Theory of Disordered Systems

Prof. Dr. Igor Sokolov

Lecture 4: Duality relations in 2d

In two dimensions there exist several exact results which solely base on the statistical isotropy of the system (J.B. Keller, J. Math. Phys. **5**, 548 (1964), was the first to discuss the problem in a relatively restricted setup; A.M. Dykhne, 1971, gave its independent and more general analysis; our discussion follows to a large extent the work of K.S. Mendelson, J. Appl. Phys., **46** 4740 (1975)).

Our starting point are the usual relations

$$\mathbf{j}(\mathbf{r}) = \hat{\sigma}(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{1}$$

where $\hat{\sigma}(\mathbf{r})$ is the tensor of local conductivity. At the beginning we concentrate on a scalar case $\hat{\sigma}(\mathbf{r}) = \sigma(\mathbf{r})\hat{I}$, where $\hat{I}$ is a unit matrix. As always,

$$\mathrm{rot}\mathbf{E} = 0 \tag{2}$$

and

$$\mathrm{div}\mathbf{j} = 0. \tag{3}$$

On the macroscopic level we have

$$\mathbf{J} = \sigma^* \mathbf{E}_0, \tag{4}$$

with σ^* being isotropic no matter what $\sigma(\mathbf{r})$ is. We note that $\mathbf{J}$ and $\mathbf{E}_0$ are the volume means of the corresponding $\mathbf{j}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$.

We consider the case of a two-component medium with phases 1 (relative volume $v_1 = p$) and 2 (relative volume $v_2 = 1 - p$) with conductivities σ_1 and σ_2. The effective conductivity of the system is then depends on tree parameters σ_1, σ_2 and p and is given by the function $\sigma^* = \sigma^*(\sigma_1, \sigma_2, p)$.

Let us introduce an orthogonal operator of 90° rotation (i.e. simply rename the axes $x \to y$ and $y \to -x$)

$$\hat{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and its inverse

$$\hat{R}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us now introduce some (yet free) constant σ_* of the dimension of conductivity and introduce the auxiliary fields

$$\begin{aligned} \mathbf{j}'(\mathbf{r}) &= \sigma_* \hat{R}\mathbf{E}(\mathbf{r}) \\ \mathbf{E}'(\mathbf{r}) &= \frac{1}{\sigma_*}\hat{R}\mathbf{j}(\mathbf{r}). \end{aligned}$$

Since for any vector in 2d div$\mathbf{V} = \partial V_x/\partial x + \partial V_y/\partial y$ and rot$\mathbf{V} = \partial V_y/\partial x - \partial V_x/\partial y$ (rotor in 2d is perpendicular to the plane in which the field acts, and can be considered as a scalar), Eqs. (2) and (3) are mapped onto

$$\text{rot}\mathbf{E}' = 0$$

and

$$\text{div}\mathbf{j}' = 0.$$

This means that $\mathbf{j}'$ and $\mathbf{E}'$ can be considered as realizations of a current and of an electric field in some system, the one *dual* to the initial one (one still must show that they fulfill reasonable boundary conditions, but the really do). Inverting the relation Eq.(1)

$$\mathbf{E}(\mathbf{r}) = \sigma^{-1}(\mathbf{r})\mathbf{j}(\mathbf{r}),$$

and acting with the operator $\hat{R}$ on the both sides of the previous equation we get

$$\begin{aligned} \mathbf{j}'(\mathbf{r}) &= \sigma_* \hat{R}\mathbf{E}(\mathbf{r}) = \sigma_* \hat{R}\sigma^{-1}(\mathbf{r})\mathbf{j}(\mathbf{r}) \\ &= \sigma_* \hat{R}\sigma^{-1}(\mathbf{r})\hat{R}^{-1}\hat{R}\mathbf{j}(\mathbf{r}) \\ &= \sigma_* \hat{R}\sigma^{-1}(\mathbf{r})\hat{R}^{-1}\sigma_*\mathbf{E}'(\mathbf{r}) \end{aligned}$$

i.e.

