{z}{\eta}}, \tag{12}$$

$$s = \frac{\ln\eta}{\ln\frac{\eta}{z}}. \tag{13}$$

For positive temperatures T, negative values of the line stiffness s correspond to values of the bending rigidity $\eta > 1$ (i.e., bending preferred) and positive values of s to $\eta < 1$ (i.e., bending is costly). The Jacobian $\mathcal{F}$ of this coor-

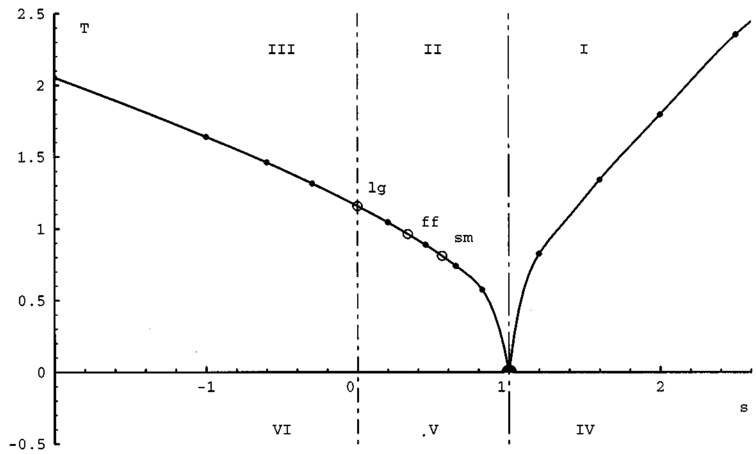

FIG. 2. The critical line of the self-avoiding loop model with a variable bending rigidity as found by MC calculations on a 64×64 lattice by Müser and Rys [33,34]. The solid line interpolates (using a polynomial interpolation scheme) between those points for which C results are available (given by black dots and hollow circles, the degeneracy point $s=1$, $T=0$ is specially emphasized by a black half disk) [34]. The domains I–VI are mapped to the correspondingly labeled domains in the $\{z,\eta\}$ plane (see Fig. 3, boundary lines are plotted in the same style in both figures). ff denotes the exactly known critical point $\{T_{cr}=2/(3\ln 2)\approx 0.962,\ s_{cr}=1/3\}$ $(z_{cr}=2)$ of the free-fermion model ($\eta=1/\sqrt{2}$) [40–42]. lg stands for the ordinary self-avoiding loop model ($\eta=1$, $s=0$) with the critical point $T_{cr}=1.157$ ($z_{cr}=2.373$) [45,46]. sm denotes the critical point $\kappa_{cr}(0)=0.38$ ($T_{cr}=0.81$, $s_{cr}=0.56$, $z_{cr}=1.73$) of the strong coupling Schwinger model ($\eta=1/2$) as found in [4–6].

dinate transformation from $\{(z,\eta)\in(0,\infty)\times(0,\infty)\backslash[(r,r):r\in(0,\infty)]\}$ to $\{(T,s)\in(0,\infty)\times\mathbf{R}\}$ reads

$$\mathcal{F}=z\,\eta\left[\ln\frac{z}{\eta}\right]^3 \tag{14}$$

$$=\frac{1}{T^3}e^{(1-2s)/T}, \tag{15}$$

and has, consequently, singular lines beyond the given range of the map. Figure 2 displays the result of Müser and Rys [33,34] for the critical line of the loop model with line stiffness and Fig. 3 displays the same information in $\{z,\eta\}$ coordinates (for further comments see the figure captions). In regions II and III the system at criticality is found to exhibit Ising-like behavior while in region I some nonuniversal behavior is seen. For better orientation, the ordinary loop gas ($\eta=1$) result is specially shown in Figs. 2 and 3 [45,46]. One immediately recognizes that the results found for the strong coupling Schwinger model [4–6] fit well onto the critical line given by Müser and Rys. Moreover, the MC result of Müser and Rys also well agrees with the exactly known critical point for the free-fermion model ($\eta=1/\sqrt{2}$) to be discussed in Sec. IV. For the ordinary loop gas it has also been found numerically [47] that critical exponents (and amplitudes, in part) agree well with those of the Ising model. The free-fermion model ($\eta=1/\sqrt{2}$, see Sec. IV), of course, also lies in the Ising universality class. This immediately suggests (for a further discussion see Sec. V) that in general the SALM with a variable bending rigidity at criticality lies in the Ising universality class (in the parameter regions II, III). From this we immediately infer that the strong coupling lattice Schwinger model ($\eta=1/2$) also belongs to this class. In [5], however, a critical exponent $\nu\simeq 0.63$ has been reported for the strong coupling Schwinger model which is quite off the Ising result $\nu=1$. The discrepancy very likely stems from finite size effects of the small lattices with nonsquare geometry investigated. These nonsquare geometry lattices, however, can be exploited in other ways as we will see in Sec. V. In agreement with the findings of Müser and Rys [33,34], the recent high precision MC study by Gausterer and Lang yields $\nu=1$ [6].

