$O(\alpha)$ Radiative Correction to the Casimir Energy for Penetrable Mirrors

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The leading radiative correction to the Casimir energy for two parallel penetrable mirrors (realized by δ-function potentials) is calculated within QED perturbation theory. It is found to be of order α like the known radiative correction for ideally reflecting mirrors from which it differs, for a mirror distance much larger than the electron Compton wavelength, only by a monotonic, powerlike function of the frequency at which the mirrors become transparent. This shows that the $O(\alpha^2)$ radiative correction calculated recently by Kong and Ravndal for ideally reflecting mirrors on the basis of effective field theory methods remains subleading even for the physical case of penetrable mirrors. [S0031-9007(98)07449-3]

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The existence of zero point fluctuations of all quantum fields is shaping our modern view of the physical vacuum which is thought to be a complicated medium [1]. As these vacuum fluctuations cannot be made to disappear completely, their modification by means of external fields or boundary conditions as they occur at conducting surfaces plays a significant role in studying the vacuum properties. The Casimir effect [2,3], i.e., the mutual attraction of two parallel uncharged conducting plates (mirrors) *in vacuo*, represents a key phenomenon in the modern study of the vacuum (in recent experiments it has become well established quantitatively [4,5]). For ideally reflecting parallel mirrors the distance-dependent part of the vacuum energy (per unit area of the mirrors) reads ($\hbar = c = 1$)

$$E_0 = -\frac{\pi^2}{720}\frac{1}{d^3}, \tag{1}$$

where d denotes the distance between the mirrors. Within quantum electrodynamics (QED), this result can be calculated by means of free field theory and the Casimir pressure p can be derived from it using the relation $p = -\partial E_0/\partial d$.

Radiative corrections to the Casimir energy have been studied for QED in [6–12] and for scalar field theory in [13–17]. Although they are of no experimental significance in QED (but possibly within the bag model in QCD) their correct calculation is a challenge for the ability to understand the underlying physical structures and the field theoretic methods applied in their study. Within standard QED perturbation theory the calculation of the (leading) $O(\alpha)$ radiative correction to the Casimir energy (1) was performed in [6] (and has been confirmed by an independent method in [7]). Recently, the analogous calculation has successfully been completed for the single sphere geometry in [8]. The $O(\alpha)$ radiative correction originates from the two-loop vacuum diagram shown in Fig. 1. Qualitatively, the result for the leading correction ΔE_0 to the ground state energy E_0 turns out to be of the order

$$\frac{\Delta E_0}{E_0} \sim \alpha\,\frac{\lambda_c}{d}, \tag{2}$$

where α is the fine structure constant, $\lambda_c = 1/m$ is the electron Compton wave length (m is the electron mass), and $d \gg \lambda_c$ is the characteristic geometric length (distance between the mirrors or radius of the sphere, respectively). As in the calculation leading to Eq. (1), it was assumed that for the photon vacuum fluctuations conductor boundary conditions apply at the ideally reflecting mirror surfaces and, in addition, that there are no boundary conditions for the spinor field (which would only contribute geometry-dependent terms which are exponentially suppressed for $d \gg \lambda_c$ [18]).

Effective field theory methods have widely been used in various branches of quantum field theory, including QED, in recent years ([19,20] and references therein). They are thought to be able to correctly encapture the relevant low energy information. As the Casimir effect is a true infrared phenomenon, it appears natural to employ these methods also to the calculation of Casimir energies. This idea has been pursued by Kong and Ravndal [11,12] for two parallel ideally reflecting mirrors. In contrast to Eq. (2), they find for the leading radiative correction ΔE_0 to the Casimir energy (1)

$$\frac{\Delta E_0}{E_0} \sim \alpha^2\left(\frac{\lambda_c}{d}\right)^4, \tag{3}$$

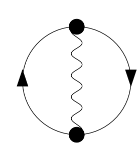

FIG. 1. Two-loop vacuum diagram from which the leading $O(\alpha)$ correction to the Casimir energy originates.