$$\mathbf{j}'(\mathbf{r}) = \sigma'(\mathbf{r})\mathbf{E}'(\mathbf{r})$$

with

$$\sigma'(\mathbf{r}) = \sigma_*^2\sigma^{-1}(\mathbf{r}).$$

Note that the same can be done for the general tensor form of local conductivities:

$$\mathbf{j}'(\mathbf{r}) = \sigma_* \hat{R}\hat{\sigma}^{-1}(\mathbf{r})\hat{R}^{-1}\sigma_*\mathbf{E}'(\mathbf{r})$$

i.e.

$$\mathbf{j}'(\mathbf{r}) = \hat{\sigma}'(\mathbf{r})\mathbf{E}'(\mathbf{r})$$

with

$$\hat{\sigma}'(\mathbf{r}) = \sigma_*^2 \hat{R}\hat{\sigma}^{-1}(\mathbf{r})\hat{R}^{-1}. \tag{5}$$

The new conductivity $\sigma'(\mathbf{r})$ can be considered as a conductivity of the geometrically dual medium, where the local conductivities are defined yet up to an arbitrary constant. For this medium the volume mean electric field and the volume mean current density are connected by

$$\mathbf{J}' = \sigma^{*\prime}\mathbf{E}_0',$$

where, clearly, $\mathbf{J}' = \sigma_* \hat{R}\mathbf{E}_0$ and $\mathbf{E}_0' = (1/\sigma_*)\hat{R}\mathbf{J}$, so that $\sigma_*\hat{R}\mathbf{E}_0 = \sigma^{*\prime}(1/\sigma_*)\hat{R}\mathbf{J}$ or

$$\mathbf{J} = \frac{(\sigma_*)^2}{\sigma^{*\prime}}\mathbf{E}_0, \tag{6}$$

which has to be compared with Eq.(4).

Let us now consider a two-component medium with phases 1 and 2 with conductivities σ_1 and σ_2, and let us take $\sigma_*^2 = \sigma_1\sigma_2$, so that the dual medium has the conductivity σ_2 in phase 1 and σ_1 in phase 2, i.e. the medium where the conductivities in the two phases are interchanged. For this medium $\sigma^{*\prime} = \sigma^*(\sigma_2, \sigma_1, p) = \sigma^*(\sigma_1, \sigma_2, 1-p)$, so that the equations (4) and (6) read

$$\mathbf{J} = \sigma^*(\sigma_1, \sigma_2, p)\mathbf{E}_0, \qquad \mathbf{J} = \frac{\sigma_1\sigma_2}{\sigma^*(\sigma_1, \sigma_2, 1-p)}\mathbf{E}_0$$

from which it follows that

$$\sigma^*(\sigma_1, \sigma_2, p)\sigma^*(\sigma_1, \sigma_2, 1-p) = \sigma_1\sigma_2$$

for any p.

In general, since we work in rotated geometries, the first conductivity is measured when the electric filed $\mathbf{E}_0$ is applied, say in x-direction, while in the dual (rotated) medium it acts in y-direction, i.e. the first $\sigma_{xx}^*(p) = \sigma^*(\sigma_1, \sigma_2, p)$ is measured in x-direction, and the second one, $\sigma_{yy}^*(1-p) = \sigma^*(\sigma_2, \sigma_1, 1-p)$ in y-direction.

Note: To appreciate the general character of this statement let us consider a layered system with layers parallel to x-axis. In this case σ_{xx}^* corresponds to the parallel switching of the layers,

$$\sigma_{xx}^*(p) = p\sigma_1 + (1-p)\sigma_2$$

and

$$\sigma^*_{yy}(1-p) = \left(\frac{1-p}{\sigma_1} + \frac{p}{\sigma_2}\right)^{-1}$$

to the sequential switching of the layers in a system with interchanged components 1 and 2. The product of these expressions,

$$[p\sigma_1 + (1-p)\sigma_2]\left[\frac{p\sigma_1 + (1-p)\sigma_2}{\sigma_1\sigma_2}\right]^{-1} = \sigma_1\sigma_2.$$