III. THE INDEPENDENT LOOP APPROXIMATION

Inasmuch as exact expressions for the partition function (9) for general η are not available, analytical attempts to understand the phase structure of the self-avoiding loop model with a variable bending rigidity have to rely on certain approximations. A method also applied in related situations

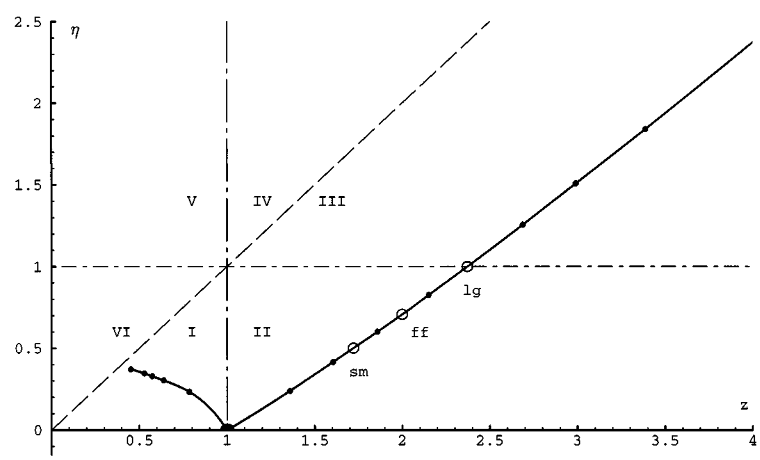

FIG. 3. This is the equivalent of Fig. 2 shown here for the $\{z,\eta\}$ coordinate system. For further explanations refer to Fig. 2. Although MC results so far are not available for region VI, it seems reasonable to expect that the critical line drawn in region I will continue in region VI and end at $z=0$, $\eta=1/2$ where it would meet the end of the critical domain of the six-vertex model ($z=0$, $\eta\geqslant 1/2$) [23].

in condensed matter physics and cosmic string physics is the so-called ''independent loop approximation'' [17–22,48,49]. The approximation is approached by writing the partition function $Z_\Lambda[z,\eta]$ as a sum over partition functions with a fixed number l of (polymer) loops:

$$Z_\Lambda[z,\eta]=z^{|\Lambda|}\sum_{l=0}^{\infty} Z_\Lambda[l,z,\eta]. \tag{16}$$

The approximation now made is to express the l-loop partition function $Z_\Lambda[l,z,\eta]$ exclusively by means of the single loop partition function $Z_\Lambda[1,z,\eta]$:

$$Z_\Lambda[l,z,\eta]=\frac{1}{l!}(Z_\Lambda[1,z,\eta])^l, \tag{17}$$

leading to

$$Z_\Lambda[z,\eta]=z^{|\Lambda|}e^{Z_\Lambda[1,z,\eta]}. \tag{18}$$

This approximation can be expected to give reasonable results for those parameter regions where the loop system is sufficiently dilute [filling on average only a certain (small) fraction (say, <0.5) of the lattice Λ]. According to Eq. (18) the investigation now may concentrate on the single loop partition function $Z_\Lambda[1,z,\eta]$. One can easily convince oneself that in the independent loop approximation the average number of loops in the system is given by the value of the single loop partition function [18, Eq. (56)]. The free energy density f reads, in the independent loop approximation ($\beta_T=1/T$),

$$\beta_T f(z,\eta)=-\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\ln Z_\Lambda[z,\eta]$$
$$=-\ln z-\lim_{|\Lambda|\to\infty}\frac{Z_\Lambda[1,z,\eta]}{|\Lambda|}. \tag{19}$$

To proceed further, the single loop partition function can now be written as sum over the loop length:

$$Z_\Lambda[1,z,\eta]=\sum_{k=1}^{\infty} z^{-2k}\mathcal{Z}_\Lambda[2k,\eta], \tag{20}$$

where $\mathcal{Z}_\Lambda[2k,\eta]$ is the conformational partition function of a single loop of length 2 [21]. (Here, we already have taken into account that on a square lattice the length of a loop is always even; unless, of course, toroidal boundary conditions on a lattice with an odd number of sites in a given direction are used.) The conformational partition function is then represented as sum over all single loop configurations L with length $2k$:

$$\mathcal{Z}_\Lambda[2k,\eta]=\sum_{L,|L|=2k}\eta^{C(L)}=\sum_{C=0}^{2k}N(2k,C)\eta^C, \tag{21}$$

where $N(2k,C)$ is the number of self-avoiding loops with length $2k$ and C corners.