which is suppressed relative to Eq. (2) by one power of α and three powers of λ_c/d. Kong and Ravndal observe that Eq. (2) cannot be obtained by means of effective field theory methods [in contrast, Eq. (3) can, of course, be obtained within full QED as a nonleading correction to Eq. (1)]. Although these authors have chosen not to comment on the problem, looking at [6,7] one quickly recognizes the reason for this failure. Effective field theory is always based on a derivative expansion of the effective action. On the other hand, the calculation of the correction (2) technically relies on the consideration of the discontinuity of the (one-loop) photon polarization operator [see Eq. (18), below] which is part of the QED effective action and which the Uehling terms discussed by Kong and Ravndal derive from. It is rather clear that this discontinuity, which only starts at the pair-production threshold and which, therefore, has rather to be viewed as a high energy feature, cannot be seen in any finite order of the derivative expansion of the effective action (it is a nonperturbative phenomenon with respect to the derivative expansion). Consequently, effective field theory methods are unable to reproduce Eq. (2). From a formal point of view, all this is perfectly clear and would not require any further study, if not for the following question with respect to the physical adequacy of the calculations performed in [6–8]: To what extent does the result of the formal calculation within full QED, which relies on the discontinuity of the polarization operator, encapture real physics if one takes into account that the frequency at which the mirrors become transparent is much smaller than the pair-production threshold $2m$? Could it be that the correction (2) calculated for ideally reflecting mirrors, although formally the leading one relative to (3), is suppressed for realistic mirrors in such a way that the effective field theory result (3) turns out to be the leading correction to the Casimir energy (1) from a physical point of view? To answer this question, in the present paper we consider a mathematical model for penetrable mirrors. We show that the leading radiative correction to the Casimir energy continues to be of the same order as given in Eq. (2). It depends in a monotonic and powerlike manner on the parameter describing the threshold at which the mirrors become transparent. From an effective field theory point of view, this result seems to teach an interesting lesson.

The photon propagator in the covariant gauge with boundary conditions on a surface S composed of ideally reflecting mirrors was initially calculated in [6], and a more accessible discussion is given in [8]. It reads

$$ {}^S D^{c}_{\mu\nu}(x,y) = D^{c}_{\mu\nu}(x-y) - \bar{D}_{\mu\nu}(x,y), \qquad (4) $$

where $D^{c}_{\mu\nu}(x-y)$ is the free-space causal propagator and $\bar{D}_{\mu\nu}(x,y)$ depends on the boundary. In plane geometry it can be represented as

$$ \bar{D}_{\mu\nu}(x,y) = \sum_{s=1,2} E^{s}_{\mu}(x)\bar{D}(x,y)E^{s}_{\nu}(y), \qquad (5) $$

where $E^{s}_{\mu}(x)$ are the two suitably chosen photon polarizations which are affected by the boundaries (see [8] for details). Here, $\bar{D}(x,y)$ is the mirror-dependent part of the corresponding scalar Green's function ${}^S D^{c}(x,y)$, which fulfills Dirichlet boundary conditions at the mirrors. It appears to be reasonable to apply the formalism which was used for the derivation of the representation (5) also in the case of penetrable mirrors. From Eq. (5) it is clear that it is sufficient to concentrate in the following on a massless scalar field which fulfills appropriate boundary conditions.