Now let us consider a symmetric composition, $p = 1/2$ which is isotropic on the average. In this case $\sigma^*_{xx}(p)$ and $\sigma^*_{yy}(1-p)$ which give different components of the conductivity tensor of our system, have to be equal. Therefore $\sigma^*(\sigma_1, \sigma_2, 1/2) = \sigma^*(\sigma_1, \sigma_2, 1/2) = \sigma^*$ and

$$(\sigma^*)^2 = \sigma_1\sigma_2,$$

i.e.

$$\sigma^* = \sqrt{\sigma_1\sigma_2}.$$

This relation looks like the perturbative result in 2d (see Homework 1) but is *exact*.

There is another situation leading to an exact (and the same!) result, namely on of a polycrystalline medium with anisotropic crystallites of random orientation. Here we have to return to Eq.(5) and assume that the local conductivity tensor has a form

$$\hat{\sigma}(\mathbf{r}) = \hat{\Theta}(\mathbf{r}) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \hat{\Theta}(\mathbf{r})^{-1}$$

where $\hat{\Theta}(\mathbf{r})$ is an orthogonal matrix giving the orientation of the main axis of a crystallite at position $\mathbf{r}$,

$$\hat{\Theta}(\mathbf{r}) = \begin{pmatrix} \cos\phi(\mathbf{r}) & -\sin\phi(\mathbf{r}) \\ \sin\phi(\mathbf{r}) & \cos\phi(\mathbf{r}) \end{pmatrix}$$

with $\phi(\mathbf{r})$ denoting angle between the main axis of the crystallite at $\mathbf{r}$ and the x-direction, i.e.

$$\hat{\sigma}(\mathbf{r}) = \begin{pmatrix} \sigma_1\cos^2\phi(\mathbf{r}) + \sigma_2\sin^2\phi(\mathbf{r}) & (\sigma_1-\sigma_2)\cos\phi(\mathbf{r})\sin\phi(\mathbf{r}) \\ (\sigma_1-\sigma_2)\cos\phi(\mathbf{r})\sin\phi(\mathbf{r}) & \sigma_1\sin^2\phi(\mathbf{r}) + \sigma_2\cos^2\phi(\mathbf{r}) \end{pmatrix}.$$

Returning to Eq.(5) we get

$$\begin{aligned}\hat{\sigma}'(\mathbf{r}) &= \sigma_*^2 \hat{R}\hat{\sigma}^{-1}(\mathbf{r})\hat{R}^{-1} \\ &= \sigma_*^2 \frac{1}{\sigma_1\sigma_2} \begin{pmatrix} \sigma_1 \cos^2\phi(\mathbf{r}) + \sigma_2 \sin^2\phi(\mathbf{r}) & (\sigma_1 - \sigma_2)\cos\phi(\mathbf{r})\sin\phi(\mathbf{r}) \\ (\sigma_1 - \sigma_2)\cos\phi(\mathbf{r})\sin\phi(\mathbf{r}) & \sigma_1 \sin^2\phi(\mathbf{r}) + \sigma_2 \cos^2\phi(\mathbf{r}) \end{pmatrix}\end{aligned}$$

Taking now $\sigma_*^2 = \sqrt{\sigma_1\sigma_2}$ we see that $\hat{\sigma}'(\mathbf{r}) = \hat{\sigma}(\mathbf{r})$, i.e. is the tensor of the same polycrystalline medium (measured, however, in the perpendicular direction). For a polycrystalline medium which is isotropic on the average (the angles ϕ are equidistributed on $[0, 2\pi)$) the effective conductivity tensor $\hat{sigma}^*$ has to be diagonal and to have equal diagonal components, i.e. can be represented as a scalar. This means that if we take $\sigma_*^2 = \sqrt{\sigma_1\sigma_2}$, the mean conductivity $\sigma^{*\prime}$ of the dual system has to be the same as for the initial one. From equations (4) and (6) it follows again that

$$\sigma^* = \sqrt{\sigma_1\sigma_2}.$$