Let us start with the consideration of the ordinary loop model ($\eta=1$). In this case $\mathcal{Z}_\Lambda[2k,1]$ denotes the total number of possibilities to place a self-avoiding loop with length $2k$ on the lattice Λ. It can be expressed by means of the number p_{2k} of $2k$-step self-avoiding loops per lattice site which is a standard quantity that has been investigated in the literature:

$$p_{2k}=\lim_{|\Lambda|\to\infty}\frac{\mathcal{Z}_\Lambda[2k,1]}{|\Lambda|} \tag{22}$$

The $n\to 0$ limit of the lattice O(n) spin model provides us now just with the information necessary to study the critical behavior [24,26] (see also, e.g., [28], Sec. 2). For large k, p_{2k} reads [29]

$$p_{2k}\underset{k\to\infty}{\sim}B\mu^{2k}[2k]^{-2\nu-1}+\cdots. \tag{23}$$

Here, μ denotes the connective constant (effective coordination number) for the self-avoiding walk problem on the given lattice Λ [50] and B is some lattice-dependent constant. The (universal) critical exponent ν is believed to be given in two dimensions by $\nu=\frac{3}{4}$ (obtained on a honeycomb lattice) [51,52]. Inserting Eq. (23) into Eq. (20), one finds

$$Z_\Lambda[1,z,1]=|\Lambda|B\sum_{k=1}^{\infty}[2k]^{-5/2}\left(\frac{\mu}{z}\right)^{2k}. \tag{24}$$

This is a justified approximation because we are mainly interested in the critical domain which is related to the $k\to\infty$ behavior. From Eq. (24) one easily recognizes that the critical point is given by $z_{\rm cr}=\mu$. Most recent (precise) estimates for μ on the square lattice can be found in [53–55]. We keep here only a few digits and write $\mu=2.638$. Consequently, we have $z_{\rm cr}=2.638$ ($T_{\rm cr}=1.031$) which is to be compared with the numerical result $z_{\rm cr}=2.373$ ($T_{\rm cr}=1.157$) [45,46]. Equation (24) inserted into Eq. (19) gives immediately the free energy and one recognizes that the phase transition at $z_{\rm cr}=\mu=2.638$ found within the independent loop approximation is of second order. Using ([56,57])

$$F(x,k)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\Gamma(1-k)(-\ln x)^{k-1}$$
$$+\sum_{n=0}^{\infty}\zeta(k-n)\frac{(\ln x)^n}{n!}, \tag{25}$$

one reobtains for the critical exponent of the specific heat α the hyperscaling relation $\alpha=2-2\nu$ entailing in the independent loop approximation $\alpha=1/2$ which is to be confronted with the expected Ising result $\alpha=0$ [47].

We are now prepared to study the general case with a variable bending rigidity η. First, we have to find an appropriate expression for the number $N(2k,C)$ of self-avoiding loops with length $2k$ and C corners. Let us count first the number of random nonbacktracking walks (i.e., nonclosed paths) of length $2k$ with C corners [21]. It should be stressed that the following argument does not depend on the dimension of the lattice. There are $2k-1$ vertices available the C corners can be placed at, i.e., there are $\binom{2k-1}{C}$ possibilities to do so. To each prospective corner exist $h=(2d-1)-1$ ways of bending where d is the dimension of a (hyper)cubic lattice (in our case of a square lattice $d=2$). $(2d-1)$ here is the nonbacktracking dimension of the lattice and it has to be diminished by 1 corresponding to the straight line choice. Consequently, we find

$$N_{\mathrm{NB}}(2k,C)=|\Lambda|\binom{2k-1}{C}h^{C}. \tag{26}$$

Using Eq. (21), the corresponding (nonbacktracking) conformational partition function reads then

$$\mathcal{Z}_{\Lambda}[2k,\eta]_{\mathrm{NB}}=|\Lambda|[1+h\eta]^{2k-1}. \tag{27}$$

We now obtain an approximation to the $2k$-step self-avoiding walk (SAW) conformational partition function by simply replacing $h=(2d-1)[1-1/(2d-1)]$ by $h=\overline{\mu}[1-1/(2d-1)]$ (this is based on the assumption that the self-avoidance constraints effectively encoded in $\overline{\mu}$ are independent of whether propagation is straight or bent) where $\overline{\mu}$ is a certain effective coordination number to be determined in a moment. This yields in our case

$$\mathcal{Z}_{\Lambda}[2k,\eta]_{\mathrm{SAW}}=|\Lambda|U(2k)\left[1+\frac{2\overline{\mu}\eta}{3}\right]^{2k-1}. \tag{28}$$

The additional factor $U(2k)$ also to be determined below takes care of some additional length dependence which might show up in the transition from nonbacktracking to SAW. Specializing Eq. (28) to $\eta=1$ we obtain the total number of SAW of length $2k$ on the square lattice which has to be confronted with the standard expectation [29] for large k

$$c_{2k}=\lim_{|\Lambda|\to\infty}\frac{\mathcal{Z}_{\Lambda}[2k,1]_{\mathrm{SAW}}}{|\Lambda|}\underset{k\to\infty}{\sim}A\mu^{2k}[2k]^{\gamma-1}+\cdots, \tag{29}$$

where A is some lattice-dependent constant and $\gamma=43/32$ [51,52]. From Eq. (29) we immediately find