Penetrable mirrors can be modeled by means of delta function potentials with support at the parallel mirror planes $x_3 = d_i$ $(i = 1,2)$ ([18,21], see [22] for a related study). For a scalar field of mass μ the wave equation reads

$$ \left[\Box + \mu^2 - 2a\sum_{i=1,2}\delta(x_3 - d_i)\right]\varphi(x) = 0\,. \qquad (6) $$

The potential is attractive for $a > 0$ (and the spectrum contains bound states), and it is repulsive for $a < 0$. The limit $a \to -\infty$ corresponds to imposing Dirichlet boundary conditions at $x_3 = d_1, d_2$. Although all subsequent formulas remain valid for arbitrary values of a we restrict ourselves to $a \le 0$. The delta function potential can be reformulated in terms of matching conditions

$$ \varphi(x)|_{x_3=d_i-0} = \varphi(x)|_{x_3=d_i+0}\,, \qquad (7) $$

$$ \frac{\partial}{\partial x_3}\varphi(x)|_{x_3=d_i-0} = \frac{\partial}{\partial x_3}\varphi(x)|_{x_3=d_i+0} + 2a\varphi(x)|_{x_3=d_i}\,. \qquad (8) $$

The parameter a sets the scale for the (smeared) threshold above which the mirrors become transparent. For the moment, consider Eq. (6) with just one delta function potential (at $x_3 = 0$). The part of its solution depending on x_3

$$ \varphi(x_3) = [e^{ikx_3} + r(k)e^{-ikx_3}]\Theta(-x_3) + t(k)e^{ikx_3}\Theta(x_3) \qquad (9) $$

has a transmitted and a reflected wave with the coefficients $r(k) = ia/(k - ia)$, $t = 1 + r$. The propagator for a scalar field in the presence of one delta function potential can be written as

$$ {}^S D^{c}(x,y) = \int \frac{d^3k}{(2\pi)^3}\frac{e^{ik_\beta(x^\beta - y^\beta)}}{-2i\Gamma} \times \{e^{i\Gamma|x_3-y_3|} - r(\Gamma)e^{i\Gamma(|x_3|+|y_3|)}\}, \qquad (10) $$

with $\Gamma = \sqrt{k_0^2 - k_1^2 - k_2^2 - \mu^2 + i0}$, $\beta = 0,1,2$ [cf. Eqs. (3.10) and (3.12) in [18]]. When the reflection coefficient $r(\Gamma)$ approaches zero for $|\Gamma| \gg |a|$, this propagator turns into the free-space propagator (in Fourier space). Note, that Eq. (10) also applies for a general choice of the reflection coefficient $r(\Gamma)$ which describes dispersive mirrors [entailing $a \to -i\Gamma r(\Gamma)/[1 + r(\Gamma)]$ in the Fourier transformed Eqs. (6) and (8)].

The Casimir energy for two parallel planes with the reflection coefficient $r(\Gamma)$ corresponding to delta function potentials has been calculated for scalar and spinor fields in [18,21] and with a more general choice of $r(\Gamma)$ for a scalar field in [22]. The propagator for a scalar field in the background of two partly transmitting mirrors modeled by delta function potentials [cf. Eq. (6)] can be written as ${}^S D^c(x,y) = D^c(x-y) - \bar{D}(x,y)$ where $D^c(x-y)$ is the free-space causal propagator and

$$\bar{D}(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{\frac{a}{2} e^{ik_\beta(x^\beta - y^\beta)}}{(\Gamma - ia)^2 + a^2 \exp(2i\Gamma d)} \times \left\{\left[\left(1 - \frac{ia}{\Gamma}\right) e^{i\Gamma(|x_3 - d_1| + |y_3 - d_1|)} + \frac{ia}{\Gamma} e^{i\Gamma(|x_3 - d_1| + |y_3 - d_2| + d)}\right] + (d_1 \leftrightarrow d_2)\right\} \tag{11}$$

is the boundary-dependent part, $d = |d_2 - d_1|$ [18,21]. [Equations (3.11) and (3.13) in Ref. [18] contain misprints, and the correct expression is given in Ref. [21], Eq. (7).] It is clear that we need to set $\mu = 0$ in the following.