$$\overline{\mu}=\tfrac{3}{2}(\mu-1), \tag{30}$$

$$U(2k)=A\mu[2k]^{\gamma-1}. \tag{31}$$

Now, we need to know the conformational partition function for the self-avoiding loop problem. In order to be able to make further progress let us assume that $N(2k,C)$ for self-avoiding loops is just a certain fraction $M(2k)$ of $N_{\mathrm{SAW}}(2k,C)$ at least for large k independent of the number of corners C. According to Eq. (21), we can then write

$$\mathcal{Z}_{\Lambda}[2k,\eta]=M(2k)\mathcal{Z}_{\Lambda}[2k,\eta]_{\mathrm{SAW}}, \tag{32}$$

which reads, after having taken into account Eqs. (28), (30), (31),

$$\mathcal{Z}_{\Lambda}[2k,\eta]=|\Lambda|M(2k)A\mu[1+(\mu-1)\eta]^{2k-1}[2k]^{\gamma-1}. \tag{33}$$

$M(2k)$ is the fraction to be determined. We here simply ignore the fact that for any loop the number of corners is even, necessarily. This is justified for the study of the $k\to\infty$ behavior we are primarily interested in. For $\eta=1$ we have

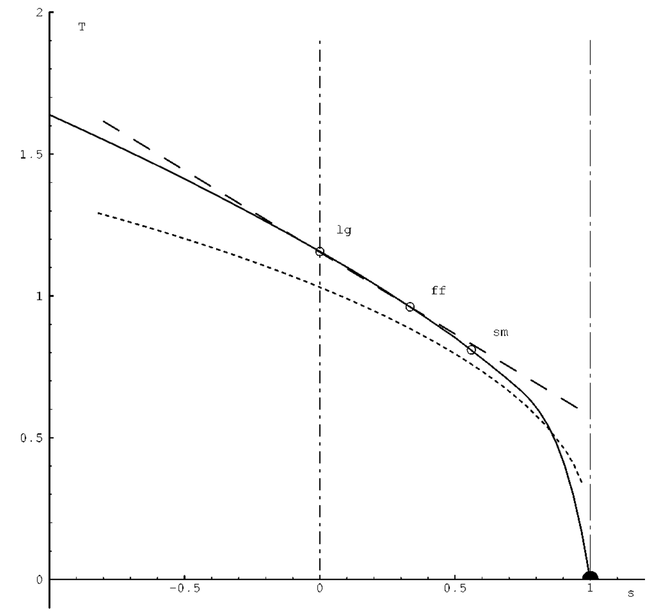

FIG. 4. The critical line according to the results of the independent loop approximation (37) (dotted line) and of the free-fermion model related approach (52) (dashed line) in comparison with the MC result (solid line) of Müser and Rys [33,34]. For further explanations refer to Fig. 2.

already displayed an expression [Eq. (23)] which now serves as reference expression to determine $M(2k)$. We obtain

$$M(2k)=\frac{B}{A}\,[2k]^{-2\nu-\gamma} \tag{34}$$

leading to

$$Z_{\Lambda}[1,z,\eta]=|\Lambda|\frac{B\mu}{[1+(\mu-1)\eta]}\times\sum_{k=1}^{\infty}[2k]^{-5/2}\left(\frac{1+(\mu-1)\eta}{z}\right)^{2k}. \tag{35}$$

Consequently, the critical line is found to be

$$\eta_{\mathrm{cr}}(z_{\mathrm{cr}})=\frac{(z_{\mathrm{cr}}-1)}{(\mu-1)}. \tag{36}$$

This translates into the $\{T,s\}$ coordinate system as

$$s_{\mathrm{cr}}(T_{\mathrm{cr}})=T_{\mathrm{cr}}\ln[e^{1/T_{\mathrm{cr}}}-\mu+1]. \tag{37}$$

Applying Eq. (25) to Eq. (35) yields, for arbitrary η, $\alpha=1/2$. From the above equations we obtain for the strong coupling Schwinger model $z_{\mathrm{cr}}=1.819$ ($T_{\mathrm{cr}}=0.774$, $s_{\mathrm{cr}}=0.537$). We find for the critical hopping parameter

$$\kappa_{\mathrm{cr}}(0)=\frac{1}{\sqrt{2(\mu+1)}}=0.371, \tag{38}$$

which is to be compared with the result of computer studies $\kappa_{\mathrm{cr}}(0)\simeq0.38-0.39$ [33,34, 4–6].

It should be emphasized that the result of our approximate consideration [Eq. (37)] entails $s_{\mathrm{cr}}\to1$ for $T_{\mathrm{cr}}\to0$ ($\eta_{\mathrm{cr}}\to0$

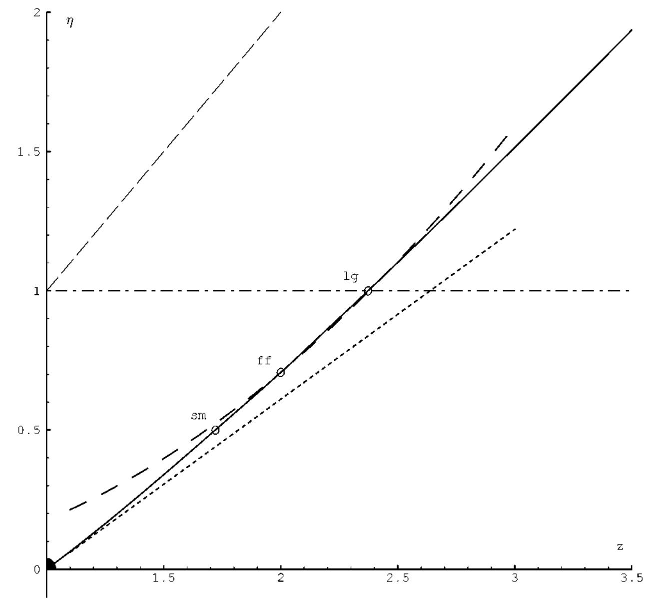

FIG. 5. This is the equivalent of Fig. 4 shown here for the $\{z,\eta\}$ coordinate system and relating to Eqs. (36) (dotted line) and (51) (dashed line).

for $z_{\rm cr}\to 1$). This is well in line with the expectation spelled out in [33]. In the limit $\eta\to 0$, the SALM degenerates into an ensemble of straight lines (on a torus) and it can, therefore, be compared with certain limits of other models encompassing the same limit. In Sec. 2 of [58], it has been found that for the straight line system $z=1$ is the critical point. This is supported by numerical studies reported in [59] (see, in particular, Fig. 3 therein). Corresponding exact information is furthermore available from the solution of the five-vertex model [60] [see Sec. IV, in particular, Eq. (30) and Fig. 1 therein]. Finally, it seems to be interesting that the independent loop approximation delivers the correct result for the critical point of the SALM in the limit $\eta\to 0$.

The critical line [Eqs. (36) and (37)] obtained within the independent loop approximation is plotted in Figs. 4 and 5. One recognizes that the critical line found analytically agrees qualitatively quite well with the result of the MC calculation of Müser and Rys [33,34]. However, it is clear that the validity of the independent loop approximation is confined to the low (polymer) loop density domain. The high density result of Müser and Rys [33,34] displayed in region I of Figs. 2 and 3 cannot be obtained within the present scheme. It is also well known [17,48] that while within the independent loop approximation the critical line can be determined in a qualitatively correct way, results for critical exponents are less accurate. This also applies to our case as we have seen above. While we would have expected, e.g., for the critical exponent α the Ising result ($\alpha=0$; at least for $\eta=1$ [47]) within the independent loop approximation we see $\alpha=1/2$. Finally, it should also be mentioned that the relative simplicity of the independent loop approximation has its price because so far no way of systematically improving it is known and one consequently has no quantitative control over the approximation made. Perhaps this drawback is offset by the applicability of the approximation to systems in any dimension.

IV. THE EXACT SOLUTION OF THE SELF-AVOIDING LOOP MODEL WITH A BENDING RIGIDITY $\eta=1/\sqrt{2}$ AND ITS USE

While in general the self-avoiding loop model (SALM) with a variable bending rigidity (which is equivalent to a seven-vertex model due to $\omega_2=0$) cannot be studied exactly so far, there exists an exact solution to it for $\eta=1/\sqrt{2}$ first investigated by Priezzhev [40] (see also [41]). This solution was later rediscovered by Blum and Shapir [42] who apparently were unaware of the earlier work of Priezzhev. The solution relies on the general study of the eight-vertex model by Fan and Wu [38,39]. They found that the eight-vertex model is exactly solvable if the free-fermion condition

$$\omega_1\omega_2+\omega_3\omega_4=\omega_5\omega_6+\omega_7\omega_8 \tag{39}$$

is satisfied (cf. Fig. 1 for the labeling of the vertices). Inserting Eqs. (4)–(7) into Eq. (39) (η taken arbitrary here), one immediately finds that the free-fermion condition is satisfied for $\eta=1/\sqrt{2}$ ($T=2s/\ln 2$). The partition function for the SALM with a bending rigidity $\eta=1/\sqrt{2}$ has been found in [40,42] by standard methods [61,62]. The free energy density f reads

$$\beta_T f(z,1/\sqrt{2})=-\frac{1}{8\pi^2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\,\ln[2+z^2+2z\cos\theta+2z\cos\phi+2\cos\theta\cos\phi]. \tag{40}$$

A second order phase transition occurs for $z_{\rm cr}=2$ [$T_{\rm cr}=2/(3\ln 2)=0.962$, $s_{\rm cr}=1/3$] which will be of main interest to us. There is, of course, also a critical point at $z=0$ related to the exactly solvable six-vertex model [23,7]. Because the system can be represented by means of free fermions [62] the SALM with a bending rigidity $\eta=1/\sqrt{2}$ lies in the Ising universality class [42]. In accordance with this it has been shown (for ω_2 chosen arbitrarily) that the partition function of the free-fermion model can be expressed in terms of that of the regular Ising model [63]. It finally deserves mention that the result of the MC calculation of Müser and Rys [33,34] is in complete agreement with the exact solution of the free-fermion model (cf. Figs. 2 and 3).