The $O(\alpha)$ radiative correction to Eq. (1) has to be derived from the vacuum diagram shown in Fig. 1. The corresponding (divergent) shift in the vacuum energy (per unit area of the mirrors, TV_2 is the infinite space-time volume in the directions $\beta = 0, 1, 2$) reads

$$\Delta E_0 = \frac{i}{2TV_2} \int d^4x \int d^4y [D^c_{\mu\nu}(x-y) - \bar{D}_{\mu\nu}(x,y)] \times \Pi^{\mu\nu}(x-y), \tag{12}$$

where $\Pi_{\mu\nu}(x) = [g_{\mu\nu}\Box - \partial_\mu\partial_\nu]\Pi(x^2)$ is the standard one-loop photon polarization tensor. The boundary-independent (divergent) first term connected with the free-space propagator $D^c_{\mu\nu}(x)$ can be disregarded in the following. Any effect from the renormalization of the photon polarization tensor $[\Pi(x^2) \sim \text{const}\ \delta^{(4)}(x)]$ can also be disregarded as it also leads to boundary-independent terms [by virtue of the defining equation of ${}^S D^c_{\mu\nu}(x,y)$]. We can now insert Eqs. (5) and (11) into Eq. (12). The sum over the Lorentz indices which involves the sum over the polarization vectors E^s_μ can be performed immediately and simply results in a factor of 2. This corresponds to the known fact that in a plane geometry the two photon polarizations have to obey the same boundary conditions (this is true for Dirichlet boundary conditions as well as for the applied penetrable mirrors model). To proceed further, it is useful to introduce the Fourier transform of $\Pi(x^2)$

$$\Pi(x^2) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\Pi}(k^2). \tag{13}$$

Then, in Eq. (12) the integrations over x and y can be carried out explicitly. We arrive at

$$\Delta E_0 = i \int \frac{d^4k}{(2\pi)^4} k^2 \tilde{\Pi}(k^2) \tilde{\bar{D}}(k), \tag{14}$$

where

$$\tilde{\bar{D}}(k) = -\frac{4\Gamma a}{(k^2)^2} \frac{[\Gamma - ia + ia\cos(k_3 d) e^{i\Gamma d}]}{[(\Gamma - ia)^2 + a^2 \exp(2i\Gamma d)]}. \tag{15}$$

In deriving Eq. (15) we have made use of the integral ($\Im\Gamma > 0$)

$$\int_{-\infty}^{\infty} dx_3\, e^{i\Gamma|x_3 - d| + ik_3x_3} = \frac{2i\Gamma}{k^2} e^{ik_3 d}. \tag{16}$$

It is useful to further transform the representation (14) by making use of the analytic properties of $\tilde{\Pi}(k^2)$. It has a cut starting at $k^2 = 4m^2$ with the discontinuity (disc) ($k^2 \geq 4m^2$)

$$\text{disc}\, \tilde{\Pi}(k^2) = \tilde{\Pi}(k^2 + i0) - \tilde{\Pi}(k^2 - i0) = -\alpha \frac{2i}{3} \sqrt{1 - \frac{4m^2}{k^2}} \left(1 + \frac{2m^2}{k^2}\right). \tag{17}$$

The integration path along the real k_3 axis can be deformed upwards into the complex k_3 plane in such a way that it encloses the cut (cf. Fig. 2 on p. 045003-8 in [8]). By means of

$$\int_{-\infty}^{\infty} dk \frac{\tilde{\Pi}(\Gamma^2 - k^2)}{\Gamma^2 - k^2} [A + B\cos(kd)] = -i \int_{2m}^{\infty} dq \frac{\text{disc}\, \tilde{\Pi}(q^2)}{q\sqrt{q^2 - \Gamma^2}} [A + Be^{-k_3 d}] \tag{18}$$

($k_3 = \sqrt{q^2 - \Gamma^2}$) which entails a change of variable we obtain

$$\Delta E_0 = -4a \int \frac{d^3k}{(2\pi)^3} \int_{2m}^{\infty} \frac{dq}{2\pi} \frac{\text{disc}\, \tilde{\Pi}(q^2)}{q\sqrt{q^2 - \Gamma^2}} \times \frac{\Gamma(\Gamma - ia + iae^{-k_3 d} e^{i\Gamma d})}{(\Gamma - ia)^2 + a^2 \exp(2i\Gamma d)}. \tag{19}$$