The above exact solution lying on the critical line of the SALM with a variable bending rigidity is quite useful because this way one may take advantage of universality arguments to draw conclusions about the model at criticality for a fairly wide range of the bending rigidity η. This will be discussed further in Sec. V. Here, we will study the approximate calculation of the critical line in a neighborhood of the model for $\eta=1/\sqrt{2}$. This discussion is in a certain sense a generalization of that given in [41,40]. Let us write the partition function (9) as

$$Z_\Lambda[z,\eta]=\sum_{l=0}^{|\Lambda|} z^{|\Lambda|-2l}\sum_{L,|L|=2l}\eta^{C(L)} \tag{41}$$

$$=\sum_{l=0}^{|\Lambda|} z^{|\Lambda|-2l}\sum_{L,|L|=2l}\sum_{k=0}^{\infty}\frac{1}{k!}[C(L)\ln\eta]^k \tag{42}$$

$$= \sum_{l=0}^{|\Lambda|} z^{|\Lambda|-2l} \sum_{k=0}^{\infty} \frac{[\ln \eta]^k}{k!} \langle C(L)^k \rangle_{2l}. \quad (43)$$

Here,

$$\langle C(L)^k \rangle_{2l} = \sum_{L,|L|=2l} C(L)^k. \quad (44)$$

One may now express $\langle C(L) \rangle_{2l}$ as

$$\langle C(L) \rangle_{2l} = \overline{C_{2l}} N(2l), \quad (45)$$

$$\overline{C_{2l}} = 2l n_C(2l). \quad (46)$$

$N(2l) = \langle 1 \rangle_{2l}$ denotes the number of (multi)loop configurations of total length $2l$ on the lattice Λ and $n_C(2l)$, $0 \leqslant n_C(2l) \leqslant 1$, stands for the average relative density of corners in the considered loop ensemble of total length $2l$. It is a purely geometrical quantity as it does not depend on η. The following, of course, holds for the higher moments of C:

$$0 \leqslant \langle C(L)^k \rangle_{2l} \leqslant (2l)^k \ N(2l). \quad (47)$$

One can now write

$$Z_\Lambda[z,\eta] = z^{|\Lambda|} \sum_{l=0}^{|\Lambda|} N(2l) \left[\frac{\eta^{n_C(2l)}}{z} \right]^{2l} \times \left[1 + \frac{\langle [C(L) - \overline{C_{2l}}]^2 \rangle_{2l}}{2N(2l)} (\ln \eta)^2 + \cdots \right], \quad (48)$$

where the ellipsis stands for a series in higher order correlation functions of the corner number C and $\ln \eta$. The critical behavior of the system is related to large l. For $l \to \infty$, n_C tends to some value $\overline{n_C}$ and consequently points (z_1, η_1), (z_2, η_2) on the critical line not too far away from each other should obey to leading approximation the equation

$$\frac{\eta_1^{\overline{n_C}}}{z_1} = \frac{\eta_2^{\overline{n_C}}}{z_2}. \quad (49)$$

The contribution of correlation functions of C should be expected to be of minor importance in Eq. (48), leading to corrections to the leading behavior only. Inserting into Eq. (49) the exactly known critical point $(z,\eta) = (2, 1/\sqrt{2})$ of the free-fermion model leads in a neighborhood of it to the equation for the critical line:

$$\eta_{\rm cr}(z_{\rm cr}) = 2^{-(\overline{n_C}+2)/2\overline{n_C}} z_{\rm cr}^{1/\overline{n_C}}. \quad (50)$$

The only unknown quantity in this expression is the average relative corner density $\overline{n_C}$. Its value $n_C(2l \to \infty)$ is related to the high density polymer limit which is reached for $z \to 0$. $\overline{n_C}$ has been calculated in [40,41] for the free-fermion model and found to have the value 1/2. Expressions for the correlation functions of C for $l \to \infty$ can also be obtained along the same lines by tedious, but standard methods ([64,65]; the latter is the English original of Ref. 16 in [40]). So, we end up with the following equation for the critical line of the SALM with a variable bending rigidity in the neighborhood of the free-fermion point (cf. also Figs. 4 and 5):

$$\eta_{\rm cr}(z_{\rm cr}) = 2^{-5/2} z_{\rm cr}^2. \quad (51)$$

This equation reads, in $\{T,s\}$ coordinates,

$$T_{\rm cr}(s_{\rm cr}) = \frac{2(2 - s_{\rm cr})}{5 \ln 2}. \quad (52)$$

Consequently, we obtain for the strong coupling Schwinger model $z_{\rm cr} = 2^{3/4} \approx 1.682$ $[T_{\rm cr} = 4/(7\ln 2) \approx 0.824$, $s_{\rm cr} = 4/7 \approx 0.571]$. This yields, for the critical hopping parameter,

$$\kappa_{\rm cr}(0) = 2^{-11/8} \approx 0.386. \quad (53)$$

We see (cf. also Figs. 4 and 5) that the approximation based on the exactly solvable free-fermion model yields a numerical value of the critical hopping parameter fairly close to the result found in computer studies $\kappa_{\rm cr}(0) \simeq 0.38 - 0.39$ [33,34,4–6]. As mentioned above, systematic improvements can be obtained by taking into account correlation functions of C. This apparently is necessary as one learns from Figs. 4 and 5 if one wants to find the critical line beyond the region defined by the critical points of the ordinary loop model and the strong coupling Schwinger model, respectively.