Furthermore, the Wick rotation $k_0 \to ik_4$, $\Gamma \to i\gamma = i\sqrt{k_4^2 + k_1^2 + k_2^2}$ can be performed. The resulting expression for ΔE_0 still contains ultraviolet divergencies which, however, do not depend on the distance between the mirrors (these divergencies arise from immersing single mirrors into the vacuum). After subtracting these divergent terms the finite, distance-dependent part of ΔE_0 finally reads

$$\Delta E_0 = -4ia \int \frac{d^3k_E}{(2\pi)^3} \int_{2m}^{\infty} \frac{dq}{2\pi} \frac{\text{disc}\, \tilde{\Pi}(q^2)}{q\sqrt{q^2 + \gamma^2}} \times \frac{\gamma a e^{-\gamma d}[ae^{-\gamma d} + (\gamma - a)e^{-k_3 d}]}{(\gamma - a)[(\gamma - a)^2 - a^2 \exp(-2\gamma d)]}. \tag{20}$$

In the limit $a \to -\infty$, which can be performed in the integrand, the expression for impenetrable mirrors, with Dirichlet boundary conditions, is reobtained.

We are interested in the leading term of Eq. (20) for $d/\lambda_c = md \gg 1$. Then, in Eq. (20) the contribution containing $\exp(-k_3 d)$ [$\leq \exp(-2md)$] in the nominator can be neglected at leading order. Also, in the denominator we can approximate $\sqrt{q^2 + \gamma^2}$ by q. So the integral related to the fermion loop decouples in this approximation and with

$$\int_{2m}^{\infty} dq \, \frac{\text{disc } \tilde{\Pi}(q^2)}{q^2} = -\frac{3i\pi}{32} \frac{\alpha}{m} \tag{21}$$

we obtain to leading order in $1/md$

$$\Delta E_0 = -\frac{3}{32\pi^2} \frac{\alpha a^3}{md} \int_0^{\infty} d\gamma \times \frac{\gamma^3 e^{-2\gamma}}{(\gamma - ad)[(\gamma - ad)^2 - (ad)^2 \exp(-2\gamma)]} . \tag{22}$$

It looks structurally quite similar to the analogous expression (4.4) in [18] for the Casimir energy E_0 itself. As a function of a it is monotonic (for $a < 0$). The limiting cases are

$$\Delta E_0 \overset{ad \to 0}{=} \frac{\alpha}{md} \frac{3}{64\pi^2} (-ad)^3 + \ldots \tag{23}$$

and

$$\Delta E_0 \overset{ad \to -\infty}{=} \frac{\alpha}{md} \frac{\pi^2}{2560} \frac{1}{d^3} \left(1 - \frac{\sigma}{-ad} + \ldots \right) \tag{24}$$

with $\sigma = 4[1 + 45\zeta(5)/\pi^4] \sim 5.92$.

Using the expression for the Casimir energy E_0 obtained in [18] [in Eq. (4.7) in [18] a factor of $1/2$ is missing], the relative weight of the radiative correction ΔE_0, i.e., the ratio (2), can be shown to be a monotonic function of a with the limiting cases

$$\frac{\Delta E_0}{E_0} \overset{ad \to 0}{=} \frac{\alpha}{md} \frac{3}{16} ad + \ldots \tag{25}$$

and

$$\frac{\Delta E_0}{E_0} \overset{ad \to -\infty}{=} \frac{\alpha}{md} \frac{-9}{32} \left(1 - \frac{\sigma - 3}{-ad} + \ldots \right). \tag{26}$$

From Eqs. (24) and (26), which apply in the physically interesting domain $-ad \gg 1$, we deduce the result stated above, that the $O(\alpha)$ radiative correction to the Casimir energy experiences a powerlike modification for penetrable mirrors.

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