V. DISCUSSION AND CONCLUSIONS

Let us first have a look at the larger picture emerging for the critical behavior of the self-avoiding loop model (SALM) with a variable bending rigidity. There is one point on the critical line known exactly from the solution of the free-fermion model ($\eta = 1/\sqrt{2}$, $z_{\rm cr} = 2$) [40–42]. For this model it is established that the phase transition is Ising-like, i.e., the model experiences a second order phase transition with exactly the same critical exponents as the regular Ising model. By the argument of universality we may conclude that neighboring models which lie on the same critical line exhibit the same behavior. This, in particular, concerns the ordinary loop model ($\eta = 1$) and the strong coupling Schwinger model ($\eta = 1/2$). For the ordinary loop model this has been confirmed by MC investigations in the past [47]. For the strong coupling Schwinger model this consideration specifies the previously unknown character of the phase transition and supports the recent suggestion that the transition might be a continuous one [5]. The very recent high precision MC study performed by Gausterer and Lang has confirmed this insight [6].

In order to extend the understanding of the SALM with a variable bending rigidity at criticality let us consider the central charge c of the corresponding conformal field theory (CFT). Helpful information can be obtained most easily for the free-fermion model considered in Sec. IV. First, it seems worthwhile mentioning that the regular Ising model can be understood as a special free-fermion model [62]. A preliminary investigation along the lines given for the Ising model in [66] indicates that the SALM with a bending rigidity $\eta = 1/\sqrt{2}$ can be represented at the critical point $z_{\rm cr} = 2$ by one massless (continuum) Majorana fermion (as in the special case of the Ising model just one-half of the fermionic modes needed to express the partition function becomes

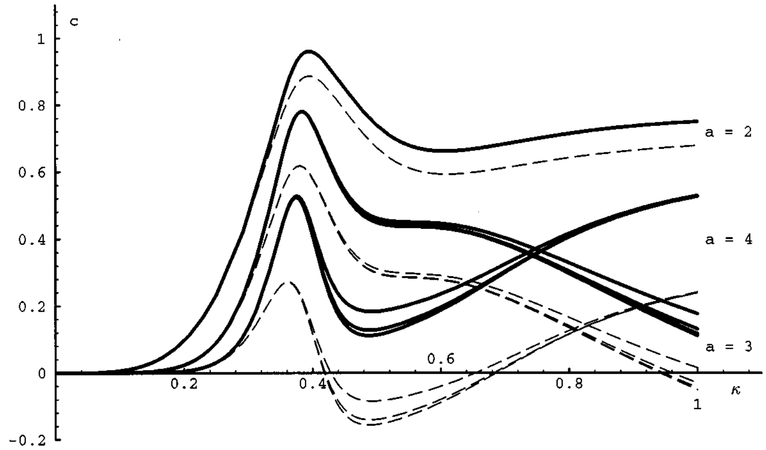

FIG. 6. The function c [see Eq. (55)] in dependence on $\kappa(0)$. The solid lines are the result related to the bulk free energy density $f[(2\kappa)^{-2},1/2]$ calculated on a 8×8 lattice while the dashed lines stand for the results related to the 6×6 lattice free energy density. The following b values for the $a\times b$ tori have been used (curves in order from top to bottom); $a=2$, $b=32,48,64$ (practically one line); $a=3$, $b=16,24,32,48$; $a=4$, $b=16,24,32$ [76]. The value of c at the critical point $\kappa_{\mathrm{cr}}(0)\simeq0.38-0.39$ has to be compared with the expectation for the central charge (for a discussion see the main text).

massless at the critical point). Consequently, this suggests that the critical SALM at the free-fermion point is equivalent to a $c=1/2$ CFT. The central charge should be expected not to change continuously on the critical line in the neighborhood of the free-fermion model; therefore CFT's corresponding to the SALM with a variable bending rigidity should all exhibit $c=1/2$ (along the critical line in the regions II and III). This of course entails that the strong coupling Schwinger model at criticality should be equivalent to a $c=1/2$ CFT. Consequently, in accordance with Zamolodchikov's c theorem [67,68], $z\neq0$ is related to a flow from the six-vertex model $(z=0)$ having $c=1$ [69,70] towards a model with $c=1/2$ (as discussed in general terms by Salmhofer [7]; more precisely, this applies for $\eta\geqslant1/2$, where the six-vertex model is critical [23]). This view is supported by still another argument stemming from the SALM with no bending rigidity (i.e., $\eta=1$). As by universality the central charge should not depend on the lattice, that the SALM is defined on, we may rely on results obtained for the SALM on the honeycomb lattice (see, e.g., [71]). The SALM on the honeycomb lattice can be viewed as a $\mathrm{O}(n=1)$ model which has a central charge $c=1/2$ [72] in agreement with the above discussion. The consideration of the $\mathrm{O}(n)$ model on the square lattice confirms this result [73].

We are now going to test the above insight by calculating the central charge for the strong coupling Schwinger model. This can be done most easily by considering the model on a strip of width a and length $b\to\infty$ [74,75]. The central charge is related to the partition function (on a torus) by the formula [up to higher order terms in $1/a$; f is the (bulk) free energy density on the infinite plane]

$$\lim_{b\to\infty}\frac{\ln Z_{\Lambda(a\times b)}[z_{\mathrm{cr}},1/2]}{b}=af(z_{\mathrm{cr}},1/2)+c\,\frac{\pi}{6}\,\frac{1}{a}. \qquad (54)$$

We, however, will approach the study of the central charge of the strong coupling Schwinger model by means of the exact partition functions calculated on finite lattices in [4,5,76]. In Fig. 6 we have plotted the function

$$c(z)=\frac{a}{b}\,\frac{6}{\pi}\{\ln Z_{\Lambda(a\times b)}[z,1/2]-abf(z,1/2)\} \qquad (55)$$

for different tori ($a=2,3,4$, for the values of b used see Fig. 6) dependent on κ [$z=(2\kappa)^{-2}$ has been inserted]. $f(z,1/2)$ has been calculated by means of the 8×8 (solid lines) [4,76] and the 6×6 (dashed lines) [76] lattice partition functions, respectively. For the moment let us concentrate on the discussion of the results obtained by means of the bulk free energy density on the 8×8 torus (solid lines). For sufficiently large b the function $c(z)$ should be expected to approach the value of the central charge at the critical point. However, one has to be aware of the fact that on the very narrow (with respect to a) tori considered, massless and massive fields can contribute comparable amounts to the Casimir energy. Inasmuch as the central charge is calculated by means of Eqs. (54) and (55) from the Casimir energy results obtained from very narrow tori may turn out to be misleading. In part, this is what we observe from Fig. 6. The result for the $a=2$ torus rather suggests $c=1$ (or some value close to it); however, the torus is so narrow that massless and massive fields contribute comparably to the Casimir energy. Consequently, in agreement with our expectation $c=1/2$ for the wider $a=3$ torus we already observe a much smaller value of $c(z)$ at the critical point and for $a=4$ some value close to 1/2 is found. However, it turns out that the sizes of the tori for which the exact partition functions have been calculated so far are too small to allow any final conclusions for the central charge of the strong coupling Schwinger model at criticality. In particular, this applies to the square lattices the bulk free energy density is derived from. From Fig. 6 one easily recognizes that the curves related to the 6×6 bulk reference system (dashed curves) differ quite significantly from those calculated for the 8×8 system. Unless numerical stability for $c(z)$ near the critical point is obtained, no final conclusion can be drawn. Therefore, only partition functions calculated on considerably larger lattices will allow us to numerically test the prediction $c=1/2$ in a reliable way.

To conclude, the study of the self-avoiding loop model with a variable bending rigidity presented in this paper enhances the understanding of the critical behavior of the strong coupling Schwinger model with Wilson fermions. We find that a second order phase transition, which lies in the Ising model universality class, takes place at some finite value of the hopping parameter $\kappa_{\mathrm{cr}}(0)$. Using certain approximate analytic methods the value of the critical hopping parameter is confirmed to lie at $\kappa_{\mathrm{cr}}\simeq0.38-0.39$ in accordance with recent numerical investigations [4–6]. Certain arguments considered suggest that the strong coupling Schwinger model at criticality is equivalent to a $c=1/2$ CFT. Finally, it should be mentioned that the discussion of the self-avoiding loop model with a variable bending rigidity seems to have a certain significance beyond the one-flavor Schwinger model. Recent investigations indicate that for the qualitative understanding of the critical behavior of the general N_f-flavor strong coupling Schwinger model with Wilson fermions the self-avoiding loop model with a bending rigidity $\eta=2^{-N_f}$ might be relevant [77].

From a technical point of view, the present paper studies the application of the independent loop approximation to the qualitative and in part quantitative exploration of the phase structure of the self-avoiding loop model with a variable bending rigidity in two dimensions. Comparison with known numerical results [33,34] shows that this method delivers a fairly correct picture for sufficiently low (polymer) loop densities. This is encouraging because the method is equally applicable to higher dimensions, while the analytic approach based on the exactly solvable free-fermion model presented in Sec. IV is at least in part specific to two dimensions. This suggests that the independent loop approximation might successfully be applied also to analogous systems in higher dimensions where it can be expected to become even more accurate (e.g., to strong coupling QCD in four dimensions where the critical hopping parameter has recently been studied by other methods [78]).

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